Journal of
Linear and Topological Algebra
Vol. 09*, No.* 04*,* 2020*,* 301- 306

A class of rings between Armendariz and Central Armendariz rings

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Received 23 August 2020; Revised 28 October 2020; Accepted 3 November 2020.
Communicated by Hamidreza Rahimi

Abstract. The purpose of this paper is to introduce a proper class of rings between Armendariz and Central Armendariz rings. In this direction, we define the concept of Idempotent Armendariz rings. We consider the closure of the Id-Armendariz rings with respect to various extensions including direct product, matrices rings, corner rings, polynomial rings and etc.

Keywords: Armendariz rings, idempotent element, abelian rings.

2010 AMS Subject Classification: 16U20, 16S36, 16W20.

1. Introduction and preliminaries

Throughout this article, R denotes an associative ring with identity. For a ring R, $Nil(R)$, $M_n(R)$, $T_n(R)$, $Id(R)$, $C(R)$ and e_{ij} denote the set of nilpotents elements in R, the $n \times n$ matrix ring over R, the $n \times n$ upper triangular matrix ring over R, the set of idempotent elements of R, the center of R and the matrix with (i, j)-entry 1 and elsewhere 0, respectively.

A ring R is called Armendariz if whenever polynomials $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$ then $a_i b_j = 0$ for each i, j (the converse is always true). The study of Armendariz ring was initiated by Armendariz [2, lemma 1] and Rege and Chhawchharia used Nagata's method of idealization to construct examples of both Armendariz rings and non-Armendariz rings in [10]. Properties, examples and counterexamples of Armendariz rings are given in [3]. So far Armendariz rings are generalized in several forms [1, 5, 9]. Liu and Zhao [9] called a ring R, weak Armendariz if

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Print ISSN: 2252-0201
Online ISSN: 2345-5934

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whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j \in Nil(R)$ for all i and j. Agayev et al. [1] called a ring R central Armendariz if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j \in C(R)$ for all i and j. They showed that the class of central Armendariz rings lies precisely between classes of Armendariz rings and abelian rings (that is, its idempotents belong to $C(R)$.)

In this paper, we introduce the concept of Idempotent Armendariz (Id-Armendariz) rings as a generalization of Armendariz rings. We show that Id-Armendariz rings are central Armendariz and so the class of Id-Armendariz rings lies between the class of Armendariz and central Armendariz rings.

2. Idempotent Armendariz Ring

Definition 2.1 A ring R is said to be Idempotent Armendaiz (Id-Armendariz) if whenever polynomials $f(x) = \sum_{i=0}^{m} a_ix^i$ and $g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j \in Id(R)$ for each i, j.

It is easy to see that subring of Id- Armendariz rings are also Id- Armendariz. Now, we have the following theorem:

Theorem 2.2 Let R_α be a ring for each $\alpha \in I$. Then any direct product of rings $\prod_{\alpha\in I} R_\alpha$ is Id-Armendariz if and only if any R_α is Id-Armendariz.

Proof. Let R_α is Id-Armendariz for each $\alpha \in I$ and $R = \prod_{\alpha\in I} R_\alpha$. Let $f(x)g(x) = 0$ for some polynomials $f(x) = \sum_{i=0}^{m} a_ix^i, g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$, where $a_i = (a_{i_1}, a_{i_2}, \ldots, a_{i_\alpha}, \cdots)$ and $b_j = (b_{j_1}, b_{j_2}, \ldots, b_{j_\alpha}, \cdots)$ are elements of the product ring R for each $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. Define $f_\alpha(x) = \sum_{i=0}^{m} a_{i_\alpha}x^i, g_\alpha(x) = \sum_{j=0}^{n} b_{j_\alpha}x^j \in R_\alpha[x]$ for any $\alpha \in I$. From $f(x)g(x) = 0$, we have $a_0b_0 = 0, a_0b_1 + a_1b_0 = 0, \ldots, a_mb_n = 0$, and this implies

$$a_{0_1}b_{0_1} = a_{0_2}b_{0_2} = \cdots = a_{0_\alpha}b_{0_\alpha} = \cdots = 0$$
$$a_{0_1}b_{1_1} + a_{1_1}b_{0_1} = a_{0_2}b_{1_2} + a_{1_2}b_{0_2} = \cdots = a_{0_\alpha}b_{1_\alpha} + a_{1_\alpha}b_{0_\alpha} = \cdots = 0$$
$$a_{m_1}b_{n_1} = a_{m_2}b_{n_2} = \cdots = a_{n_\alpha}b_{n_\alpha} = \cdots = 0.$$

This means that $f_\alpha(x)g_\alpha(x) = 0$ in $R_\alpha[x]$ for each $\alpha \in I$. Since R_α is Id-Armendariz for each $\alpha \in I$ and $a_{i_\alpha}b_{j_\alpha} \in Id(R_\alpha)$. Now the equation $\prod_{\alpha\in I} Id(R_\alpha) = Id(\prod_{\alpha\in I} R_\alpha)$ implies that $a_ib_j \in Id(R)$, and so R is Id-Armendariz.

Conversely, assume that $R = \prod_{\alpha\in I} R_\alpha$ is Id-Armendariz and $f_\alpha(x)g_\alpha(x) = 0$ for some polynomials $f_\alpha(x) = \sum_{i=0}^{m} a_{i_\alpha}x^i, g_\alpha(x) = \sum_{j=0}^{n} b_{j_\alpha}x^j \in R_\alpha[x]$ with $\alpha \in I$. Define $F(x) = \sum_{i=0}^{m} a_ix^i, G(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$, where $a_i = (0, \cdots, 0, a_{i_\alpha}, 0, \cdots)$ and $b_j = (0, \cdots, 0, b_{j_\alpha}, 0, \cdots) \in R$. Since $f_\alpha(x)g_\alpha(x) = 0$, we have $F(x)G(x) = 0$. Since R is Id-Armendariz, $a_ib_j \in Id(R)$. Therefore, $a_{i_\alpha}b_{j_\alpha} \in Id(R_\alpha)$ and so R_α is Id-Armendariz for each $\alpha \in I$. ■

For an idempotent element e, by the corner ring of R, we mean the ring eRe with identity element e.

Proposition 2.3 Let R be a ring and $e \in Id(R)$. Then the following statements are equivalent:

(1) R is Id-Armendariz.
(2) The corner rings of R (eRe and $(1-e)R(1-e)$) are Id-Armendariz and R is an abelian ring.

Proof. If R is Id-Armendariz, then eR and $(1-e)R$ are Id-Armendariz since they are the invariant subrings of R. Now, let e be an idempotent of R. Consider $f(x) = e - er(1-e)x$ and $g(x) = (1-e) + er(1-e)x$. Therefore, $f(x)g(x) = 0$. By hypothesis $er(1-e)$ is an idempotent element and so $er(1-e) = 0$. Hence, $er = ere$ for each $r \in R$. Similarly, consider $p(x) = (1-e) - (1-e)rex$ and $q(x) = e + (1-e)rex$ in $R[x]$ for all $r \in R$. Then $p(x)q(x) = 0$. As before $(1-e)re = 0$ and $ere = re$ for all $r \in R$. It follows that e is central element of R; that is, R is abelian. Conversely, suppose eRe and $(1-e)R(1-e)$ are Id-Armendariz rings and R is abelian. We use the pierce decomposition of the ring R and so $R = eRe \oplus (1-e)R(1-e)$ and so R is Id-Armendariz ring by Theorem 2.2. ■

The following example shows that abelian rings need not to be Id-Armendariz in general.

***Example* 2.4** Consider

$$R = \{ae_{11} + be_{12} + ce_{21} + de_{22} \in M_2(\mathbb{Z}) | a \equiv d(mod2), b \equiv c \equiv 0(mod2)\}.$$

The only idempotents in R are 0 and $e_{11} + e_{22}$. So R is an abelian ring. Let $f(x) = (2e_{11} + 2e_{12}) + 2e_{12}x, g(x) = 2e_{12} - 2e_{22} + 2e_{12}x \in R[x]$. Then $f(x)g(x) = 0$, but $(2e_{11} + 2e_{12})(2e_{12}) = 4e_{12}$ is not an idempotent in R. Therefore, R is not Id-Armendariz.

Corollary 2.5 [7] Armendariz rings are abelian.

Corollary 2.6 Let R be an Id-Armendariz ring. Then e_iRe_i is Id-Armendariz for each $e_i \in Id(R)$. The converse holds if $1 = e_1 + e_2 + \cdots + e_n$, where the $e_i^{,}s$ for $1 \leqslant i \leqslant n$ are orthogonal central idempotents.

Proof. We have $R \cong e_1Re_1 \oplus \cdots \oplus e_nRe_n$ and the proof is done. ■

Since Id-Armendariz rings are abelian by Proposition 2.3, then Id-Armendariz rings are central Armendariz. Next Example shows that central Armendariz rings need not to be Id-Armendariz in general. Also, this example shows factor ring of an $Id(R)$-Armendariz ring R need not to be Id-Armendariz.

***Example* 2.7** Consider the polynomial $f(x) = (\bar{4}, \bar{0}) + (\bar{4}, \bar{1})x$ over ring $R = (\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z})$. The square of $f(x)$ is zero but the product $(\bar{4}, \bar{0})(\bar{4}, \bar{1}) = (\bar{0}, \bar{4})$ is not in $Id(R)$. Thus R is not Id-Armendariz. But since R is commutative, then R is central Armendariz. In fact R is a factor ring of $(\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z})$, which is Armendariz by [10] and so is Id-Armendariz ring.

A ring R is called reversible if for any $a, b \in R$, $ab = 0$ implies $ba = 0$. Clearly, Armendariz rings are Id-Armendariz. Now we investigate when Id-Armendariz rings are Armendariz.

Theorem 2.8 Let R be an Id-Armendariz ring which is reversible. Then R is Armendariz.

Proof. Suppose that $f(x) = \sum_{i=0}^{m} a_ix^i$ and $g(x) = \sum_{j=0}^{n} b_jx^j$ are two polynomials in

$R[x]$ such that $f(x)g(x) = 0$. Then we have

$$a_0b_0 = 0 \tag{1}$$
$$a_0b_1 + a_1b_0 = 0 \tag{2}$$
$$a_0b_2 + a_1b_1 + a_2b_0 = 0 \tag{3}$$
$$\dots$$

Since R is Id-Armendariz, $a_ib_j \in Id(R)$. We show that $a_ib_j = 0$. Since R is reversible, (1) implies that $b_0a_0 = 0$. If we multiply (2) on the right side by a_0, then $a_0b_1a_0 + a_1b_0a_0 = 0$. Therefore, $a_0b_1a_0 = 0$ and hence $a_0b_1 = (a_0b_1)^2 = 0$. So $a_1b_0 = 0$ by (2). Also if we multiply (3) on the right side by a_0, then $a_0b_2a_0 + a_1b_1a_0 + a_2b_0a_0 = 0$. Therefore $a_0b_2a_0 = 0$ and so $a_0b_2 = (a_0b_2)^2 = 0$. Hence (3) reduces to $a_1b_1 + a_2b_0 = 0$. If we multiply $a_1b_1 + a_2b_0 = 0$ on the right side by a_1, then we have $a_2b_0a_1 = 0$ and so $a_1b_1 = (a_1b_1)^2 = 0$. Therefore, $a_2b_0 = 0$. Continuing this process, we have $a_ib_j = 0$ for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. Hence, R is Armendariz. ■

We conjecture that R is an Id-Armendariz ring if for any nonzero proper Ideal I of R, R/I and I are Id-Armendariz. However, we have a counterexample to this situation as in the following.

***Example* 2.9** Let F be a field and consider the ring $R = Fe_{11} + Fe_{12} + Fe_{22}$. The only nonzero proper ideals of R are $Fe_{11} + Fe_{12}$, $Fe_{12} + Fe_{22}$ and Fe_{12}. Then R/I and I is an Armendariz ring by [7, Example 14] and so is Id-Armendariz ring. If we consider $f(x) = e_{11} + e_{12} + e_{12}x$ and $g(x) = e_{12} + e_{22} + e_{12}x$, then $f(x)g(x) = 0$ but $e_{12}(e_{12} + e_{12}) = e_{12} \notin Id(R)$. Therefore, R is not Id-Armendariz ring.

The rings $M_n(R)$ and $T_n(R)$ contain non-central idempotents. Therefore, they are not abelian and so these rings are not Id-Armendariz by Proposition 2.3.
Let S be a ring and denote the ring extension

$$\left\{ \begin{pmatrix} a & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a & a_{23} & \dots & a_{2n} \\ 0 & 0 & a & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a \end{pmatrix} | a, a_{ij} \in S \right\}$$

by R_n. In [7, Example 3] proved that R_n is not Armendariz ring for $n \geqslant 4$. Now we show that R_n is not Id-Armendariz ring for $n \geqslant 4$.

***Example* 2.10** Let S be a ring and

$$R_4 = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} | a, a_{ij} \in S \right\}.$$

Also, let $f(x) = e_{12} + (e_{12} - e_{13})x$ and $g(x) = e_{34} + (e_{24} + e_{34})x$ be two polynomials in R_4. Then $f(x)g(x) = 0$, but $e_{12}(e_{24} + e_{34}) \notin Id(R_4)$. So R_4 is not Id-Armendariz. Similarly, for the case of $n \geqslant 5$, we have the same result.

Now we have an equivalence between Id-Armendarizness and related concepts through R_3.

Proposition 2.11 For a ring S and R_3 over S the following conditions are equivalent:

(1) S is reduced;
(2) R_3 is Armendariz;
(3) R_3 is Id-Armedariz;
(4) R_3 is weak Armendariz;
(5) R_3 is semicommutative.

Proof. $1 \Rightarrow 2$ [7, Proposition 2], $2 \Rightarrow 3$ is clear, $1 \Rightarrow 5$ is proved in [8, Proposition 1.2], $5 \Rightarrow 4$ [9, Corollary 3.4] and $4 \Rightarrow 1$ is proved in [6, Proposition 2.8].
$3 \Rightarrow 1$. Let R_3 be Id-Armendariz, and assume on the contrary that there is a nonzero $a \in S$ with $a^2 = 0$ and $a \neq 0$. Put $u = a(e_{11} + e_{22} + e_{33})$ and $v = e_{12}$ in R_3. Then $u^2 = 0 = v^2$ and $uv = vu$ doesn't belong to $Id(R_3)$. Hence, R_3 is not Id-Armendariz from $(u+vx)(u-vx) = 0$, where x is an indeterminate over R_3. We get a contradiction. ■

Theorem 2.12 Let R be a ring. Then we have the following assertions:
(1) R is Id-Armendariz if and only if $R[x]$ is Id-Armendariz.
(2) R is Id-Armendariz if and only if $R[[x]]$ is Id-Armendariz.

Proof. Let R be an Id- Armendariz ring. $R[x]$ is a subring of $R[[x]]$ and so it is enough to prove (2). We have

$$R \cong \{(a_i) : a_i \in R, \forall i \geqslant 0\} = \prod_{i \geqslant 0} R.$$

Hence, by this fact and Theorem 2.2, $R[[x]]$ is Id-Armendariz. ■

Recall that for a ring R with an endomorphism α of R, the skew polynomial ring of R, denoted by $R[x, \alpha]$, is the ring obtained by giving the polynomial ring over R with the new multiplication $xr = \alpha(r)x$ for all $r \in R$. There exists an Id-Armendariz ring R over which the skew polynomial rings is not an Id-Armendariz ring as in the following.

***Example* 2.13** Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2$. Since R is a reduced ring, it is Id-Armendariz. Now let $\alpha : R \to R$ be defined by $\alpha((a, b)) = (b, a)$. Then α is an automorphim of R. Let $f(y) = (1, 0) + [(1, 0)x]y$ and $g(y) = (0, 1) + [(1, 0)x]y$ be elements in $R[x; \alpha][y]$. Then $f(y)g(y) = 0$, but $(1, 0)[(1, 0)x] \notin Id(R[x, \alpha])$. Therefore, $R[x; \alpha]$ is not Id-Armendariz.

Proposition 2.14 Let R be a ring which 2 is invertible and $G = \{1, g\}$ be a group. Then RG is Id-Armendariz if and only if R is Id-Armendariz.

Proof. Since 2 is invertible, we have $RG \cong R \times R$ via the map $\theta : a + bg \to (a + b, a - b)$. Then the result follows by Theorem 2.2. ■

Let I be an ideal of R, the amalgamated duplication of a commutative ring R along the ideal is defined to be the subring $R \bowtie I = \{(r, r + i) | r \in R, i \in I\}$ of $R \times R$. That containing R as a subring with unit element $(1, 1)$.

Proposition 2.15 Let R be a commutative ring with unit element 1 and let I be a proper ideal of R. Then R is Id-Armendariz if and only if $R \bowtie I$ is Id-Armendariz.

Proof. It is clear by definition of $R \bowtie I$. ■

Acknowledgement

This paper is supported by Islamic Azad University Central Tehran Branch (IAUCTB). The authors want to thank the authority of IAUCTB for their support to complete this research.

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