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Some applications of basic operations in Clifford algebra

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Abstract. Geometric algebra provides intuitive and easy description of geometric entities (encoded by blades) along with different operations and orthogonal transformations. Grassmann's Exterior and Hamilton's quaternions lead to the existence of Clifford (Geometric) algebra. Clifford or geometric product has its significant role in whole domain of Clifford algebra, while as contraction (anti outer product or analogous to dot product) is grade reduction operation. The other operations can be derived from the former one. The paper explores elucidation of Clifford algebra and Clifford product with some salient features and applications.

Keywords: Bivector, CA (GA), contraction, dualization, multivector, g-numbers, versor.

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1. Introduction and preliminaries

William Kingdon Clifford, a British mathematician cum philosopher born in 1845 and expired in 1879. The essay "The Ethics in Belief" opens new vistas in Clifford's mathematical philosophy. Umpteen articles have delineated the biography of this great mathematician [9, 11]. Mathematicians have enunciated distinguished branches of mathematics and their relationships with other fields of knowledge accordingly. The present world is familiar with algebra and its different types, such as algebra of real number system, complex numbers, hyperbolic numbers, quaternions, dual quaternions, and Dirac algebra. Some of them are either sub algebras or embeddings of Clifford algebras. Grassmann's ground breaking publication 'Ausdenungslehre' (2000). English version [34] is fully loaded with the concepts of exterior algebra. The translated version of the book

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is more mathematical as compared to its original edition and diverted the attention of whole mathematical community towards him. Clifford algebras are directed number systems or extensions of complex numbers $\mathbf{C}$ and quaternions $\mathbf{H}$, while keeping preserved the anti commutation rule ($ij = -ji$), i, j are unit quaternions, adjoining of additional square roots of -1 is made possible through these algebras. Hestenes' Introduction of Geometric Algebra for Physicists [11] illustrates the copious evidences about the evolution of this subject. Clifford's prime motive was to amalgamate Grassmann's and Hamilton's works in order to unfold novel dimensions in algebra. This is unique from other areas of mathematics most probably, because of its containment of mixed grade elements. New associative geometric product (coalesce of inner and outer products) paved a track to Clifford to discover this algebra. His idea was to generalize this product to arbitrary dimension by replacing outer product's imaginary term. Hestenes [20] highlighted the mathematical framework for physics especially in description of Clifford language for Pauli and Dirac equations, and along with other collaborators [24, 35]. They carried out the job of Clifford to new peaks from the stage where he left to serve. Vector algebra without geometrical representation of scalars and vectors is a sort of "hermaphrodite monster" [41]. So far this algebra dealt with algebra of oriented subspaces through the origin. Such spaces can be reflected, rotated, projected and, even intersected with insertion of generic manipulations and equations. Other kinds of algebras or models of Clifford algebra are homogeneous models that usually isolated the emergence of algebra from origin of represented subspace, and is effective in blade notation of points, lines and planes. Updated version of this model is conformal model [16] which preserves angle. Invertible multivectors are helpful in representation of all conformal transformations. The applications of these algebras are vast running from engineering to geometric algebra software. Engineering branch includes electrical engineering and optical fibers [30, 33], robotics and control [5, 42], computer graphics and modelling [19, 23, 46], software libraries [7, 22] and computer algebra systems [2, 31].

This paper is composed of six sections and subsections including introduction and conclusion. Section 2 contains description of geometric algebra with some important definitions, section 3 is about geometric product, section 4 comprising g- numbers, and section 5 is application part.

2. Geometric algebra

Geometric algebra term was proposed by pre-eminent mathematician Artin [3] of 20th century, while discussing algebras of symplectic and orthogonal groups. The subject is also known by Clifford algebra or Clifford's geometric algebra, Grassmann algebra is its backbone. Some basic definitions are necessary to explore geometric algebra and derivation of results. Elements in geometric algebra are scalars, vectors, bivectors, trivectors, quadvectors ..., where bivectors are pseudoscalars, vectors are pseudovectors in $\mathcal{C}l_2$, trivectors are pseudoscalars in $\mathcal{C}l_3$ and bivectors are its pseudovectors. It can be mentioned here that pseudovectors and pseudoscalars are grade $(n-1)$ and grade (n) elements in a Clifford algebra. Apart from defining geometric product, we have other unary operations and properties without which various computations and manipulations of results will be cumbersome.

Definition 2.1 A mapping $\mathcal{B} : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{F}$ is bilinear form if

- $\mathcal{B}(ax_1 + bx_2, y) = a\mathcal{B}(x_1, y) + b\mathcal{B}(x_2, y)$,
- $\mathcal{B}(x, ay_1 + by_2) = a\mathcal{B}(x, y_1) + b\mathcal{B}(x, y_2)$.

$\mathbb{V}$ is a finite dimensional vector space over field $\mathbb{F}$, $a, b \in F$ and $x_i, y_i \in \mathbb{V}$.

Definition 2.2 $q : \mathbb{V} \longrightarrow \mathbb{F}$ is quadratic form if $q(x) = B(x, x)$ for some symmetric bilinear form. Equivalently, for a given finite dimensional vector space $\mathbb{V}$ over field $\mathbb{F}$,

$$Q : \mathbb{V} \longrightarrow \mathbb{F} \implies Q(\lambda v) = \lambda Q(v) \ \ \forall \lambda \in \mathbb{F}, \quad v \in \mathbb{V}.$$

It implies that a quadratic form is homogeneous polynomial of degree 2 in a number of variables. Both quadratic and bilinear forms have matrix representation [26]. It became important in comparison of lengths of non parallel line segments.

Definition 2.3 [17] The quotient algebra $\mathcal{T}(\mathbb{V})/\{\mathbb{R}\}$ is exterior algebra, where $\mathbb{V}$ is vector space of dimension n, $\mathbb{K}$ is field and

$$\mathcal{T}(\mathbb{V}) = \bigoplus_{m \geqslant 0} \mathbb{V}^{\otimes m} \quad \text{or}$$
$$\mathcal{T}(\mathbb{V}) = (u_1 \otimes u_2 \otimes ... \otimes u_p) \cdot (v_1 \otimes v_2 \otimes ... \otimes v_q) = u_1 \otimes u_2 \otimes ... \otimes u_p \otimes v_1 \otimes v_2 \otimes ... \otimes v_q$$

is tensor algebra, $\{\mathbb{R}\}$ is two sided ideal in $\mathcal{T}(\mathbb{V})$ generated by the relations $\mathbb{R}$ and this $\mathbb{R}$ is actually a vector subspace of $\mathbb{V} \otimes_k \mathbb{V}$.
The product in $\mathcal{T}(\mathbb{V})/\{\mathbb{R}\}$ is $\wedge$ and equivalent to $\otimes$. As for basis vectors

$$e_1 \wedge e_2 \wedge ... \wedge e_n, \quad e_i \wedge e_i = 0 \Longrightarrow e_i \otimes e_i$$

is relation in $\mathbb{R}$. The associative, linear and anti commutative properties

- $x \wedge (y \wedge z) = (x \wedge y) \wedge z$,
- $\alpha(x \wedge y) + \beta(x \wedge z) = x \wedge (\alpha y + \beta z)$,
- $e_i \wedge e_i = 0$ and $e_i \wedge e_j = -e_j \wedge e_i$,

provides algebraic structure to exterior algebra $E(n)$. Dimension of $E(n) = 2^n$.

Definition 2.4 Clifford algebra: $\mathcal{C}l_{p,q}^F$ of n-dimensional vector space E over some field F with signature $p + q = n$ and basis $\{e_0, e_1, ..., e_p, e_{p+1}, e_{p+2}, ..., e_q\}$ endowed with Clifford/geometric multiplication AB on E such that for all $A, B \in E \ \Rightarrow AB \in E$ according to the following (for all $A, B, C \in E$):

- $\forall \alpha, \beta \in \mathbb{F}$,

$$(\alpha A + \beta B)C = \alpha AC + \beta BC \ \text{ and } \ A(\alpha B + \beta C) = \alpha AB + \beta AC;$$

- The associativity is $(AB)C = A(BC)$;
- Unitality is $Ae = A = eA$;
- $e_i e_j + e_j e_i = 2\epsilon_{ij} e$.

(or)

$\mathcal{C}l(\mathcal{V}_{n,q})$ is geometric algebra of n-dimensional vector space $\mathcal{V}_n$ over some field K generated by all $x \in \mathcal{V}_n$ with quadratic form $q : \mathcal{V}_n \rightarrow K$ and $x^2 = q(x)$, dim $\{\mathcal{C}l(\mathcal{V}_{n,q})\} = 2^n$. $\mathcal{C}l(\mathcal{V}_{n,q})$ reduces to exterior algebra $\{\wedge(\mathcal{V}_n)\}$ for $q = 0$.

Definition 2.5 Clifford map:A linear mapping ψ defined between a linear space $\mathcal{V}$ with quadratic map $\mathbf{Q}$ and an associative algebra $\mathcal{M}$ over some field $\mathbb{F}$.

$$\Longrightarrow \psi(x)^2 = \mathbb{Q}(x) \cdot 1_{\mathcal{A}} \ \ \forall x \in \mathcal{V}.$$

Thus in Clifford map sense, Clifford algebra $\mathcal{C}l(\mathbb{Q})$ is quadratic algebra along with Clifford map $\psi : \mathcal{V} \longrightarrow \mathcal{C}l(\mathbb{Q}) \implies x \longrightarrow \alpha x$.

$\ni$ for any Clifford map $\phi : \mathcal{C}l(\mathbb{Q}) \longrightarrow \mathcal{M}$, there exists a unique algebra homomorphism $\psi : \mathcal{C}l(\mathbb{Q}) \longrightarrow \mathcal{M}$ and $\phi x = \psi(\alpha x)$. All maps here can be deduced from $\alpha : V \longrightarrow \mathcal{C}l(\mathbb{Q})$ which is universal.

Definition 2.6 Simplest Euclidean geometric algebra and unified geometric algebra for plane:Generalized geometric algebra notation

$$\mathbb{G}_{p,q} \equiv \mathcal{C}l_{p,q} \equiv \mathbb{R}\{a_1, a_2, ..., a_p, b_1, b_2, ..., b_q\}$$

as an associative algebra with $a_i^2 = \{1; 1 \leqslant i \leqslant p\}$ and $b_j^2 = \{-1; 1 \leqslant j \leqslant q\}$ of dimension

$$\binom{n}{0} + \binom{n}{1} + ... + \binom{n}{n} = 2^n.$$

Note that

$$\mathbb{G}_1 \equiv \mathcal{C}l_1 \equiv \{g; g = x + ye : x, y \in \mathbb{R}\} \equiv \mathbb{R}(e)$$

with $e^2 = 1$ is considered as the simplest Euclidean GA and defines a hyperbolic plane.

$$\mathbb{G}_{1,1} \equiv \{g; g = g_0 + g_1e_1 + g_2e_2 + g_{12}e_{12} : g_i \in \mathbb{R}\} \equiv \mathbb{R}(e_1, e_2), \quad \mathbb{G}_{1,1} \cong M_2(\mathbb{R})$$

In matrix notation of standard basis [37]

$$\mathbb{G}_{1,1} \equiv span_{\mathbb{R}}(1, e_1, e_2, e_1e_2) \equiv \begin{bmatrix}1 \ e_1\end{bmatrix}^T \begin{bmatrix}1 \ e_2\end{bmatrix} = \begin{pmatrix} 1 & e_2 \\ e_1 & e_1e_2 \end{pmatrix}.$$

Any element $g \in \mathbb{G}_{1,1} \equiv \begin{pmatrix}1 \ e_1\end{pmatrix} \begin{pmatrix} g_0 & g_2 \\ g_1 & g_{12} \end{pmatrix} \begin{bmatrix}1 \ e_2\end{bmatrix}^T$.
$\mathbb{G}_2 = \mathbb{G}_2^0 + \mathbb{G}_2^1 + \mathbb{G}_2^2 = \mathbb{G}_2^+ + \mathbb{G}_2^-$ is unified GA system for plane, or $\mathbb{G}_2 = \mathbb{G}_2^+ \oplus \mathbb{G}_2^- = \mathbb{R} \oplus \mathbb{R}^2 \oplus \wedge^2\mathbb{R}^2$ with even constituent $\mathbb{G}_2^+ \equiv \mathbb{G}_{2,0}^{+2} = \{x \mid x = x + ye_{12} \ni x, y \in \mathbb{R}\} \cong \mathbb{C}$ and odd constituent $\mathbb{G}_{\not{2}}^- \equiv \mathbb{G}_2^1 = \{x \mid x = xe_1 + ye_2\} \equiv \mathbb{R}^2$.

Other geometric algebras in higher dimensions are

$$\mathbb{G}_3 \equiv \mathcal{C}l_{3,0} \equiv span_{\mathbb{R}}\{1, e_1, e_2, e_3, e_{12}, e_{23}, e_{13}, e_{123}\} \equiv \mathbb{R}(e_1, e_2, e_3)$$

with scalars, vectors, bivectors and trivectors as its elements. Its odd and even constituents are $\mathcal{C}l_{3,0}^- = \mathbb{R}^3 \oplus \wedge^3\mathbb{R}^3$ and $\mathcal{C}l_{3,0}^+ = \mathbb{R} \oplus \wedge^2\mathbb{R}^3$.

Remark 1 *Subalgebra generated by $F = F \cdot 1_M$ and vector space V. $\mathbb{G}_{p,q}$ is graded linear space and universal, because no relations between new square roots are assumed [36, 40]. For any r-vector of GA, $A_r = \{a_1 \wedge a_2 \wedge a_3 \wedge ... \wedge a_r\} \Rightarrow A_r^* = (-1)^r A_r$, so inversion is generalization of complex conjugation.*

Moreover, $\mathbb{G}_3^+ \cong \mathbb{H}$ is also known by spinor algebra [4, 8, 15] in order to emphasize the geometric significance of its elements. The decomposition of complex numbers into their real and imaginary parts paves a way for decomposition of CA into even and odd constituents, which in turn lead to an involution operation of CA. The main involution is inversion mapping that distinguishes even and odd constituents in Clifford algebra.
$\mathbb{G}_{p,q}^* = \{\mathbb{G}_{p,q}^+ + \mathbb{G}_{p,q}^-\}^* = \mathbb{G}_{p,q}^+ - \mathbb{G}_{p,q}^-$ is inversion of $\mathbb{G}_{p,q} = \mathbb{G}_{p,q}^+ + \mathbb{G}_{p,q}^-$.

2.1 *Blade, multivector, versor and their grade*

(1). Geometric product of various elements in an algebra gives rise to blades of varying grades. From the canonical/Clifford basis $\{1, e_1, e_2, e_{12}\}$ of $\mathcal{C}l_2$, scalar is zero grade blade, vectors are of grade 1 and bivector is grade 2 blade. Blades can be both simple (some authors like Baylis prefer to call k-vectors that square to a real number) as well as compound. Simple ones are those which can be reduced to the outer product of basis vectors upto the real factor in some basis and vice versa case is in compound ones. Grade of blades made order to scalars, vectors, bivectors, ... tangible to some degree. Scalars are of grade 0, vectors of grade 1, planar elements of grade 2, space elements of grades $3, 4, \ldots.$

For instance $e_{12} \in \mathbb{G}_3$ is simple, and $\{e_1e_2 + e_2e_3\} = e_2\{e_1 + e_3\}$ is compound. Grade of a basis blade $e_k \in \mathbb{G}_{p,q} = |k|$ or $\mathrm{gr}(e_k) = |k|$ in $\mathbb{G}_{p,q}$ $gr_+(e_k) = 1\{a \in k; 1 \leqslant a \leqslant p\}$ and $gr_-(e_k) = -1\{a \in k; p < a \leqslant p+q\}$ as defined in [43].

(2). Multivector or geometric (hypercomplex) number is linear combination of basis elements of GA. Standard notation for a generic/general multivector of $\mathbb{G}_{p,q}$ is $\alpha_i E_i = \sum\limits_{i=1}^{2^n} \alpha_i E_i$ for $\alpha_i \in \mathbb{R}, E_i = \overline{G}_{p,q}[i]$. A multivector in $\mathcal{C}l_3$ is of the form

$$\begin{gathered} M = a_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_{12}e_{12} + a_{13}e_{13} + a_{23}e_{23} + a_{123}e_{123} \\ \implies M = M_0 + M_1 + M_2 + M_3, \end{gathered}$$

where $0, 1, 2, 3$ represent grade of different elements. Due to antisymmetric property of outer product, arbitrary multivector in case of orthonormal basis reduces to the linear combination of basis elements. Generic multivector is a structure comprising of linear combination of elements with different grades and simple multivector can be of any grade $(0, 1, 2, ..., n)$ depends on the dimension of space.

(3). Versors are invertible multivectors and are factorizable in geometric product. Invertibility of a multivector means that the quadratic form of each of its components is non zero. i.e, $\mathbb{Q}\big[a_i, a_i\big] \neq 0$. Versors are even (factors are even) and odd (with odd factors).

$$A = a_1a_2a_3, ..., a_n \text{ is } n \text{ versor.}$$

2.2 *Reversion, inversion, grade involution and conjugation*

(1). Reversing the order of factors in a blade is reversion like $b = a \wedge b \wedge c \Rightarrow B^{\sim} = -c \wedge b \wedge a$. Reversion (anti involution) effects the orientation of a blade as per its sign change and for a k- blade, it is denoted by $\mathbf{B}^{\sim}_k = (-1)^{\left(\frac{k!}{2(k-2)!}\right)}$.

(2). Positive scalar magnitue of a blade A_k is $|A_k| = \sqrt{|A_k^2|} = \sqrt{|A_k^{\sim} A_k|}$ and $A_k^{\sim} A_k = A_k * A_k^{\sim}$. Then inversion for A_k is

$$A_K^{-1} = |A_k|^2 A_k^{\sim} \implies A_K^{-1} A_k = A_k^{\sim} A_K^{-1}.$$

Let $A_3 = a_1 \wedge a_2 \wedge a_3$ is a 3 blade, $A_3^{\sim} = -a_3 \wedge a_2 \wedge a_1$, then

$$\begin{aligned} &|A_3| = \sqrt{|(a_1 \wedge a_2 \wedge a_3) * (-a_3 \wedge a_2 \wedge a_1)|}, \\ &A_3^{-1} = |a_1 \wedge a_2 \wedge a_3|^2 (-a_3 \wedge a_2 \wedge a_1), \end{aligned}$$

where $(a_1 \wedge a_2 \wedge a_3) * (-a_3 \wedge a_2 \wedge a_1)$ can be evaluated using determinant.

(3). If $A_2 = m_0+m_1e_1+m_2e_2+m_{12}e_{12}$ is a multivector in $\mathcal{C}l_2$, then its grade involution $A_3^{\wedge} = m_0 - m_1e_1 - m_2e_2 + m_{12}e_{12}$ grade involution clamps the orientation of odd grade blades as here in vectors. $A_r^{\wedge} = (-1)^{\frac{r^2}{r}}$ for any r blade.

(4). Conjugate of a blade $A_4 = a_1 \wedge a_2 \wedge a_3 \wedge a_4$ is $A_4^* = a_4 \wedge a_3 \wedge a_2 \wedge a_1$ and $A_4^* = (-1)^{2(5)}A_4 = A_4$. In general, $A_k^* = (-1)^{\frac{k(k+1)}{2}}A_k$

Thus original blade is obtained on applying anti involution, involution and conjugation operations successively two times.

3. Geometric product

William Kingdon Clifford's [10] algebraic framework that he founded from integration of inner product to Grassmann's outer/exterior product, generalization of Grassmann's algebra [6] and Hamilton's quaternions is behind the evolution of geometric algebra. Geometric product unites inner and outer products as scalar and vector products are united by quaternion product. In formal sense, geometric product for vectors $a, b \in \mathbb{R}^4$ with orthonormal basis $\{1, e_1, e_2, e_3, e_4\}$, where $a = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4$ and $b = b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4$, is

$$\begin{aligned}
ab &= \sum_{i=1}^{4}(a_ib_i) + (a_1b_2 - a_2b_1)e_{12} + (a_1b_3 - a_3b_1)e_{13} + (a_1b_4 - a_4b_1)e_{14} \\
&\quad + (a_2b_3 - a_3b_2)e_{23} + (a_2b_4 - a_4b_2)e_{24} + (a_3b_4 - a_4b_3)e_{34}, \\
ab &= a \cdot b + a \wedge b.
\end{aligned}$$

$ba = a \cdot b - b \wedge a$ as $a \wedge b = -b \wedge a$.
Symmetric part of ab is $a \cdot b = \frac{1}{2}(ab + ba)$, anti symmetric one is $a \wedge b = \frac{1}{2}(ab - ba)$, anti-commuatator product is $\frac{1}{2}(ab+ba) = \{a, b\}$ and commutator product is $\frac{1}{2}(ab-ba) = \{a, b\}$. Also, $a \wedge A_k$ is $(k+1)$ vector, $\alpha \wedge A_k$ is A_k vector multiplied by scalar α, $a \cdot A_k = \frac{1}{2}\{aA_k+(-1)^{k-1}A_ka\}$ is grade lowering inner product and $a \wedge A_k = \frac{1}{2}\{aA_k-(-1)^{k+1}A_ka\}$ is grade raising outer product.

It is not flexible to decompose two higher grade multivectors into symmetric and anti symmetric constituents. Geometric product ab is dyad if written in simple product operation (exclusion of exterior product), the branch of mathematics concerned to dyads is dyadics [44].

Inner and outer product names are supposed to be proposed first by Grassmann, while as scalar and outer product by Hamilton. The only crucial difference between the two is that Grassmann's outer product is used in computation of areas for basis bivectors and vector product in creation of axial vector perpendicular to original vectors.

3.1 *Scalar (inner) product*

This operation is known by inner, dot or scalar product reliable on nature of context. Inner product of vectors is common in linear algebra, but here it is defined for blades. If $a = a_1e_1+a_2e_2+a_3e_3$ and $b = b_1e_1+b_2e_2+b_3e_3$ are two vectors in $\mathbb{R}^3$ then $a \cdot b = \sum_{i=1}^{3} a_ib_i$ in coordinate sense and its corresponding geometrical notation is $a \cdot b = \|a\|\|b\|cos\theta$ with $0 \leqslant \theta \leqslant \pi$ between a and b. The key role of inner product in formation of Euclidean plane $\mathbb{R}^2$ is determined by its bilinear $\{(a \cdot b)\alpha = a \cdot (\alpha b)$ and $a \cdot b + a \cdot c = a \cdot (b + c)\}$, symmetric $(a \cdot b = b \cdot a)$ and positive definite $(a \neq 0, \Rightarrow a \cdot a. > 0)$ features. Its linear feature enables to compare parallel line segments and orthogonality for standard basis vectors $x_1 = (1, 0, 0),\ x_2 = (0, 1, 0),\ x_3 = (0, 0, 1)$ forms orthonormal basis [25] for $\mathbb{R}^3$.

This operation can be denoted by a mapping $\cdot\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ which is bilinear (linear and symmetric). Like exterior product, it has its own domain of applications such as in computation of norm and cosine of angle, and cosine of angle is possible between blades of same grade. In [13] inner product of vectors is special case of scalar product $* : \wedge^k\mathbb{R}^n \times \wedge^k\mathbb{R}^n \longrightarrow \mathbb{R}$.

***Example* 3.1** For blades $X = e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5$ and $Y = m_1 \wedge m_2 \wedge m_3 \wedge m_4 \wedge m_5$,

$$X * Y = det \begin{pmatrix} e_1 \cdot m_5 & e_1 \cdot m_4 & e_1 \cdot m_3 & e_1 \wedge m_2 & e_1 \wedge m_1 \\ e_2 \cdot m_5 & e_2 \cdot m_4 & e_2 \cdot m_3 & e_2 \wedge m_2 & e_2 \wedge m_1 \\ e_3 \cdot m_5 & e_3 \cdot m_4 & e_3 \cdot m_3 & e_3 \wedge m_2 & e_3 \wedge m_1 \\ e_4 \cdot m_5 & e_4 \cdot m_4 & e_4 \cdot m_3 & e_4 \wedge m_2 & e_4 \wedge m_1 \\ e_5 \cdot m_5 & e_5 \cdot m_4 & e_5 \cdot m_3 & e_5 \wedge m_2 & e_5 \wedge m_1 \end{pmatrix}$$

Reversion in blade Y is taken in order to overcome negation in norm and angle, $X * Y$ vanishes for blades of unequal grade. In orthonormal basis blades, atleast two rotations are needed to align disjoint sub blades of grade 2, because geometrically single scalar angle can't be defined. Inner product of multivectors $a.M = a \cdot (b \wedge c) = (a \cdot b)c - (a \cdot c)b$ is a vector not a scalar. The extra step in computation of angle between two blades is to find a common factor whose norm must be $= 1$ and this becomes a component along with a vector. i.e, $X = x_1 \wedge x_2 = x \wedge b;\ x \cdot b = 0$ and $Y = y_1 \wedge y_2 = y \wedge b;\ y \cdot b = 0$,

$$X \wedge Y = (x \wedge b) * (b \wedge y) \quad \text{and} \quad \|X\| = \|x\| \Longrightarrow X * Y = \|X\|\|Y\|cos\theta.$$

If $a \cdot A_k = \frac{1}{2}\{aA_k - (-1)^k A_k a\} = (-1)^{k+1} A_k \cdot a$, $a \cdot A_k$ is a (k-1) vector.

3.2 *Exterior (outer) product*

Cross product $a \times b$ is familiar, considered as axial vector in Gibb's vector algebra [18], and replaced outer product after publication of Grassmann's algebra of extension. Such an operation has its own applications like in description of angular velocity, force acting on charge in magnetic field and torque about origin of force. Yet, it is used in 3 dimensional and 7 dimensional cases and, analogously in higher dimensions [28]. The point of interest is exterior product and relation between these two products. In two dimensions, exterior product [45] of two vectors a and b is $a \wedge b$ has both magnitude as well as orientation, and is completely defined by anti symmetry and bilinearity. It is called by bivetor, directed number from Grassman's inception and shape of plane is immaterial in its representation. Exterior product vanishes for two parallel or norm zero vectors, as no bivector is formed or this operation between an element of grade n and another element of grade $n + 1$ reduces to zero in n dimensional space.

***Example* 3.2** Let $x = 3e_1 + 2e_2$ and $y = -3e_1 + e_2$ be two elements in $\mathcal{C}l_2$. Then $x \wedge y = (3e_1 + 2e_2) \wedge (-3e_1 + e_2) = 3e_1 \wedge e_2 + 6e_1 \wedge e_2 = 9e_1 \wedge e_2 = 9e_1e_2$ and $y \wedge x = -9e_1 \wedge e_2 = -9e_1e_2$, or

$$x \wedge y = \det \begin{pmatrix} 3 & 2 \\ -3 & 1 \end{pmatrix} e_1 \wedge e_2 \quad \text{and} \quad y \wedge x = \det \begin{pmatrix} -3 & 1 \\ 3 & 2 \end{pmatrix} e_1 \wedge e_2,$$

where e_1, e_2 are orthonormal vectors and $e_1 \wedge e_2$ is unit bivector. Analogous of bivectors in 3 and 4 dimensions are trivectors and quad vectors. Components of bivectors are areas of parallelogram in coordinate planes, whereas trivector components are volume projections of a parallelopiped onto coordinate spaces.

Definition 3.3 Linear Bivector and Trivector SpacesBivector space consists bivectors as elements and trivector space has trivectors. A map $\wedge : \mathbb{R}^n \times \mathbb{R}^n \to \wedge^2\mathbb{R}^n$ determines linear bivector space with elements of the form $A = x \wedge y,\ y = B = y \wedge z$, additive identity is $e \wedge e = 0$. It has scaling $a \wedge (\rho b) = \rho(a \wedge b),\ \rho \in \mathbb{R}$ and distributive $a \wedge (b+c) = (a \wedge b) + (a \wedge c)$ properties, where $a \wedge b, a \wedge c \in \wedge^2\mathbb{R}^n$. A map $\wedge : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \wedge^3\mathbb{R}^n$ defines linear trivector space with scaling $a \wedge (\rho b \wedge c) = \rho(a \wedge b \wedge c)$ and distributive $a \wedge (b \wedge c + d \wedge f) = a \wedge b \wedge c + a \wedge d \wedge f$ for all $a \wedge b \wedge c,\ a \wedge d \wedge f\ \in \wedge^3\mathbb{R}^n$ properties.

$$a \wedge b \wedge c = \frac{1}{3!}\{(abc - cba) + (bca - acb) + (cab - bac)\} = \frac{1}{6}det[a\ b\ c],$$

as abc consists 6 permutations [44] because of inversions. Factor 1/6 appears, because $a \wedge b \wedge c$ has six combinations $abc, cba, bca, acb, cab, bac$, and can be summed as $(abc - cba) + (bca - acb) + (cab - bac)$. $x \wedge M = (-1)^l M \wedge x$ is general rule for symmetry of exterior product with multivector M and l is dimension dependent. Cross product has its significant role in Lie algebras as $\mathbb{R}^3$ with Jacobi identity $x \times (y \times z) + y \times (z \times x) + z \times (x \times y) = 0$ is one instance.

3.3 *Decomposition of blades in terms of exterior product*

Subordinate product (outer product) of geometric product paves a smooth way in elimination of coefficients of basis vectors of an arbitrary vector in $\mathbf{R}^2$. In fact, this leads to a concept that lengths are length ratios along same line, areas are ratios of areas in same plane through origin. In the similar manner, volumes are ratios of volumes and their analogous (hyper volumes) in higher dimensional cases are so.

A vector $W = \{\alpha_1 k_1 + \alpha_2 k_2 + ... + \alpha_n k_n\} \in \mathbb{R}^n$ with basis $(k_1, k_2, k_3, ..., k_n)$, where

$$\alpha_1 = \frac{W \wedge k_2 \wedge ... \wedge k_n}{k_1 \wedge ... \wedge k_n},\ \alpha_2 = \frac{k_1 \wedge W \wedge k_3 \wedge ... \wedge k_n}{k_1 \wedge ... \wedge k_n}\ ,\ ...\ ,\ \alpha_n = \frac{k_1 \wedge k_2 \wedge ... \wedge k_{n-1} \wedge W}{k_1 \wedge ... \wedge k_n}.$$

Then

$$W = \Big(\frac{W \wedge k_2 \wedge ... \wedge k_n}{k_1 \wedge k_2 \wedge ... \wedge k_n}\Big)(k_1) + \quad ... \quad + \Big(\frac{k_1 \wedge k_2 \wedge ... \wedge k_{n-1} \wedge W}{k_1 \wedge k_2 \wedge ... \wedge k_{n-1} \wedge k_n}\Big)(k_n)$$

(1). $n = 3$, $W \in \mathbb{R}^3$ with basis (k_1, k_2, k_3);

$$W = \Big(\frac{W \wedge k_2 \wedge k_3}{k_1 \wedge k_2 \wedge k_3}\Big)(k_1) + \Big(\frac{k_1 \wedge W \wedge k_3}{k_1 \wedge k_2 \wedge k_3}\Big)(k_2) + \Big(\frac{k_1 \wedge k_2 \wedge W}{k_1 \wedge k_2 \wedge k_3}\Big)(k_3),$$

(2). $n = 4$, $W \in \mathbb{R}^4$ with basis (k_1, k_2, k_3, k_4);

$$W = \Bigg\{ \left(\frac{W \wedge k_2 \wedge k_3 \wedge k_4}{k_1 \wedge ... \wedge k_4}\right)(k_1) + \left(\frac{k_1 \wedge W \wedge k_3 \wedge k_4}{k_1 \wedge ... \wedge k_4}\right)(k_2) + ...,$$

(3). $n = 5,\ W \in \mathbb{R}^5$ with basis $(k_1, k_2, k_3, k_4, k_5)$;

$$W = \Big\{ \Big(\frac{W \wedge k_2 \wedge k_3 \wedge k_4 \wedge k_5}{k_1 \wedge ... \wedge k_5}\Big)(k_1) + \Big(\frac{k_1 \wedge W \wedge k_3 \wedge k_4 \wedge k_5}{k_1 \wedge ... \wedge k_5}\Big)(k_2) + \Big(\frac{k_1 \wedge k_2 \wedge W \wedge k_4 \wedge k_5}{k_1 \wedge ... \wedge k_5}\Big)(k_3) + ... \Big\}.$$

***Example* 3.4** For a vector $B = \alpha a + \beta b + \gamma c \in \mathbb{R}^3$ α, β and γ are to be eliminated.

$$\alpha = \frac{B \wedge b \wedge c}{a \wedge b \wedge c},\quad \beta = \frac{a \wedge B \wedge c}{a \wedge b \wedge c},\quad \gamma = \frac{a \wedge b \wedge B}{a \wedge b \wedge c}.$$

Let

$$B = \big[1\ 2\ 3\big]^T,\ a = \big[2\ {-1}\ 0\big]^T,\ b = \big[-1\ 0\ 2\big]^T,\ c = \big[0\ {-3}\ 2\big]^T.$$

Then

$$\alpha = \frac{B \wedge b \wedge c}{a \wedge b \wedge c} = 1.9,\ \beta = \frac{a \wedge B \wedge c}{a \wedge b \wedge c} = 2.8,\ \gamma = \frac{a \wedge b \wedge B}{a \wedge b \wedge c} = -1.3$$

(or)

$$\alpha = \left(\frac{B \wedge b \wedge c}{a \wedge b \wedge c}\right) = det\frac{[B\ b\ c]}{[a\ b\ c]},\ \beta = \left(\frac{a \wedge B \wedge c}{a \wedge b \wedge c}\right) = det\frac{[a\ B\ c]}{[a\ b\ c]},\ \gamma = \left(\frac{a \wedge b \wedge B}{a \wedge b \wedge c}\right) = det\frac{[a\ b\ B]}{[a\ b\ c]}$$

and $B = (1.9)a + (2.8)b - (1.3)c$.
The outer product of column vectors is taken in terms of basis vectors (e_1, e_2, e_3).

***Example* 3.5** A vector $S = \alpha a + \beta b + \gamma c + \delta d + \rho e \in \mathbb{R}^5$, coefficients of variables a, b, c, d, e can be eliminated by taking ratio of their wedge product with the original vector.

$$\alpha = \frac{S \wedge b \wedge ... \wedge e}{a \wedge ... \wedge e},\ ...\ \ \rho = \frac{a \wedge ... \wedge d \wedge S}{a \wedge ... \wedge e}.$$

Let

$$S = \big[1\ 0\ 2\ {-1}\ 9\big]^T,\ a = \big[-2\ 3\ 0\ 1\ {-4}\big]^T,\ b = \big[3\ {-2}\ 4\ {-5}\ 0\big]^T,$$
$$c = \big[-4\ 2\ 0\ 3\ 1\big]^T,\ d = \big[-3\ 2\ 0\ {-6}\ 1\big]^T,\ e = \big[0\ 0\ {-2}\ {-3}\ 1\big]^T.$$

Then

$$\alpha = -0.53642\ ,\ \beta = -0.25920\ ,\ \gamma = -0.17666\ ,\ \delta = 0.21745\ ,\ \rho = -0.11473$$

and

$$S = -\left\{\frac{53642 \times 10^{-1}}{10^4}\right\}a + \big(-25920 \times 10^{-3} \times 10^{-2}\big)b - \left\{\frac{17666 \times 10^{-}2}{10^2 \times 10}\right\}c + \left\{\frac{(21745) \times -1000}{100}\right\}d + (-0.11473)e.$$

Determinants are used to determine the ratio in linear algebra and outer product provides geometrical representation of these ratios without inclusion of coordinates. That is why geometric algebra is coordinate free.

Remark 2 *Dual mapping develops a connection between exterior and cross products. Let $a = a_1e_1 + a_2e_2 + a_3e_3$ and $b = b_1e_1 + b_2e_2 + b_3e_3$ are two vectors in $\mathbb{R}^3$. Then*

$$a \wedge b = det\begin{pmatrix} a_2 & a_3 \\ b_2 & b_3 \end{pmatrix} e_2 \wedge e_3 + det\begin{pmatrix} a_3 & b_3 \\ a_1 & b_1 \end{pmatrix} e_3 \wedge e_1 + det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} e_1 \wedge e_2,$$

$$(a \wedge b)^* = a \times b = det\begin{pmatrix} a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}^T e_1 + det\begin{pmatrix} a_3 & a_1 \\ b_3 & b_1 \end{pmatrix}^T e_2 + det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}^T e_3.$$

Coefficients of both vectors and bivectors in determinant form are scalar quantities. Cross product of two vectors is natural dual to their outer product and vice versa, so this concept can be easily extended to higher dimensional geometric algebras of both Euclidean and non-Euclidean spaces [12, 14, 21].

3.4 *Contraction and duality*

Arbitrary blades can't be endowed with inner product, inner product of a vector can be computed with a two blade, but unfortunately that lacks orientation concept. To preserve or maintain orientation, contraction is important and more efficient than inner product. $\rfloor : \wedge^a\mathbb{R}^n \times \wedge^b\mathbb{R}^n \to \wedge^{(b-a)}\mathbb{R}^n$ is bilinear and distributive over addition and is grade reduction operation. Blades of same grade made it identical to inner product. Contraction can be either implicit or explicit, where former is applicable to only degenerate metrics and latter one is for both degenerate as well as non degenerate metrics. We are using explicit case frequently. For any X, Y and Z blades, $X\rfloor Y = \{\tilde{Y}\rfloor\tilde{X}\tilde{\}} = (-1)^{y(x+1)}\{Y\rfloor X\}$, where $X\rfloor Y$ is a blade contained in Y and is orthogonal to Z.

$Y\rfloor X$ and $Y\lfloor X$ are distinguished only by grade dependent sign. Therefore, geometric algebra allows us to perform all sorts of product operations such as (line×line), (line×area), (area×area), (line×volume), (area×volume), (volume×volume) and (hypervolumes × hypervolumes) reliable on dimension. All the above products or operations can be derived from geometric product which is generally neither commutative nor anti-commutative and is without geometric intuition, so it acts as a fundamental operation in geometric algebra.

Dual of a blade $A_k \in \mathcal{C}l_{n,0}$ is $A_k^* = A_K\lfloor I_n^{-1}$, where I_n is pseudoscalar in $\mathcal{C}l_n$ and the process of finding dual to a blade is dualization. Consider a blade $A_2 = 3e_1 + 2e_2 \in \mathcal{C}l_2$. Then

$$A_2^* = (3e_1 + 2e_2)\rfloor I_2^{-1} = 3\big\{e_1\rfloor(e_2 \wedge e_1)\big\} + 2\big\{e_2\rfloor(e_2 \wedge e_1)\big\} = 2e_1 - 3e_2.$$

Dual is always orthogonal to other blades and requires no coordinates to its notion. Sign of vector changes while computing its dual twice, original blade or element can be retrieved by undualization. i.e, $A_k^{-^*} = A_k\rfloor I_n$. More information and applications of contraction and dual can be accessed in [1, 29].

4. Introduction to g-numbers

The number system $\mathcal{N} = R(a, b)$ ([32, 38, 39]) has following unique features irrespective of associativity and distributivity

(1) $a^2 = 0 = b^2$ with a and b are nilpotents,
(2) $ab + ba = 1$ with $ab \neq ba$ (partition unity),

where $R \subset \mathcal{N}$ and a and b are also termed as g-numbers since of their geometric interpretation.

Canonical basis of $\mathcal{N} = R(a, b)$ over reals is $\begin{pmatrix} ab & a \\ b & ba \end{pmatrix}$. Every g number has the form $[g] = g_{11}ab + g_{12}a + g_{21}b + g_{22}ba$ which implies that $[g] = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$, $g_{ij} \in \mathbb{R}$.

Multiplication table with mutual annihilation property $ab(1 - ba) = (ba)^{-1}$ can be constructed for such a number system. Addition and multiplications can be easily performed in such a number system and each g-number can be decomposed into even and odd components:

$$g = g_+ + g_- = \big(g_{11}ab + g_{22}ba\big) + \big(g_{12}a + g_{21}b\big),$$

where g_+ is even part and g_- is odd. The operations like reversion, inversion and mixed conjugation of multivectors can be applied here too.

Definition 4.1 Reversion and Mixed Conjugation in g numbers:
$\tilde{g} = \widetilde{(g_- + g_+)} = g_- + \tilde{g_+}$, $g_+ = g_{11}ba + g_{22}ba$ as $\widetilde{(ab)} = ba$ $\widetilde{(fg)} = \tilde{g}\tilde{f}$. $\bar{g} = g_+ - g_-$ is inversion, $\bar{g_-} = -g_{12}a - g_{21}b = -g_-$, $g^* = \bar{(\tilde{g})} = \overline{(g_- + \tilde{g_+})} = -g_- + \tilde{g_+}$, $(f + g)^* = f^* + g^* = (f_+ + g_+) - (f_- + g_-)$, $(fg)^* = g^* f^* = (fg)^*_- + (fg)^*_+$ as $(fg)^*_+ = (g_- f_- + \tilde{g_+}\tilde{f_+})$
g numbers can be similarly decomposed into symmetric and anti-symmetric parts like in geometric product of vectors and multivectors. The recursive formula for a vector m with a k-vector A_k is $m \rfloor A_k = m \rfloor (l_1 \wedge l_2 \wedge ... \wedge l_k) = \sum_{i=1}^{k} (-1)^{i-1} l_1 \wedge l_2 \wedge ... \wedge (m \rfloor l_i) \wedge ... \wedge l_k$.

The relation between reversion, scalar product, left and right contractions, and outer product is given by

$$\begin{aligned}
\tilde{E} \star \widetilde{(D \lfloor C)} &= E \star (D \lfloor C) = (D \lfloor C) \star E, \\
P \star (Q \rfloor S) &= (P \wedge Q) \star S, \\
m \rfloor (m_1 \wedge m_2) &= (m \cdot m_1) m_2 - m_1 (m.m_2), \\
m \rfloor (m_1 \wedge m_2 \wedge m_3) &= (m \cdot m_1) m_2 m_3 - (m \cdot m_2) m_1 m_3 + (m \cdot m_3) m_1 m_2.
\end{aligned}$$

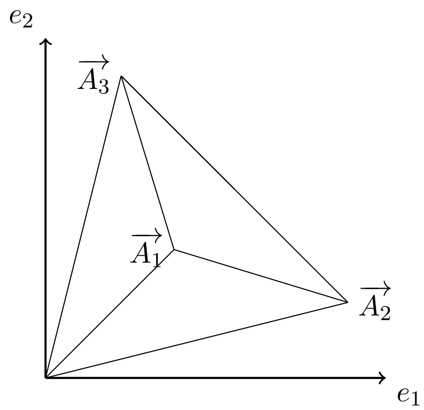
The symmetric product of any three arbitrary odd g- numbers $f_1, f_2, f_3 \in \mathcal{N}$ is

$$f_1 \otimes (f_2 \cdot f_3) = (f_1 \cdot f_2) f_3 - (f_1 \cdot f_3) f_2$$

Similar procedure is applicable to $n \geqslant 3$ odd g numbers.

5. Applications of geometric algebra

***Example* 5.1** Area of triangle in exterior product:



Let $\overrightarrow{A_1}\overrightarrow{A_3} = \overrightarrow{C}$, $\overrightarrow{A_1}\overrightarrow{A_2} = \overrightarrow{A}$ and $\overrightarrow{A_2}\overrightarrow{A_3} = \overrightarrow{B}$, and $\overrightarrow{A} = \overrightarrow{A_2} - \overrightarrow{A_1}$ and $\overrightarrow{B} = \overrightarrow{A_3} - \overrightarrow{A_2}$.
In coordinate form $\overrightarrow{A_1} = (\alpha_1, \beta_1)$, $\overrightarrow{A_2} = (\alpha_2, \beta_2)$ and $\overrightarrow{A_3} = (\alpha_3, \beta_3)$. Area of triangle $M = \frac{1}{2}\{\overrightarrow{A} \wedge \overrightarrow{B}\}$ is a tensor. $M = \frac{1}{2}\{(A_2 - A_1) \wedge (A_3 - A_2)\}$ and

$$\frac{4}{2}M = \{\alpha_1\beta_2 - \alpha_2\beta_1 + \alpha_2\beta_3 - \alpha_3\beta_2 + \alpha_3\beta_1 - \alpha_1\beta_3\}e_1e_2;$$

or $M = \frac{2\hat{B}}{4}\{A_1 \wedge A_2 + A_2 \wedge A_3 - A_1 \wedge A_3\}$, where $\hat{B} = e_1e_2$ and therefore, $\frac{1}{2}\{\overrightarrow{A} \wedge \overrightarrow{B}\} = \hat{B}$ (Area).

Therefore, each bivector formed by any two basis vectors represents area element and this wedge composition is easily accessible to compute area of planar elements as well as volume of trivectors in space, irrespective of design of two dimensional and three dimensional elements.

Theorem 5.2 Let $U_1, U_2, ..., U_m$ and V be the vector spaces. If $T : U_1 \otimes U_2 ... \otimes U_m \longrightarrow V$ is a mapping, then there exists corresponding map $T^{'} : U_1 \wedge U_2 ... \wedge U_m \longrightarrow V$.

Proof. $U\big(U_1 \wedge U_2 \wedge ... \wedge U_m\big)$ possess basis elements $(u_1 \wedge u_2 \wedge ... \wedge u_m)$ and extension of mapping T is $T_1 : U\big(U_1 \wedge U_2 \wedge ... \wedge U_m\big) \longrightarrow V$ defined by

$$T_1\big(U_1 \wedge U_2 \wedge ... \wedge U_m\big) = T\big(U_1 \wedge U_2 \wedge ... \wedge U_m\big)$$

The map $T_1 = 0$. For any basis element λu_i

$$T_1\Big\{\Big(u_1 \wedge u_2 \wedge ... \wedge \lambda u_i \wedge ... \wedge u_m\Big) - \lambda\Big(u_1 \wedge u_2 \wedge ... \wedge u_m\Big)\Big\}$$

$$\begin{aligned}
&= T_1\Big(u_1 \wedge u_2 \wedge ... \wedge \lambda u_i \wedge ... \wedge u_m\Big) - \lambda T_1\Big(u_1 \wedge u_2 \wedge ... \wedge u_m\Big)\\
&= T\Big(u_1 \wedge u_2 \wedge ... \wedge \lambda u_i \wedge ... \wedge u_m\Big) - \lambda T\Big(u_1 \wedge u_2 \wedge ... \wedge u_m\Big)\\
&= \lambda T\Big(u_1 \wedge u_2 \wedge ... \wedge u_i \wedge ... \wedge u_m\Big) - \lambda T\Big(u_1 \wedge u_2 \wedge ... \wedge u_m\Big),\\
&\Longrightarrow T_1\Big\{\Big(u_1 \wedge u_2 \wedge ... \wedge \lambda u_i \wedge ... \wedge u_m\Big) - \lambda\Big(u_1 \wedge u_2 \wedge ... \wedge u_m\Big)\Big\} = 0.
\end{aligned}$$

Other generators can be proved synonymously. From fundamental theorem of factor spaces, there exists a map

$$T^{'}: U\Big(U_1 \otimes U_2 \otimes ... \otimes U_m\Big)\Big/U_0\Big(U_1 \otimes U_2 \otimes ... \otimes U_m\Big) \longrightarrow V$$

defined by

$$T^{'}\Big(u_1 \wedge u_2 \wedge ... \wedge u_m\Big) = T\Big(u_1 \otimes u_2 \otimes ... \otimes u_m\Big).$$

■

Mapping $T^{'}$ is clearly unique, because elements of the form $u_1 \wedge u_2 \wedge ... \wedge u_m$ generate $U_1 \wedge U_2 \wedge ... \wedge U_m$ and this theorem shows connection between tensor products and exterior products.

Theorem 5.3 The rotation generated by reflections of arbitrary vector X along the unit vectors m and n is

$$(mXm)(nXn) = (mXm)(nXn) = 2(m \cdot X)\Big\{2(n \cdot X)n - X\Big\}m - \Big\{2(n \cdot X)n - X\Big\}X.$$

Proof. Geometric product of

$$mX = m \cdot X + m \wedge X = \frac{1}{2}\big(mX + Xm\big) + \frac{1}{2}\big(mX - Xm\big),$$
$$mX = mX + \frac{1}{2}\big(Xm - Xm\big) = \frac{1}{2}\big(mX + mX\big) + \frac{1}{2}\big(Xm - Xm\big).$$

Bilinear form $Q[m, X] = \frac{1}{2}\big\{Q[m, X] + Q[X, m]\big\}$ is symmetric part of mX.

$$Q[m, X]m = mXm + \frac{1}{2}\big\{Q[m, m]X - mXm\big\},$$
$$\frac{1}{2}\Big\{(m + X)(m + X) - mm - XX\Big\} = \frac{1}{2}\big(mX + Xm\big)$$
$$= \frac{1}{2}\Big\{Q[m + X, m + X] - Q[m, m] - Q[X, X]\Big\}$$
$$\Rightarrow \frac{1}{2}\Big\{Q[m, m] + Q[m, X] + Q[X, m] + Q[X, X] - Q[m, m] - Q[X, X] = \frac{1}{2}\big(mX + Xm\big).$$

$$mXm = 2(m \cdot X)m - (m \cdot m)X = 2Q[m, X]m - X,\ nXn = 2(n \cdot X)n - X$$
$$(mXm)(nXn) = 4(m \cdot X)(n \cdot X)(mn) - 2(n \cdot X)(nX) - 2(m \cdot X)(mX) + XX$$
$$(mXm)(nXn) = 2(m \cdot X)\Big\{2(n \cdot X)n - X\Big\}m - \Big\{2(n \cdot X)n - X\Big\}X.$$

■

Lemma 5.4 For $n \in \mathbf{N}$ and arbitrary vectors X and a, the expression for addition of reflections along direction a is $n(aXa) = 2nQ[a, X]a - nQ[a, a]X$.

Proof. From the above theorem $(aXa) = 2Q[a, x]a - Q[a, a]X$ and

$$(aXa) + (aXa) = 2Q[a, x]a - Q[a, a]X + 2Q[a, x]a - Q[a, a]X = 4Q[a, x]a - 2Q[a, a]X$$
$$(aXa)+(aXa)+(aXa) = 2Q[a, x]a - Q[a, a]X + 2Q[a, x]a - Q[a, a]X + 2Q[a, x]a - Q[a, a]X$$
$$\vdots$$

$$(aXa) + (aXa) + ... + (aXa) =$$
$$2Q[a, x]a - Q[a, a]X + 2Q[a, x]a - Q[a, a]X + ... + 2Q[a, x]a - Q[a, a]X$$
$$\Rightarrow (aXa) + (aXa) + ... + (aXa) = 2nQ[a, x]a - nQ[a, a]X$$
$$n(aXa) = 2nQ[a, X] - nQ[a, a]X$$

If a is unit vector, then $n(aXa) = 2nQ[a, X] - nX$. ■

Theorem 5.5 Formation of trivector from an arbitrary bivector, two unit bivectors and a vector.

$$x \wedge y \wedge z = \Big\{(x \wedge y)e_{12} \cdot z)\Big\}e_{12}$$

Proof. Let $C = x \wedge y$ and using associativity $(Ce_{12})z = C(e_{12}z)$ as Ce_{12} is product of C and unit bivector e_{12}, and $(\mathcal{C}e_{12}) \cdot z = C(e_{12}z)$, where inner product [27] is $\big((Ce_{12}) \cdot z\big) = C(e_{12}z)$. Anti-commutative property appears in product of vector with a bivector, and it is obvious that both C and z are parallel. By using anti commutative nature of bivector $(a \wedge b = -b \wedge a)$, we have $(Ce_{12}) \cdot z = C(e_{12}z) = C(-ze_{12})$ and $(Ce_{12}) \cdot z = \big(Cz(-e_{12})\big) = \big(Cz(-1e_{12})\big)$. Geometric product of $Cz = C \cdot z + C \wedge z$ and inner product is symmetric. So, $\{C \wedge z + z \cdot C\}(-1e_{12}) = (Ce_{12}) \cdot z$ Since e_1 and e_2 are orthonormal and $-1 = (e_{12})^2$,

$$C \cdot z\Big((e_{12})^2 e_{12})\Big) - \Big(z \wedge C\big[(e_{12})^2 e_{12})\big]\Big) = \Big(C \wedge z\big[(e_{12})^2 e_{12})\big]\Big).$$

As we know $e_{12}^2 = -1$ implies that $e_{12}^3 = -e_{12}$ and $(-e_{12})^2 = 1$. $\Big\{(Ce_{12}) \cdot z\Big\}e_{12} = (C \wedge z)\{(e_{12}^3)e_{12}\}$ and $\Big\{(x \wedge y)e_{12} \cdot z)\Big\}e_{12} = x \wedge y \wedge z$. ■

Thus, this result can be elongated to form quad-vectors and their analogous in higher dimensional cases.

6. Conclusion

The applications of geometric algebra cited above are not enough and doesn't impart ample about it. Efforts are on to develop and elaborate this field in each and every sub domain of Mathematics, Physics and Computer Science. Special cases of geometric algebra, Outer Product Null Space (OPNS) and Inner Product Null Space (IPNS) can be used to reconnoitre this with vital role played by Hodge dual in them and, in GA softwares, GAALOP (Geometric Algebra Algorithm Optimizer) and CluViz.

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