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A closure operator versus purity

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Abstract. Any notion of purity is normally defined in terms of solvability of some set of equations. To study mathematical notions, such as injectivity, tensor products, flatness, one needs to have some categorical and algebraic information about the pair ($\mathcal{A}$,$\mathcal{M}$), for a category $\mathcal{A}$ and a class $\mathcal{M}$ of monomorphisms in a category $\mathcal{A}$. In this paper we take $\mathcal{A}$ to be the category **Act-S** of S-acts, for a semigroup S, and $\mathcal{M}_{sp}$ to be the class of C_I^{sp}-pure monomorphisms and study some categorical and algebraic properties of this class concerning the closure operator C_I^{sp}.

Keywords: Semigroup, closure operator, purity, *S-act*.

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1. Introduction and preliminaries

One usually takes a subclass $\mathcal{M}$ of monomorphisms in a category $\mathcal{A}$, members of which may be called $\mathcal{M}$-morphisms.

In this paper we take $\mathcal{A}$ to be the category **Act-S** of S-acts and homomorphism between them, for a semigroup S, and introduce a new class, denoted by $\mathcal{M}_{sp}$, to be the class of C_I^{sp}-pure monomorphisms and study some algebraic properties of them. In section 2 we introduce a new closure operator, denoted by C^{sp}, which has a closely related by purity.

Let us first recall the definition and some ingredients of the category **Act-S** needed in the sequel. For more information and other related notions, see [2, 4, 11].

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Recall that, for a semigroup S, a set A is an S-*act* (or an S-*set*) if there is a, so called, *action* $\mu : A \times S \to A$ such that $\mu(a, s) := as$, $a(st) = (as)t$ and if S is a monoid with 1, $a1 = a$.

Each semigroup S can be considered as an S-act with the action given by its multiplication. Notice that adjoining an external left identity 1 to a semigroup S an S-act $S^1 := S \cup \{1\}$ is obtained.

Also, recall that an element $a \in A$ is said to be *fixed* (or a *zero*) if $as = a$ for all $s \in S$. The S-act $A \cup \{0\}$ with a zero adjoined to A is denoted by A^0. A fixed element of the semigroup S as an S-act is called a *left zero* of the semigroup.

The definitions of a *homomorphism of S-acts* or *S-maps*, *subact* A of B, written as $A \leqslant B$, an *extension* of A, a *congruence* ρ on A, a *quotient* A/ρ of A, and a *homomorphism* (S-maps) between S-acts are all clear.

Since the class of S-acts is an equational class, the category **Act-S** is complete and cocomplete (has all products, equalizers, pullbacks, coproducts, coequalizers, and pushouts). In fact, these are calculated in the category **Set** of sets and are equipped with a natural action.

In particular, the terminal object of **Act-S** is the singleton $\{0\}$ with the obvious S-action. Also, for S-acts A, B, their cartesian product $A \times B$ with the S-action defined by $(a, b)s = (as, bs)$ is the *product* of A and B in **Act-S**.

All colimits in **Act-S** exist and are calculated as in **Set** with the natural action of S on them. In particular, $\emptyset$ with the empty action of S on it, is the initial object of **Act-S**. Also, the *coproduct* of S-acts A, B is their disjoint union $A \sqcup B = (A \times \{1\}) \cup (B \times \{2\})$ with the obvious action, and coproduct injections are defined naturally.

Recall that for a family $\{A_i : i \in I\}$ of S-acts, each with a unique fixed element 0, the *direct sum* $\bigoplus_{i\in I} A_i$ is defined to be the subact of the product $\prod_{i\in I} A_i$ consisting of all $(a_i)_{i\in I}$ such that $a_i = 0$ for all $i \in I$ except a finite number of indices.

2. C^{sp}-purity

In this section we introduce a closure operator which is closely related to one kind of purity (see [12]). First recall the following definition of a categorical closure operator from [6] (also, see [5, 10]). Denoting the lattice of all subacts of an S-act B by $Sub(B)$, we have:

Definition 2.1 The family $C = (C_B)_{B\in \mathbf{Act-S}}$ with $C_B : Sub(B) \to Sub(B)$, taking any subact $A \leqslant B$ to a subact $C_B(A)$ (or $C(A)$, if no confusion arises) is called a *closure operator* on **Act-S** if it satisfies the followings:
(c_1) (*Extension*) $A \leqslant C(A)$,
(c_2) (*Monotonicity*) $A_1 \leqslant A_2 \leqslant B$ implies $C(A_1) \leqslant C(A_2)$,
(c_3) (*Continuity*) $f(C_B(A)) \leqslant C_C(f(A))$ for all morphisms $f : B \to C$.

Now, one has the usual two classes of monomorphisms related to any closure operator as follows (also, see [3]).

Definition 2.2 Let $A \leqslant B$ be in **Act-S**. We say that A is C-*closed* in B if $C(A) = A$, and it is C-*dense* in B if $C(A) = B$. Also, an S-map $f : A \to B$ is said to be C-dense (C-closed) if $f(A)$ is a C-dense (C-closed) subact of B.

Dikranjan and Tholen in [6] state some properties of a closure operator in general. Here we investigate those for the closure operator C satisfy or not.

Definition 2.3 The closure operator C is said to be:

(1) idempotent, if for $A \subseteq B$, $C_B(A) = C_B(C_B(A))$.
(2) hereditary, if for $A_1 \subseteq A_2 \subseteq B$, $C_{A_2}(A_1) = C_B(A_1) \cap A_2$.
(3) weakly hereditary, if for every $A \subseteq B$, $C_{C_B(A)}(A) = C_B(A)$.
(4) grounded, if $C_B(\emptyset) = \emptyset$.
(5) additive, if for subacts A, C of B, $C_B(A \bigcup C) = C_B(A) \bigcup C_B(C)$.
(6) productive, if for every family of subacts A_i of B_i, taking $A = \prod_i A_i$ and $B = \prod_i B_i$, $C_B(A) = \prod_i C_{B_i}(A_i)$.
(7) fully additive, if for $A_i \subseteq B$, $C_B^{sd}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} C_B^{sd}(A_i)$.
(8) discrete, if $C_B(A) = A$ for every S-act B and $A \subseteq B$.
(9) trivial, if $C_B(A) = B$ for every B and $A \subseteq B$.
(10) minimal, if for $C \subseteq A \subseteq B$ one has $C_B(A) = A \cup C_B(C)$.

Now, we recall the following closure operator needed in the sequel and has been studied in [1, 7] to study a kind of injectivity. The references [8, 9] can be also used to study a specific case of projectivity.

Definition 2.4 For any ideal I of S and subact A of an S-act B, define a closure operator C_I^d by

$$C_I^d(A) = \{b \in B : bI \subseteq A\}.$$

Now, note that A is C_I^d-dense (or simply s-dense) in an extension B of A if $C_I^d(A) = B$, that is, for every $b \in B$, $bI \subseteq A$.

We now introduce another closure operator on **Act-S**, denoted by C_I^{sp}-clsure operator, and study some algebraic and categorical properties of them.

Definition 2.5 For any ideal I of S and subact A of an S-act B, the C_I^{sp} *closure operator* on **Act-S** is defined as

$$C_I^{sp}(A) = \{b \in B : \exists a \in A, \lambda_b^I = \lambda_a^I\},$$

where $\lambda_x^I : I \to A$ is defined by $\lambda_x^I(s) = xs$.

Now, note that A is C_I^{sp}-dense in an extension B of A if $C_I^{sp}(A) = B$ (this means that for every $b \in B$ there is an $a \in A$ with $\lambda_b^I = \lambda_a^I$; that is, $bs = as$ for every $s \in I$). Also A is C_I^{sp}-closed in B if $C_I^{sp}(A) = A$; that is, for every $b \in B - A$ and $a \in A$ there is an $s \in I$ with $bs \neq as$.

Lemma 2.6 C_I^{sp}-closedness is preserved by inverse image of S-maps and C_I^{sp}-denseness is preserved by images of onto S-maps.

Proof. let $f : B \to D$ be a homomorphism and X be a C_I^{sp}-closed subact of D. By definition $f^{-1}(X) \subseteq C_I^{sp} f^{-1}(X)$. Consider $b \in C_I^{sp} f^{-1}(X)$, so there exists $a \in f^{-1}(X)$ such that $\lambda_a^I = \lambda_b^I$. Thus $\lambda_{f(a)}^I = \lambda_{f(b)}^I$. Since X is a C_I^{sp}-closed subact of D, $f(b) \in X$, which implies $b \in f^{-1}(X)$ and hence $f^{-1}(X)$ is a C_I^{sp}-closed subact of B.

For the second part, let $f : B \to D$ be an epimorphism and A be a C_I^{sp}-dense subact of B. Let $d \in D$. Since f is onto and A is C_I^{sp}-dense subact of B, there exists $b \in B$ and $a \in A$ such that $d = f(b)$ and $\lambda_a^I = \lambda_b^I$. So $\lambda_{f(a)}^I = \lambda_d^I$ and hence $f(A)$ is C_I^{sp}-dense subact of D. ■

Lemma 2.7 If $A_1 \subseteq A_2 \subseteq B$, then $(C_{A_2})_I^{sp}(A_1) \subseteq (C_B)_I^{sp}(A_1)$.

Some easily proved properties of this last closure operator is stated in the following:

Lemma 2.8 C_I^{sp} is: (1) a closure operator, (2) idempotent, (3) hereditary, (4) weakly hereditary, (5) grounded, (6) additive, (7) fully additive, (8) productive.

Also, some of the properties that C_I^{sp} does not satisfy in general are:

Example **2.9** For any semigroup S, C_I^{sp} is not: (1) discrete, (2) trivial, (3) minimal.

Proof. Let $0 \in A$ be a fixed element of A, and adjoin two elements θ, ω to A with actions $\omega s = \omega$ and $\theta s = 0$. Then $(C_B)_I^{sp}(A) = A \cup \{\theta\}$ where $B = A \cup \{\theta, \omega\}$. Hence C_I^{sp} is neither discrete nor trivial. Also, it is not minimal. Because, adjoining two elements θ, ω to an S-act C with actions $\omega s = \theta$ and $\theta s = \theta$, and taking $A = C \cup \{\theta\}$, $B = C \cup \{\theta, \omega\}$, we get $C \subset A \subset B$, and $(C_B)_I^{sp}(A) = B$ while $(C_B)_I^{sp}(C) = C$. ■

Another monomorphism which corresponds to this closure operator, and is the main interest of this paper, is defined as follows:

Definition 2.10 An S-act A is said to be C_I^{sp}*-pure* in an extension B of A if $C_I^{sp}(A) = C_I^d(A)$.

Definition 2.11 A monomorphism $f : A \to B$ is called a C_I^{sp}*-pure* if f(A) is a C_I^{sp}*-pure* subact of B.

Proposition 2.12 Let A_i be a family of subacts of A,
(i) $(C_A)_I^{sp}(\bigcap A_i) \subseteq \bigcap (C_A)_I^{sp}(A_i)$.
(ii) If $\bigcap A_i$ is $(C_A)_I^{sp}$-pure in A, then $(C_A)_I^{sp}(\bigcap A_i) = \bigcap (C_A)_I^{sp}(A_i)$.
(iii) If each A_i is $(C_A)_I^{sp}$-pure in A and $(C_A)_I^{sp}(\bigcap A_i) = \bigcap (C_A)_I^{sp}(A_i)$, then $\bigcap A_i$ is $(C_A)_I^{sp}$-pure in A.

Proof. (i) This is trivial.
(ii) Let $b \in \bigcap (C_A)_I^{sp}(A_i)$. Then for each $A_i \in I$, there exists $a_i \in A_i$ such that $\lambda_b^I = \lambda_{a_i}^I$ and hence $bI \subseteq \bigcap A_i$. Now, since $\bigcap A_i$ is $(C_A)_I^{sp}$-pure in A, there exists $a \in \bigcap A_i$ that $\lambda_b^I = \lambda_{a_i}^I$, which deduces that $b \in (C_A)_I^{sp}(\bigcap A_i)$.
(iii) Let $bI \subseteq \bigcap A_i$. So for each A_i, since $\bigcap A_i$ is $(C_A)_I^{sp}$-pure in A, $b \in (C_A)_I^{sp}(A_i)$. Now, by hypothesis, $b \in (C_A)_I^{sp}(\bigcap A_i)$. ■

Remark 1 *For* $A \leqslant B$, *we have* $A \leqslant C_I^{sp}(A) \leqslant C_I^d(A) \leqslant B$. *So, if* A *is* C_I^{sp}*-dense in* B, *then* $C_I^{sp}(A) = C_I^d(A) = B$ *and so* A *is* C_I^d*-dense as well as* C_I^{sp}*-pure. Similarly, if* A *is* C_I^d*-closed in* B, *then* $A = C_I^{sp}(A) = C_I^d(A)$ *and hence* A *is* C_I^{sp}*-closed as well as* C_I^{sp}*-pure.*

Proposition 2.13 (1) Any retraction is $(C_A)_I^{sp}$-pure.
(2) Any C_S^{sp}-dense monomorphism is a retraction.

Proof. : (1) Let $A \hookrightarrow B \xrightarrow{\pi} A = id_A$ be a retraction, I a right ideal of S and $bI \subseteq A$ for some $b \in B$. It is clear that for each $s \in I$, $bs = \pi(bs) = \pi(b)s$.
(2) Let $A \hookrightarrow B$ be a C_S^{sp}-dense subact. Then it is C_S^{sp}-pure as well as C_S^d-dense subact. So, for every $b \in B$ there exists $a_b \in A$ such that $\lambda_b = \lambda_{a_b}$. Now, for every $b \in B - A$ choose and fix such an $a_b \in A$. Define $\pi : B \to A$ by

$$\pi(x) = \begin{cases} x, & \text{if } x \in A \\ a_x, & \text{if } x \notin A \end{cases}$$

Then, clearly π is a retraction. It is a homomorphism because it is a homomorphism on A, and for $x \in B - A$ and $s \in S$, we have $xs \in A$ and so $\pi(xs) = xs = a_x s = \pi(x)s$. ■

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