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A Chebyshev functions method for solving linear and nonlinear fractional differential equations based on Hilfer fractional derivative

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Abstract. The theory of derivatives and integrals of fractional in fractional calculus have found enormous applications in mathematics, physics and engineering so for that reason we need an efficient and accurate computational method for the solution of fractional differential equations. This paper presents a numerical method for solving a class of linear and nonlinear multi-order fractional differential equations with constant coefficients subject to initial conditions based on the fractional order Chebyshev functions that this function is defined as follows:

$$\overline{T}^{\alpha}_{i+1}(x) = (4x^{\alpha} - 2)\overline{T}^{\alpha}_{i}(x)\overline{T}^{\alpha}_{i-1}(x),\ i = 0, 1, 2, \ldots,$$

where $\overline{T}^{\alpha}_{i+1}(x)$ can be defined by introducing the change of variable x^{α}, $\alpha > 0$, on the shifted Chebyshev polynomials of the first kind. This new method is an adaptation of collocation method in terms of truncated fractional order Chebyshev Series. To do this method, a new operational matrix of fractional order differential in the Hilfer sense for the fractional order Chebyshev functions is derived. By using this method we reduces such problems to those of solving a system of algebraic equations thus greatly simplifying the problem. At the end of this paper, several numerical experiments are given to demonstrate the efficiency and accuracy of the proposed method.

Keywords: Chebyshev polynomials, fractional-order differential equation, operational matrix, Hilfer fractional derivative.

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1. Introduction

In this paper, we focus on a direct solution technique for solving the linear and nonlinear multi-order fractional differential equations in the sense of the Hilfer fractional derivative using a collocation method and we develop a new and efficient approach to obtain the numerical solution of the general linear and nonlinear FDEs. For this multi-order fractional differential equation, we consider two cases as follows:
Case 1. Consider the linear multi-order fractional system FDE as follows:

$$\mathbf{D}_{0^+}^{\mu,\nu} y(x) = a_1 \mathbf{D}_{0^+}^{\mu_1,\nu_1} y(x) + \cdots + a_k \mathbf{D}_{0^+}^{\mu_k,\nu_k} y_k(x) + a_{k+1} y(x) + a_{k+2} g(x), \tag{1}$$

with initial conditions

$$y^i(0) = d_i, \quad i = 0, 1, 2, \cdots, m, \tag{2}$$

where $a_j,\ j = 1, 2, \cdots, k+2$, are real constant coefficients and also $m < \mu < m+1,\ m \leqslant \nu \leqslant m+1,\ 0 < \mu_1 < \mu_2 < \cdots < \mu_k < \mu,\ 0 < \nu_1 < \nu_2 < \cdots < \nu_k < \nu$.
Case 2. Consider the nonlinear multi-order fractional system FDE as follows:

$$F(x, y(x), \mathbf{D}_{0^+}^{\mu_1,\nu_1} y(x), \cdots \mathbf{D}_{0^+}^{\mu_k,\nu_k} y(x)) = 0, \tag{3}$$

with boundary or supplementary conditions

$$\begin{gathered} \mathcal{H}_i\Big(y(\zeta_i) y'(\zeta_i), \ldots, y^{(q)}(\zeta_i)\Big) = d_i, \quad i = 0, 1, 2, \cdots, q, \\ \zeta_i \in [0,1],\ 0 \leqslant q < \max\{\mu_i, \nu_i,\ i = 1, 2, \ldots, k\} \leqslant q+1, \end{gathered} \tag{4}$$

where $m < \mu < m+1,\ m \leqslant \nu \leqslant m+1,\ 0 < \mu_1 < \mu_2 < \cdots < \mu_k < \mu,\ 0 < \nu_1 < \nu_2 < \cdots < \nu_k < \nu$ and $\mathcal{H}_i$ are linear combinations of $y(\zeta_i) y'(\zeta_i), \ldots, y^{(q)}(\zeta_i)$.
According to the above relations $\mathbf{D}_{0^+}^{\mu,\nu}$ is the Hilfer fractional derivative of order μ and type ν and for an absolutely integrable function $y(t)$ is defined by

$$\begin{gathered} \Big({}_{0^+}\mathbf{D}_t^{\mu,\nu} y\Big)(t) = \Big({}_{0^+}\mathbf{I}_t^{\nu(1-\mu)} \frac{d}{dt} {}_{0^+}\mathbf{I}_t^{(1-\nu)(1-\mu)} y\Big)(t), \\ 0 < \mu < 1,\ 0 \leqslant \nu \leqslant 1, \end{gathered} \tag{5}$$

where ${}_{0^+}\mathbf{I}_t^{\nu(1-\mu)}$ denotes the fractional integral of order $\nu(1-\mu)$ with starting point 0, the so-called Riemann-Liouville introduced in Section 2.
In recent years, the fractional calculus (FC) draws increasing attention due to its applications in science , engineering, physicists, biologists, engineers and economists and it the branch of Mathematics that generalizes the derivative and the integral of a function to a non-integer order and many applications of fractional calculus amount to replacing the time derivative in an evolution equation with a derivative of fractional order [4, 16, 17, 20, 21, 29, 30]. The theory of the fractional calculus have been playing important role in analytical and numerical solutions for the fractional problems and fractional differential equations (FDEs). Fractional differential equations (FDEs) powerful tools for modeling phenomena in mathematical and interest due to their important applications in science and engineering . The increasing applicability of FDEs has required efficient algorithms for calculating their solutions and the analytical solutions of most

FDEs are not easy to obtain, therefore, seeking numerical solutions of these equations becomes more and more important an widely used in science and engineering. In this paper, first we introduce a novel generalization of derivatives of both Riemann-Liouville and Caputo types, which is called Hilfer fractional derivative and indicated by the symbol $\mathbf{D}_{0^+}^{\mu;\nu}$ and as specific cases for $\nu = 0$ coincides, with the Riemann-Liouville derivative and for $\nu = 1$ with the Caputo derivative and second, we propose an accurate numerical algorithm for calculating the solutions of different classes of FDEs with initial and also using the collocation method with the Hilfer fractional derivative of the fractional-order Chebyshev functions and we reduce the FDEs to systems of algebraic equations. During the last decades, several methods have been used to solve fractional differential equations for example, finite difference method [11, 13, 27], spline and B-spline collocation methods [14, 18], homotopy analysis method [22], Taylor collocation method [25], Wavelet method [15], Legendre wavelet method, collocation method based on Jacobi, Laguerre and Legendre polynomials [2, 3, 19, 26], Boubaker polynomial approach [23], Bernoulli polynomial approach [6, 28]. In the present paper, we construct the fractional order Chebyshev functions (FCFs) and then derive the operational matrix of fractional order FCFs and apply it to solve FDEs. The rest of the paper is organized as follows. In Section 2, we introduce some necessary definitions and give some relevant properties FCFs which are required for establishing our results and also, in this section, we make a new operational matrix for fractional derivative by FCFs. An application of the method for linear and nonlinear differential equation is are given in Section 3. In Section 4 the proposed methods are applied to several examples.

2. Preliminaries

In this section, we begin with some preliminary definitions of fractional calculus and properties of Chebyshev and shifted Chebyshev polynomials which we use later.

Definition 2.1 [12, 17] The Riemann-Liouville fractional integral of order α for an absolutely integrable function $f(t)$ is defined by

$$\mathbf{I}_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad \alpha > 0, \tag{6}$$

and also, the Riemann-Liouville fractional derivative of order α is defined by

$$\mathbb{D}_{0^+}^{\alpha} f(t) = \frac{d^m}{dt^m} \mathbf{I}_{0^+}^{m-\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\alpha-1} f(\tau) d\tau, \tag{7}$$

where $\Gamma(.)$ is the gamma function and $m-1 \leqslant \alpha \leqslant m$, $m \in \mathbb{N}$.

Definition 2.2 [12] The Caputo fractional integral of order α is defined by

$$^{C}\mathbf{D}_{a^+}^{\alpha} u(t) = \mathbf{I}_{a^+}^{m-\alpha} \frac{d^m}{dt^m} f(t), \ \alpha > 0, \ m = [\alpha]. \tag{8}$$

Definition 2.3 [12, 24] Let $0 < \mu < 1, \ 0 < \nu < 1, \ f \in L^1[a,b], \ I_{a^+}^{(1-\nu)(1-\mu)} f \in AC[a,b]$, where $AC[a,b]$ is the space of absolutely continuous functions. The Hilfer fractional derivative is defined by

$$\mathbf{D}_{a^+}^{\mu,\nu} f(t) = \left(\mathbf{I}_{a^+}^{\nu(1-\mu)} \frac{d}{dt} (\mathbf{I}_{a^+}^{(1-\nu)(1-\mu)} f)\right)(t). \tag{9}$$

Similar to integer-order differentiation, Hilfer fractional differentiation is a linear operator

$$\mathbf{D}_{0^+}^{\mu,\nu}(\lambda f(x) + \gamma g(x)) = \lambda \mathbf{D}_{0^+}^{\mu,\nu} f(x) + \gamma \mathbf{D}_{0^+}^{\mu,\nu} g(x), \tag{10}$$

where λ and γ are constants. Also for the Hilfer derivative we have

$$\begin{aligned} \mathbf{D}_{0^+}^{\mu,\nu} C &= 0, \ C = constant, \\ \mathbf{D}_{0^+}^{\mu,\nu} x^{\alpha} &= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\mu+1)} x^{\alpha-\mu}. \end{aligned} \tag{11}$$

2.1 *A short overview on Chebyshev and shifted Chebyshev polynomials*

The Chebyshev polynomials of all kinds are widely use in approximation of functions [5, 9, 10]. The Chebyshev polynomial $T_n(t)$ of the first kind is a polynomial in t of degree n, defined by the following relation:

$$T_n(t) = \cos n\theta, \ t = \cos\theta, \ \theta \in [0, \pi], \tag{12}$$

with relation (12), we obtain the following fundamental recurrence relation [7]:

$$T_{n+1}(t) = 2tT_n(t) - T_{n+1}(t), \, T_0(t) = 1, \, T_1(t) = t, \, n = 1, 2, 3, \ldots, \, n \in \mathbb{N}, \tag{13}$$

recursively generates all the polynomials $\{T_n(t)\}$ very efficiently. Let the shifted Chebyshev polynomials $T_n\Big(2t-1\Big)$ be denoted by $T_n^*(t)$. Then can be generated with the aid of the following recurrence formula:

$$\begin{aligned} T_0^*(t) &= 1, \ T_1^*(t) = 2t - 1, \\ T_{n+1}^*(t) &= (4t-2)T_n^*(t) - T_{n-1}^*(t), \ t \in [0, 1], \ n = 1, 2, 3, \ldots. \end{aligned} \tag{14}$$

The analytic form of the shifted Legendre polynomial $T_n^*(t)$ of degree n are given by

$$\begin{aligned} T_n^*(t)(t) &= \sum_{k=0}^{n} f_{n,k} t^k, \\ f_{n,k} &= n(-1)^{n-k} \frac{(n+k-1)!}{(n-k)!} \frac{2^{2k}}{(2k!)}. \end{aligned} \tag{15}$$

The orthogonality condition of the shifted Chebyshev polynomials is

$$\int_0^1 T_i^*(t) T_j^*(t) \omega(t) dt = \delta_{ij} \rho_j, \tag{16}$$

where $\rho_j = \frac{\epsilon_j}{2}\pi,\ \epsilon_0 = 2,\ \epsilon_j = 1,\ j \geqslant 1$ and δ_{mn} is the Kronecker function. A function $f(t) \in L^2_{\omega(t)}[0, \infty)$ can be expressed in terms of the shifted Chebyshev polynomials as:

$$f(t) = \sum_{j=0}^{\infty} c_j T_j^*(t), \tag{17}$$

where the coefficients c_j are given by

$$c_j = \frac{1}{\rho_j} \int_0^1 \omega(t) f(t) T_j^*(t),\ j = 0, 1, 2, \dots. \tag{18}$$

If we approximate $f(t)$ by the first$(m + 1)$-terms of the shifted Chebyshev polynomials, then we have

$$f(t) = \sum_{j=0}^{m} c_j T_j^*(t) = \mathbf{C}^T \Theta_m(t), \tag{19}$$

where the shifted Chebyshev coefficient vecto $\mathbf{C}$ and $\Theta_m(t)$ are given by

$$\mathbf{C}^T = [c_0, c_1, \dots, c_m],\ \Theta_m(t) = [T_0^*(t), T_1^*(t), \dots, T_m^*(t)]^T. \tag{20}$$

Using the equations(20), we can write

$$\Theta_m(t) = \mathbf{F} X_m, \tag{21}$$

where $f_{n,k},\ n, k = 0, 1, 2, \dots, m$ are the matrix entries of $\mathbf{F}$ and $X_m = [1, x, x^2, \dots, x^m]^T$. Due to the orthogonality property of the shifted Chebyshev polynomials (16), the matrix $\mathbf{F}$ is invertible and the vector $X_m(t)$ can be expressed in terms of $\Theta_m(t)$ as:

$$X_m = \mathbf{F}^{-1} \Theta_m(t). \tag{22}$$

Theorem 2.4 The Hilfer fractional derivative of order μ and type ν for the shifted Chebyshev polynomials is given by

$$\mathbf{D}_{0^+}^{\mu,\nu} T_n^*(t) = \sum_{j=0}^{\infty} \mathbb{M}_{\mu,\nu}(n, j) T_j^*(t),\ j = 0, 1, 2, \dots, \tag{23}$$

where

$$\mathbb{M}_{\mu,\nu}(n, j) = \sum_{k=0}^{n} \frac{(-1)^{n-k} 2n(n + k - 1)!\Gamma(k - \mu + \frac{1}{2})}{c_j \Gamma(k + \frac{1}{2})(n - k)!\Gamma(k - \mu - j + 1)\Gamma(k + j - \mu + 1)}. \tag{24}$$

Proof. Using formula (9), (10) and taking the Hilfer fractional derivative of (15) then we have

$$\begin{aligned}\mathbf{D}_{0^+}^{\mu,\nu} T_n^*(t) &= n\sum_{k=0}^{n}(-1)^{n-k}\frac{(n+k-1)!}{(n-k)!}\frac{2^{2k}}{(2k!)}\mathbf{D}_{0^+}^{\mu,\nu} t^k \\ &= n\sum_{k=0}^{n}(-1)^{n-k}\frac{(n+k-1)!}{(n-k)!}\frac{2^{2k}\Gamma(k+1)}{(2k!)\Gamma(k-\mu+1)}t^{k-\mu}. \end{aligned} \tag{25}$$

Now, $t^{k-\mu}$ may be expressed in terms of shifted Chebyshev polynomials series, we get

$$t^{k-\mu} \simeq \sum_{j=0}^{\infty} b_{kj} T_j^*(t), \tag{26}$$

where b_{kj} is given from (18) with $f(t) = t^{k-\mu}$, then we have

$$b_{kj} = \begin{cases} \frac{1}{\sqrt{\pi}}\frac{\Gamma(k-\mu+\frac{1}{2})}{\Gamma(k-\mu+1)} & j=0 \\ \frac{j}{\sqrt{\pi}}\sum_{r=0}^{j}(-1)^{j-r}\frac{(j+r-1)!2^{2r+1}\Gamma(k+r-\mu+\frac{1}{2})}{(j-r)!(2r)!\Gamma(k+r-\mu+1)} & j=1,2,\ldots. \end{cases} \tag{27}$$

By using Eqs.(25)-(27), we obtain

$$\mathbf{D}_{0^+}^{\mu,\nu} T_n^*(t) = \sum_{j=0}^{\infty}\mathbb{M}_{\mu,\nu}(n,j)T_j^*(t),\ n=0,1,2,\ldots, \tag{28}$$

where $\mathbb{M}_{\mu,\nu}(n,j) = \sum_{k=0}^{n}\theta_{njk}$, and

$$\theta_{njk} = \begin{cases} n(-1)^{n-k}\frac{(n+k-1)!2^{2k}\Gamma(k-\mu+\frac{1}{2})}{(n-k)!(2k)!\sqrt{\pi}(\Gamma(k-\mu+1))^2} & j=0 \\ \frac{(-1)^{n-k}nj(n+k-1)!2^{2k+1}k!}{(n-k)!(2k)!\Gamma(k-\mu+1)\sqrt{\pi}} & \\ \times\sum_{r=0}^{j}(-1)^{j-r}\frac{(j+r-1)!2^{2r}\Gamma(k+r-\mu+\frac{1}{2})}{(j-r)!(2r)!\Gamma(k+r-\mu+1)} & j=1,2,\ldots. \end{cases} \tag{29}$$

After some lengthly manipulation θ_{njk}, may be put in the form

$$\theta_{njk} = \sum_{k=0}^{n}\frac{(-1)^{n-k}2n(n+k-1)!\Gamma(k-\mu+\frac{1}{2})}{c_j\Gamma(k+\frac{1}{2})(n-k)!\Gamma(k-\mu-j+1)\Gamma(k+j-\mu+1)}, \tag{30}$$

we obtain the claimed result. ■

2.2 *Fractional-order Chebyshev functions*

By introducing the change of variable $t = x^\eta$, $\eta > 0$ on the shifted Chebyshev polynomials of the first kind, we denote the fractional-order Chebyshev functions (FCSs)$T_n^*(x^\eta)$ as $\overline{T}_n^\eta(x)$ which can be obtained with the following recurrence formula:

$$\begin{aligned}&\overline{T}_0^\eta(x) = 0,\ \overline{T}_1^\eta(x) = 2x^\eta - 1, \\ &\overline{T}_{n+1}^\eta(x) = (4x^\eta - 2)\overline{T}_n^\eta(x) - \overline{T}_{n-1}^\eta(x),\ n=1,2,3,\ldots, \end{aligned} \tag{31}$$

and the analytic form of the fractional-order Chebyshev functions (FCSs) $\overline{T}_n^{\eta}(x)$ of degree $n\eta$ are given by

$$\overline{T}_n^{\eta}(x) = \sum_{k=0}^{n} g_{n,k} x^{k\eta}, \tag{32}$$

$$g_{n,k} = n(-1)^{n-k} \frac{(n+k-1)!}{(n-k)!} \frac{2^{2k}}{(2k!)},$$

which alternatively may be written in the following matrix form:

$$\overline{\Theta}_N(x) = \mathbf{G}\mathbb{X}_N(x), \tag{33}$$

where

$$\overline{\Theta}_N(x) = [\overline{T}_0^{\eta}(x), \overline{T}_1^{\eta}(x), \ldots, \overline{T}_N^{\eta}(x)]^T, \tag{34}$$

$$\mathbb{X}_N(x) = [1, x^{\eta}, x^{2\eta}, \ldots, x^{N\eta}]^T, \tag{35}$$

and $g_{n,k},\ n, k = 0, 1, 2, \ldots, m$ are the matrix entries of $\mathbf{G}$. The orthogonality condition of the fractional-order Chebyshev functions (FCSs) is

$$\int_0^1 \overline{T}_i^{\eta}(x)\overline{T}_j^{\eta}(x)\omega_{\eta}(x)dx = \delta_{ij}\varrho_j, \tag{36}$$

where $\varrho_j = \frac{\epsilon_j}{2\eta}\pi,\ \epsilon_0 = 2,\ \epsilon_j = 1,\ j \geqslant 1$ and $\omega_{\eta}(x) = \omega(x^{\eta}) = \frac{1}{\sqrt{x^{\eta}-x^{2\eta}}} = \frac{1}{x^{\eta}\sqrt{x^{-\eta}-1}}$. A function $\xi(x) \in L^2_{\omega_{\eta}(x)}[0, \infty)$ can be expressed in terms of the fractional-order Chebyshev functions as:

$$\xi(x) = \sum_{j=0}^{\infty} v_j \overline{T}_j^{\eta}(x), \tag{37}$$

where the coefficients v_j are given by

$$v_j = \frac{1}{\varrho_j} \int_0^1 \omega_{\eta}(x)\xi(x)\overline{T}_j^{\eta}(x),\ j = 0, 1, 2, \ldots. \tag{38}$$

In practice, only the first $(N+1)$-terms fractional-order Chebyshev functions are considered. Hence we can write

$$\xi_N(x) = \sum_{j=0}^{N} v_j \overline{T}_j^{\eta}(x) = \mathbf{V}^T\overline{\Theta}_N(x), \tag{39}$$

where the fractional-order Chebyshev functions coefficient vector $\mathbf{V}$ is given by $\mathbf{V}^T = [v_0, v_1, \ldots, v_m]$.

Theorem 2.5 [1] Suppose $t_j,\ j = 0, 1, 2, \ldots, i$ are the fractional-order Chebysev function nodes of $\overline{T}_i^{\eta}(x)$. Then the zeros of $\overline{T}_i^{\eta}(x)$ is given by

$$t_j = \Big(\frac{1}{2} + \frac{1}{2}\cos\Big(\frac{(2j-1)\pi}{2i}\Big)\Big)^{\frac{1}{\eta}},\ j = 1, 2, 3, \ldots, i. \tag{40}$$

2.3 *The fractional derivatives of $\overline{\Theta}_N(x)$*

The main objective of this subsection is to prove the following theorem for the fractional derivatives of the fractional-order Chebyshev functions and also in this subsection we show a operational matrix of the Hilfer fractional derivative for the fractional-order Chebyshev functions.

Theorem 2.6 The Hilfer fractional derivative of order μ and type ν for the fractional-order Chebyshev function vectors $\overline{\Theta}_N(x)$ is given by

$$\mathbf{D}_{0^+}^{\mu,\nu}\overline{\Theta}_N(x) = \mathfrak{D}_{0^+}^{\mu,\nu}\overline{\Theta}_N(x), \tag{41}$$

where $\mathfrak{D}_{0^+}^{\mu,\nu}$ is an $(N+1)\times(N+1)$ matrix of the following form

$$\mathfrak{D}_{0^+}^{\mu,\nu} = \mathbf{G}\mathfrak{B}\mathbf{G}^{-1}, \tag{42}$$

where $\mathbf{G}$ is given by Eq.(33) and $\mathfrak{B}$ is an $(N+1)\times(N+1)$ matrix and its elements, $\mathfrak{b}_{ij}, 0 \leqslant i, j \leqslant N$ are given as follows:

$$\mathfrak{b}_{ij} = \begin{cases} \frac{x^{-\mu}\Gamma(i\eta+1)}{\Gamma(i\eta-\mu+1)} & i = j,\ j = n, n+1, \ldots N, \\ 0 & \text{otherwise.} \end{cases} \tag{43}$$

Proof. Using formulas (11), (34) and taking the Hilfer fractional derivative of (33), we have

$$\begin{aligned} \mathbf{D}_{0^+}^{\mu,\nu}\overline{\Theta}_N(x) &= \mathbf{D}_{0^+}^{\mu,\nu}[\overline{T}_0^{\eta}(x), \overline{T}_1^{\eta}(x), \ldots, \overline{T}_N^{\eta}(x)]^T = \mathbf{G}\mathbf{D}_{0^+}^{\mu,\nu}\mathbb{X}_N(x) \\ &= \mathbf{G}\Big[0, \ldots, \frac{x^{n\eta-\mu}\Gamma(n\eta+1)}{\Gamma(n\eta-\mu+1)}, \frac{x^{(n+1)\eta-\mu}\Gamma((n+1)\eta+1)}{\Gamma((n+1)\eta-\mu+1)} \\ &\ldots, \frac{x^{(N+1)\eta-\mu}\Gamma((N+1)\eta+1)}{\Gamma((N+1)\eta-\mu+1)}\Big] = \mathbf{G}\mathfrak{B}\mathbb{X}_N(x), \end{aligned} \tag{44}$$

where $\mathfrak{B}$ is an $(N+1)\times(N+1)$ matrix of the following form:

$$\mathfrak{B} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & \ddots & 0 & \ldots & 0 \\ 0 & 0 & \frac{x^{-\mu}\Gamma(n\eta+1)}{\Gamma(n\eta-\mu+1)} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \ldots & \frac{x^{-\mu}\Gamma((N+1)\eta+1)}{\Gamma((N+1)\eta-\mu+1)} \end{pmatrix}. \tag{45}$$

Since $\mathbf{G}$ is invertible, then we get

$$\begin{aligned}\mathbf{D}_{0^+}^{\mu,\nu}\overline{\Theta}_N(x) &= \mathbf{G}\mathfrak{B}\mathbb{X}_N(x) = \mathbf{G}\mathfrak{B}\mathbf{G}^{-1}\mathbf{G}\mathbb{X}_N(x) = \mathbf{G}\mathfrak{B}\mathbf{G}^{-1}\overline{\Theta}_N(x)\\ &= \mathfrak{D}_{0^+}^{\mu,\nu}\overline{\Theta}_N(x). \end{aligned} \tag{46}$$

This proves the theorem. ■

3. Applications of the operational matrix of fractional derivative

In this section, in order to show some fundamental importance of the formula, we use the fractional-order Chebyshev functions as basis functions for the collocation scheme and we apply it to solve multi-order fractional differential equation.

3.1 *Linear multi-order FDEs*

Consider the linear multi-order FDE given by Eqs. (1) and (2). To solve problem Eqs. (1) and (2) we approximate $y(x)$ and $g(x)$ by FCFs as:

$$\begin{aligned} y(x) &\simeq \sum_{i=0}^{N} h_i \overline{T}_i^{\eta}(x) = \mathbf{H}^T\overline{\Theta}_N(x),\\ g(x) &\simeq \sum_{i=0}^{N} s_i \overline{T}_i^{\eta}(x) = \mathbf{S}^T\overline{\Theta}_N(x), \end{aligned} \tag{47}$$

where $\mathbf{S} = [s_0, s_1, \ldots, s_N]^T$ is known but $\mathbf{H} = [h_0, h_1, \ldots, h_N]$ is an unknown vector. By using formula (41) in the theorem , we get

$$\mathbf{D}_{0^+}^{\mu,\nu} y(x) \simeq \mathbf{H}^T\mathbf{D}_{0^+}^{\mu,\nu}\overline{\Theta}_N(x) \simeq \mathfrak{D}_{0^+}^{\mu,\nu}\overline{\Theta}_N(x), \tag{48}$$

$$\mathbf{D}_{0^+}^{\mu_j,\nu_j} y(x) \simeq \mathbf{H}^T\mathbf{D}_{0^+}^{\mu_j,\nu_j}\overline{\Theta}_N(x) \simeq \mathbf{H}^T\mathfrak{D}_{0^+}^{\mu_j,\nu_j}\overline{\Theta}_N(x),\ j = 1, 2, \ldots, k, \tag{49}$$

$$y(x)^{(i)}(0) = \mathbf{H}^T\mathfrak{D}_{0^+}^{i,i}\overline{\Theta}_N(0) = d_i,\ i = 0, 1, \ldots, m. \tag{50}$$

Employing Eqs. (47), (48) and (49), the residual $\mathbb{R}_N$ for (1) can be written as:

$$\mathbb{R}_N \simeq \Big(\mathbf{H}^T\mathfrak{D}_{0^+}^{\mu,\nu} - \mathbf{H}^T\sum_{j=1}^{k} a_j \mathfrak{D}_{0^+}^{\mu_j,\nu_j} - \mathbf{H}^T a_{k+1} - a_{k+2}\mathbf{S}^T\Big)\overline{\Theta}_N(x). \tag{51}$$

As in a typical tau method [8], we generate $N - m$ inear equations by applying

$$\langle \mathbb{R}_N, \overline{T}_j^{\eta}(x)\rangle_{\omega_\eta} = \int_0^1 \mathbb{R}_N \overline{T}_j^{\eta}(x)\omega_\eta(x)dx = 0,\ j = 0, 1, 2, \ldots, N - m - 1. \tag{52}$$

Using Eqs. (52) and (50) generate $(N-m)$ and $(N+1)$ set of linear equations, respectively. These equations give a linear system of algebraic equations which can be solved for the unknown vector $\mathbf{H}$. Then the approximate solution of Eq. (47) can be calculated.

3.2 *Nonlinear multi-order FDEs*

In this subsection, we use the collocation method to numerically solve the nonlinear multi-order FDE, namely. In order to use the fractional-order Chebyshev functions collocation method to solve Eqs. (3) and (4), we approximate $y(x)$ by Eq. (47), then we have

$$\mathbf{D}_{0^+}^{\mu_j,\nu_j} y(x) \simeq \mathbf{H}^T \mathbf{D}_{0^+}^{\mu_j,\nu_j} \overline{\Theta}_N(x) \simeq \mathbf{H}^T \mathfrak{D}_{0^+}^{\mu_j,\nu_j} \overline{\Theta}_N(x),\ j = 1, 2, \ldots, k. \tag{53}$$

Employing Eq. (53) in Eq. (3) yields

$$F\Big(x, \mathbf{H}^T \overline{\Theta}_N(x), \mathbf{H}^T \mathfrak{D}_{0^+}^{\mu_1,\nu_1} \overline{\Theta}_N(x), \ldots, \mathbf{H}^T \mathfrak{D}_{0^+}^{\mu_k,\nu_k} \overline{\Theta}_N(x)\Big) = 0. \tag{54}$$

Similarly, usuing Eq. (47) in Eq. (4) yields

$$\mathcal{H}_i\Big(\mathbf{H}^T \overline{\Theta}_N(\zeta_i), \mathbf{H}^T \mathfrak{D}_{0^+}^{i,i} \overline{\Theta}_N(\zeta_i), \ldots, \mathbf{H}^T \mathfrak{D}_{0^+}^{q,q} \overline{\Theta}_N(\zeta_i)\Big) = d_i, \quad i = 0, 1, 2, \cdots, q. \tag{55}$$

To find the solution $y(x)$, we first collocate Eq. (54) at $N - q$ points. For suitable collocation points we use the first $(N - q)$ fractional-order Chebyshev functions roots of $\overline{T}_{N+1}^{\eta}(x)$ which computed in (40). These equations together with Eq. (55) generate $(N + 1)$ nonlinear equations which can be solved using Newton's iterative method to find h_i, $i = 0, 1, 2, \ldots\ldots, N$. Consequently $y(x)$ given in Eq. (47) can be calculated.

4. Illustrative numerical examples

In this section we shall present three numerical examples to illustrate the effectiveness of the proposed methods in this paper. For Examples 4.1-4.2, we use the root mean-square error (**RMSE**) to show the accuracy of method and **RMSE** is defined by

$$\| \mathbb{E}_N(x) \|_2 = \sqrt{\frac{\sum_{i=0}^{N} \big(u(x_i) - u_N(x_i)\big)^2}{N}}, \tag{56}$$

where u and u_N are the exact and approximate solutions, respectively.

Table 1. Absolute error for different values of μ, ν for Example 4.1.

x	$\mu = 0.25,\ \nu = 0.1$	$\mu = 0.5,\ \nu = 0.25$	$\mu = 0.75,\ \nu = 0.99$	$\mu = 0.95,\ \nu = 1$
$x = 0.0025$	0.007961560886390	0.005971170664793	0.003980780443195	0.001990390221598
$x = 0.005$	0.008870539268759	0.006652904451570	0.004435269634380	0.002217634817190
$x = 0.0075$	0.009463782250430	0.007097836687822	0.004731891125215	0.002365945562607
$x = 0.01$	0.009915383829290	0.007436537871967	0.004957691914645	0.002478845957322
$x = 0.0125$	0.010284467799255	0.007713350849442	0.005142233899628	0.002571116949814
$x = 0.015$	0.010598845204928	0.007949133903696	0.005299422602464	0.002649711301232
$x = 0.0175$	0.010873991974275	0.008155493980707	0.005436995987138	0.002718497993569
$x = 0.02$	0.011119473449066	0.008339605086800	0.005559736724533	0.002779868362267
$x = 0.0225$	0.011341645945856	0.008506234459392	0.005670822972928	0.002835411486464
$x = 0.025$	0.011544964109262	0.008658723081946	0.005772482054631	0.002886241027315
$x = 0.0275$	0.011732679864200	0.008799509898150	0.005866339932100	0.002933169966050

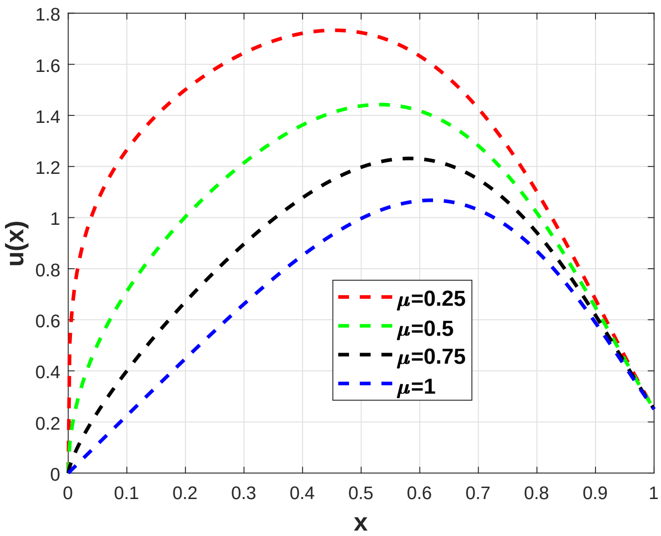

Figure 1. Numerical and exact solutions for different values of μ for example 4.1.

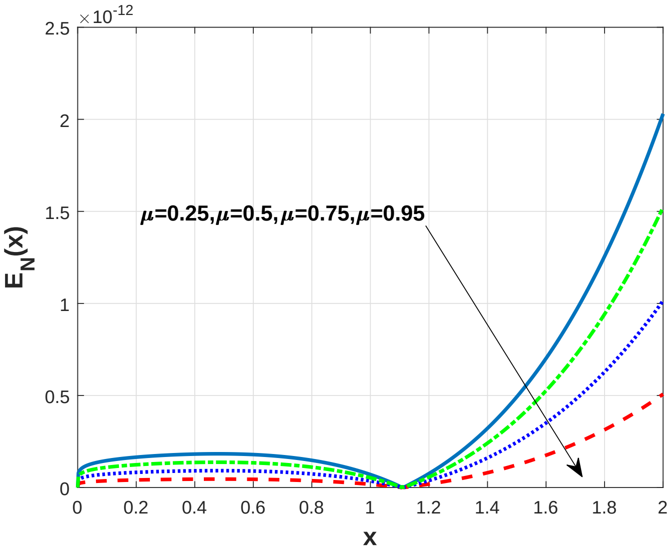

Figure 2. Plot of absolute error for different values of μ for example 4.1.

***Example* 4.1** Consider the following nonlinear fractional differential equation:

$$\mathbf{D}_{0^+}^{\mu,\nu} u(x) = \frac{40320}{\Gamma(9-\mu)} x^{8-\mu} - 3\frac{\Gamma(\frac{\mu}{2}+5)}{\Gamma(-\frac{\mu}{2}+5)} x^{4-\frac{\mu}{2}} + \frac{9}{4}\Gamma(\mu+1)$$
$$+ \left(\frac{3}{2} x^{\frac{\mu}{2}} - x^4\right) - u^{\frac{3}{2}}(x) \ 0 < x < 1,\ \mu,\ \nu \in (0,1) \tag{57}$$

with the initial conditions

$$u(0) = 0. \tag{58}$$

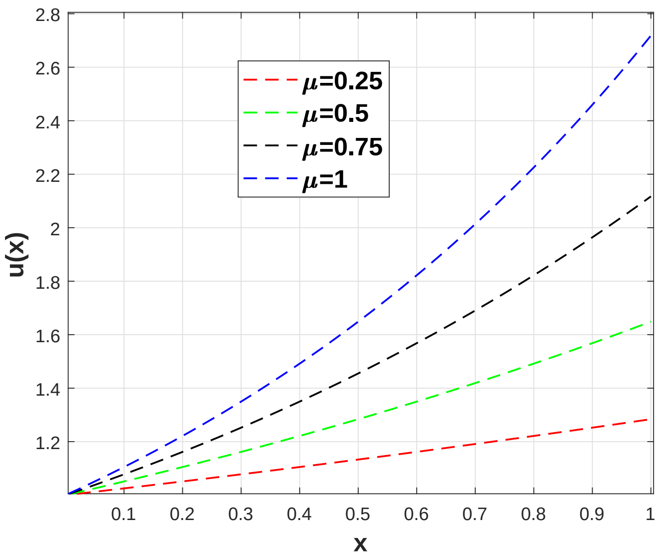

Figure 3. Numerical and exact solutions for different values of μ for example 4.2.

The exact solution of this equation is $u(x) = x^8 - 3x^{4+\frac{\mu}{2}} + \frac{9}{4}x^{\mu}$. The approximate solutions $u_N(x)$ for different values $\mu = 0.25, 0.5, 0.75, 0.95$ are plotted in Fig. 1 also, in Table 1 we introduce the absolute error for (57) and (58) using proposed method in several choices of μ. Also Fig. 2, shows plot of the absolute error for different values of μ.

Table 2. Absolute error for different values of μ, ν for Example 4.2.

x	$\mu = 0.25,\ \nu = 0.1$	$\mu = 0.5,\ \nu = 0.25$	$\mu = 0.75,\ \nu = 0.99$	$\mu = 0.95,\ \nu = 1$
$x = 0.0025$	0.138202749610853	0.103652062208139	0.069101374805426	0.034550687402713
$x = 0.005$	0.138243879653295	0.103682909739971	0.069121939826647	0.034560969913324
$x = 0.0075$	0.138284963025358	0.103713722269019	0.069142481512679	0.034571240756340
$x = 0.01$	0.138325999510666	0.103744499633000	0.069162999755333	0.034581499877667
$x = 0.0125$	0.138366988892158	0.103775241669119	0.069183494446079	0.034591747223040
$x = 0.015$	0.138407930952089	0.103805948214067	0.069203965476044	0.034601982738022
$x = 0.0175$	0.138448825472023	0.103836619104017	0.069224412736012	0.034612206368006
$x = 0.02$	0.138489672232836	0.103867254174627	0.069244836116418	0.034622418058209
$x = 0.0225$	0.138530471014710	0.103897853261033	0.069265235507355	0.034632617753678
$x = 0.025$	0.138571221597133	0.103928416197850	0.069285610798566	0.034642805399283
$x = 0.0275$	0.138611923758892	0.103958942819169	0.069305961879446	0.034652980939723

***Example* 4.2** In this example we consider the following linear boundary value problem

$$\mathbf{D}_{0^+}^{\mu,\nu} u(x) + 3u(x) - u(\frac{x}{2}) = f(x),\ \ x \in (0,1),$$
$$u(0) = 1,\ \ u(1) = e, \qquad (59)$$

where $f(x) = -e^{\frac{x}{2}} + e^{x}\left(4 - \frac{\Gamma(4-\mu)}{\Gamma(2-\mu)}\right)$. The exact solution of the boundary value problem (59), when $\mu = \nu = 1$ is $u(x) = e^x$. The approximate solutions for $u_N(x)$ for different values $\mu = 0.25, 0.5, 0.75, 0.95$ are plotted in Fig. 3 also absolute error results for different values of μ are shown in Table 2 and Fig. 4.

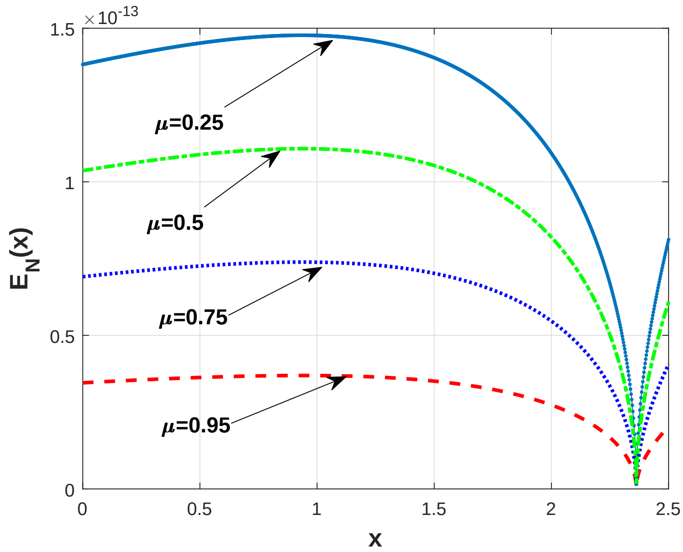

Figure 4. Plot of absolute error for different values of μ for example 4.2.

5. Conclusion

In this paper, we first proposed a new and efficient approach for numerical approximation of the linear and nonlinear multi-order FDE using the fractional-order Chebyshev functions and then we obtain a new fractional derivative operational matrix for these orthogonal functions. Implementation of the numerical approximation reduces the problem to a system of algebraic equations. Application of the method for numerical solution of fractional-order for the linear and nonlinear differential equations is also considered. Finally, numerical examples have been presented to demonstrate accuracy of the proposed method.

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