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Coupled fixed point results for T-contractions on $\mathcal{F}$-metric spaces and an application

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Abstract. The main purpose of this article is to introduce the concept of T-contraction type mappings in the function weighted metric spaces and to obtain some coupled fixed points theorems in this framework. Also, an example and an application of the existence of a solution of a system of nonlinear integral equations are considered to protect the main results.

Keywords: Function weighted metric spaces, coupled fixed point, T-contraction.

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1. Introduction and preliminaries

In 2006, Bhaskar and Lakshmikantham [7] introduced the concept of a coupled fixed point in partially ordered metric spaces. Then, other authors generalized this concept in various ordered metric spaces and obtained several fixed point results in [1, 5, 10, 15, 16], and reference therein.

On the other hand, Moradi [11] and Morales and Rojas [12] defined a T-contraction and proved some fixed point results such as Kannan contraction and Zamfirescu operator concerning this concept. Later, Filipović et al. [8] considered T-Hardy-Rogers contraction and proved some fixed and periodic point theorems. After that, many authors proved some new fixed point, coupled fixed point, tripled fixed point, and quadrupled fixed point theorems for T-contractions on various spaces in [3, 13, 14] and references therein.

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Also, in 2018, Jleli and Samet [9] defined the new concept of metric spaces, which is called function weighted metric spaces (in summary, $\mathcal{F}$-metric spaces). After that, some researchers such as Aydi et al. [4], Bera et al. [6] and Alqahtani et al. [2] discussed the structure of this space and the fixed points of mappings for such spaces.

In this paper, we define the concept of T-contraction in coupled fixed point theory in $\mathcal{F}$-metric space and establish some new fixed point theorems. Also, we consider an example and an application to the system of integral equations to support the main theorems.

We begin with some important definitions and necessary lemmas and notations.

Definition 1.1 [9] A function $f : (0, +\infty) \to \mathbb{R}$ is called a non-decreasing function if for all $s, t \in (0, +\infty)$ we have $f(s) \leqslant f(t)$, and is called logarithmic-like if every sequence $\{t_n\} \subset (0, +\infty)$ satisfies $\lim\limits_{n\to\infty} t_n = 0$ iff $\lim\limits_{n\to\infty} f(t_n) = -\infty$.

Note that we apply $\mathcal{F}$ for the set of all non-decreasing functions that are logarithmic-like.

Definition 1.2 [2] Consider a mapping $\delta : X \times X \to [0, +\infty)$, a constant $B \in [0, +\infty)$ and a $f \in \mathcal{F}$ so that for every $x_1, x_2 \in X$

(Δ_1) (self-distance axiom) $\delta(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$;
(Δ_2) (symmetry axiom) $\delta(x_1, x_2) = \delta(x_1, x_2)$;
(Δ_3) (Generalized function f-weighted triangle inequality axiom) $\delta(x_1, x_2) > 0$ implies that $f(\delta(x_1, x_2)) \leqslant f(\sum\limits_{i=1}^{N-1} \delta(v_i, v_{i+1})) + B$ for every $N \in \mathbb{N}$ with $N \geqslant 2$, and for all $(v_i)_{i=1}^{N} \subset X$ with $(v_1, v_N) = (x_1, x_2)$.

Then, the function δ is named as an $\mathcal{F}$-metric on X, and the pair (X, δ) is called an $\mathcal{F}$-metric space.

Note that any metric on X is an $\mathcal{F}$-metric on X by considering $f(t) = \ln t$ for the axiom (Δ_3). Indeed on a account of the triangle inequality, for every distinct $x_1, x_2 \in X$, for all $N \in \mathbb{N}$ with $N \geqslant 2$ and for all $(v_i)_{i=1}^{N} \subset X$ with $(v_1, v_N) = (x_1, x_2)$, we find

$$d(x_1, x_2) > 0 \Rightarrow \ln\big(d(x_1, x_2)\big) \leqslant \ln\Big(\sum_{i=1}^{N-1} d(u_i, u_{i+1})\Big).$$

Definition 1.3 [2] Consider an $\mathcal{F}$-metric space (X, δ) with a sequence $\{x_n\}$ therein. Then $\{x_n\}$ is a convergent sequence to $x \in X$ if $\lim\limits_{n\to\infty} \delta(x_n, x) = 0$.

Definition 1.4 [2] Consider an $\mathcal{F}$-metric space (X, δ) with a sequence $\{x_n\}$ therein. Then $\{x_n\}$ is a Cauchy sequence if $\lim\limits_{n,m\to\infty} \delta(x_n, x_m) = 0$.

An $\mathcal{F}$-metric space (X, δ) is complete if each Cauchy sequence in X is convergent to $x \in X$.

2. Main results

We start with the following definitions in the framework of an $\mathcal{F}$-metric space.

Definition 2.1 Let (X, δ) be an $\mathcal{F}$-metric space. An element $(x, y) \in X \times X$ is named a coupled fixed point of a mapping $F : X \times X \to X$, if $F(x, y) = x$ and $F(y, x) = y$.

Definition 2.2 Consider an $\mathcal{F}$-metric space (X, δ) with a self-mapping T on X. Then

(T_1) T is sequentially convergent if for each sequence $\{x_n\}$ that $\{Tx_n\}$ is convergent, $\{x_n\}$ also is convergent.

(T_2) T is subsequentially convergent if for each sequence $\{x_n\}$ that $\{Tx_n\}$ is convergent, $\{x_n\}$ has a convergent subsequence.

(T_3) T is a continuous mapping if $\lim_{n\to\infty} x_n = x$ induces that $\lim_{n\to\infty} Tx_n = Tx$ for each $\{x_n\}$ in X.

Theorem 2.3 Consider an $\mathcal{F}$-metric space (X, δ) with a continuous and one to one self-mapping T on X. Moreover, let $F : X \times X \to X$ be a mapping satisfying

$$\delta\big(TF(x, y), TF(x^*, y^*)\big) \leqslant \alpha\delta(Tx, Tx^*) + \beta\delta(Ty, Ty^*) \tag{1}$$

for all $x, y, x^*y^* \in X$, where $\alpha, \beta \geqslant 0$ with $\alpha + \beta < 1$; that is, F be a T-contraction. Then

i) for each $x_0, y_0 \in X$, $\{TF^n(x_0, y_0)\}$ and $\{TF^n(y_0, x_0)\}$ are Cauchy sequences;

ii) there exist $Z_{x_0}, Z_{y_0} \in X$ such that

$$\lim_{n\to\infty} TF^n(x_0, y_0) = Z_{x_0} \text{ and } \lim_{n\to\infty} TF^n(y_0, x_0) = Z_{y_0};$$

iii) if T is subsequentially convergent, then $\{TF^n(x_0, y_0)\}$ and $\{TF^n(y_0, x_0)\}$ have a convergent subsequence;

vi) there exist unique $W_{x_0}, W_{y_0} \in X$ so that

$$F(W_{x_0}, W_{y_0}) = W_{x_0} \text{ and } F(W_{y_0}, W_{x_0}) = W_{y_0},$$

that is, F has a unique coupled fixed point;

v) if T is sequentially convergent, then the sequence $\{TF^n(x_0, y_0)\}$ converges to $W_{x_0} \in X$ and the sequence $\{TF^n(y_0, x_0)\}$ converges to $W_{y_0} \in X$ for each $x_0, y_0 \in X$.

Proof. Let $x_0, y_0 \in X$ and consider

$$\begin{cases} x_1 = F(x_0, y_0) \\ y_1 = F(y_0, x_0) \end{cases} \cdots \begin{cases} x_{n+1} = F(x_n, y_n) = F^{n+1}(x_0, y_0) \\ y_{n+1} = F(y_n, x_n) = F^{n+1}(y_0, x_0) \end{cases}$$

for all $n \in \mathbb{N}\cup\{0\}$. First we shall prove that both $\{Tx_n\}$ and $\{Tx_y\}$ are Cauchy sequences. Applying (1), we obtain

$$\begin{aligned} \delta(Tx_n, Tx_{n+1}) &= \delta\big(TF(x_{n-1}, y_{n-1}), TF(x_n, y_n)\big) \\ &\leqslant \alpha\delta(Tx_{n-1}, Tx_n) + \beta\delta(Ty_{n-1}, Ty_n) \end{aligned} \tag{2}$$

and

$$\delta(Ty_n, Ty_{n+1}) \leqslant \alpha\delta(Ty_{n-1}, Ty_n) + \beta\delta(Tx_{n-1}, Tx_n). \tag{3}$$

Consider $\delta_n = \delta(Tx_n, Tx_{n+1}) + \delta(Ty_n, Ty_{n+1})$ and add (2) and (3). Then

$$\delta_n \leqslant \lambda\big(\delta(Tx_{n-1}, Tx_n) + \delta(Ty_{n-1}, Ty_n)\big) = \lambda\delta_{n-1},$$

where $\lambda = \alpha + \beta$. Therefore,

$$0 \leqslant \delta_n \leqslant \lambda\delta_{n-1} \leqslant \lambda^2\delta_{n-2} \leqslant \ldots \leqslant \lambda^n\delta_0. \tag{4}$$

If $\delta_0 = 0$, then (x_0, y_0) is a coupled fixed point. Thus, let $\delta_0 > 0$ and $m, n \in \mathbb{N}$ with $n < m$. Then

$$\begin{aligned}\sum_{i=n}^{m-1} \delta_i &= \delta_n + \delta_{n+1} + \ldots + \delta_{m-1} \\ &\leqslant (\lambda^n + \lambda^{n+1} + \ldots + \lambda^{m-1})\delta_0 \\ &\leqslant \frac{\lambda^n}{1-\lambda}\delta_0,\end{aligned}$$

where $\delta_i = \delta(Tx_i, Tx_{i+1}) + \delta(Ty_i, Ty_{i+1})$. On the other hand, assume $(f, B) \in \mathcal{F} \times [0, +\infty)$ so that (Δ_3) is complied. For an arbitrary $\epsilon > 0$ and by (Δ_3), there is a $\gamma > 0$ so that $0 < t < \gamma$ induces $f(t) < f(\epsilon) - B$. So we conclude

$$f\big(\sum_{i=n}^{m-1} \delta_i\big) \leqslant f\big(\frac{\lambda^n}{1-\lambda}\delta_0\big) < f\big(\epsilon\big) - B.$$

Then

$$f\big(\sum_{i=n}^{m-1} \delta(Tx_i, Tx_{i+1}\big) < f\big(\sum_{i=n}^{m-1} \delta_i\big) \leqslant f\big(\frac{\lambda^n}{1-\lambda}\delta_0\big) < f\big(\epsilon\big) - B, \tag{5}$$

$$f\big(\sum_{i=n}^{m-1} \delta(Ty_i, Ty_{i+1}\big) < f\big(\sum_{i=n}^{m-1} \delta_i\big) \leqslant f\big(\frac{\lambda^n}{1-\lambda}\delta_0\big) < f\big(\epsilon\big) - B. \tag{6}$$

Now, by applying (Δ_3) together with (5) and (6), we obtain

$$\delta(Tx_n, Tx_m) > 0 \ \Rightarrow f(\delta(Tx_n, Tx_m)) \leqslant f\big(\sum_{i=n}^{m-1} \delta(Tx_i, Tx_{i+1})\big) + B < f(\epsilon),$$

$$\delta(Ty_n, Ty_m) > 0 \ \Rightarrow f(\delta(Ty_n, Ty_m)) \leqslant f\big(\sum_{i=n}^{m-1} \delta(Ty_i, Ty_{i+1})\big) + B < f(\epsilon).$$

Hence, $\{Tx_n\}$ and $\{Ty_n\}$ are cauchy sequence. By completeness of X, there exist $Z_{x_0}, Z_{y_0} \in X$ such that

$$\lim_{n\to\infty} TF^n(x_0, y_0) = Z_{x_0} \text{ and } \lim_{n\to\infty} TF^n(y_0, x_0) = Z_{y_0}.$$

Now if T is subsequentially convergent, then $F^n(x_0, y_0)$ and $F^n(y_0, x_0)$ have convergent subsequences. Thus, there exist $W_{x_0}, W_{y_0} \in X$ and two sequences $\{x_{n_i}\}$ and $\{y_{n_i}\}$ so that

$$\lim_{i\to\infty} F^{n_i}(x_0, y_0) = W_{x_0} \text{ and } \lim_{i\to\infty} F^{n_i}(y_0, x_0) = W_{y_0}.$$

Because of the continuity of T, we obtain

$$\lim_{n\to\infty} TF^{n_i}(x_0,y_0) = TW_{x_0} \text{ and } \lim_{n\to\infty} TF^{n_i}(y_0,x_0) = TW_{y_0}.$$

So we have $TW_{x_0} = Z_{x_0}$ and $TW_{y_0} = Z_{y_0}$. Therefore,

$$\begin{aligned}\delta\big(TF(W_{x_0},W_{y_0}),TW_{x_0}\big) &= \lim_{i\to\infty}[\delta\big(TF(W_{x_0},W_{y_0}),TF^{n_i}(x_0,y_0)\big)]\\ &= \lim_{i\to\infty}[\delta\big(TF(W_{x_0},W_{y_0}),TF(x_{n_{i-1}},y_{n_{i-1}})\big)]\\ &\leqslant \lim_{i\to\infty}[\alpha\delta(TW_{x_0},Tx_{n_i-1}) + \beta\delta(TW_{y_0},Ty_{n_{i-1}})],\end{aligned}$$

which implies that $\delta\big(TF(W_{x_0},W_{y_0}),TW_{x_0}\big) = 0$. Thus, $TF(W_{x_0},W_{y_0}) = TW_{x_0}$. Since T is one to one, we obtain $F(W_{x_0},W_{y_0}) = W_{x_0}$. Similarly, we can obtain $F(W_{y_0},W_{x_0}) = W_{y_0}$. Thus, (W_{x_0},W_{y_0}) is a coupled fixed point of F. Now, we show that W_{x_0} and W_{y_0} are unique. Suppose that (U_{x_0},U_{y_0}) is another coupled fixed point. Then, from (1), we get

$$\begin{aligned}\delta(TW_{x_0},TU_{x_0}) &\leqslant \alpha\delta(TW_{x_0},TU_{x_0}) + \beta\delta(TW_{y_0},TU_{y_0}),\\ \delta(TW_{y_0},TU_{y_0}) &\leqslant \alpha\delta(TW_{y_0},TU_{y_0}) + \beta\delta(TW_{x_0},TU_{x_0}).\end{aligned}$$

Hence,

$$\delta(TW_{x_0},TU_{x_0}) + \delta(TW_{y_0},TU_{y_0}) \leqslant (\alpha+\beta)\big(\delta(TW_{x_0},TU_{x_0}) + \delta(TW_{y_0},TU_{y_0})\big),$$

which implies that

$$\delta(TW_{x_0},TU_{x_0}) + \delta(TW_{y_0},TU_{y_0}) = 0.$$

Therefore, $TW_{x_0} = TU_{x_0}$ and $TW_{y_0} = TU_{y_0}$. Since T is one to one, $W_{x_0} = U_{x_0}$ and $W_{y_0} = U_{y_0}$. Finally, if T is sequentially convergent, then we can replace n by n_i. Thus, we have

$$\lim_{n\to\infty} TF^{n}(x_0,y_0) = W_{x_0} \text{ and } \lim_{n\to\infty} TF^{n}(y_0,x_0) = W_{y_0}.$$

Here, the proof ends. ■

Theorem 2.4 Consider an $\mathcal{F}$-metric space (X,δ) with a continuous and one to one mapping self-mapping T on X. Moreover, let $F : X\times X \to X$ be a mapping satisfying

$$\delta\big(TF(x,y),TF(x^*,y^*)\big) \leqslant \alpha\delta\big(TF(x,y),Tx) + \beta\delta\big(TF(x^*,y^*),Tx^*\big) \tag{7}$$

for all $x,y,x^*y^* \in X$, where $\alpha,\beta \geqslant 0$ with $\alpha+\beta < 1$; that is, F be a T-contraction. Then the result of Theorem 2.3 hold.

Proof. Let $x_0,y_0 \in X$ and consider

$$\begin{cases} x_1 = F(x_0,y_0)\\ y_1 = F(y_0,x_0)\end{cases} \cdots \begin{cases} x_{n+1} = F(x_n,y_n) = F^{n+1}(x_0,y_0)\\ y_{n+1} = F(y_n,x_n) = F^{n+1}(y_0,x_0)\end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$. First we shall prove that both $\{Tx_n\}$ and $\{Tx_y\}$ are Cauchy sequences. Applying (7), we obtain

$$\begin{aligned} \delta(Tx_n, Tx_{n+1}) &= \delta\big(TF(x_{n-1}, y_{n-1}), TF(x_n, y_n)\big) \\ &\leqslant \alpha\delta\big(TF(x_{n-1}, y_{n-1}), Tx_{n-1}\big) + \beta\delta\big(TF(x_n, y_n), Tx_n\big) \end{aligned} \tag{8}$$

and

$$\delta(Ty_n, Ty_{n+1}) \leqslant \alpha\delta\big(TF(y_{n-1}, x_{n-1}), Ty_{n-1}\big) + \beta\delta\big(TF(y_n, x_n), Ty_n\big). \tag{9}$$

Let $d_n = \delta(Tx_n, Tx_{n+1}) + \delta(Ty_n, Ty_{n+1})$ and add (8) and (9). Then $d_n \leqslant \alpha d_{n-1} + \beta d_n$, which implies that $(1-\beta)d_n \leqslant \alpha d_{n-1}$. Hence,

$$d_n \leqslant \frac{\alpha}{1-\beta} d_{n-1} \leqslant \ldots \leqslant (\frac{\alpha}{1-\beta})^n d_0,$$

where $0 < \frac{\alpha}{1-\beta} = \lambda < 1$ and $n \in \mathbb{N}$. If $d_0 = 0$, then (x_0, y_0) is a coupled fixed point. Thus, let $d_0 > 0$ and $m, n \in \mathbb{N}$ with $n < m$. Then

$$\begin{aligned} \sum_{i=n}^{m-1} d_i &= d_n + d_{n+1} + \ldots + d_{m-1} \\ &\leqslant (\lambda^n + \lambda^{n+1} + \ldots + \lambda^{m-1}) d_0 \\ &\leqslant \frac{\lambda^n}{1-\lambda} d_0. \end{aligned}$$

where $d_i = \delta(Tx_i, Tx_{i+1}) + \delta(Ty_i, Ty_{i+1})$. On the other hand, assume $(f, B) \in \mathcal{F} \times [0, +\infty)$ so that (Δ_3) is complied. For an arbitrary $\epsilon > 0$ and by (Δ_3), there is a $\gamma > 0$ so that $0 < t < \gamma$ induces $f(t) < f(\epsilon) - B$. So we conclude

$$f(\sum_{i=n}^{m-1} d_i) \leqslant f(\frac{\lambda^n}{1-\lambda} d_0) < f(\epsilon) - B.$$

Then

$$f(\sum_{i=n}^{m-1} \delta(Tx_i, Tx_{i+1})) < f(\sum_{i=n}^{m-1} d_i) \leqslant f(\frac{\lambda^n}{1-\lambda} d_0) < f(\epsilon) - B, \tag{10}$$

$$f(\sum_{i=n}^{m-1} \delta(Ty_i, Ty_{i+1})) < f(\sum_{i=n}^{m-1} d_i) \leqslant f(\frac{\lambda^n}{1-\lambda} d_0) < f(\epsilon) - B. \tag{11}$$

Now, by applying (Δ_3) together with (10) and (11), we obtain

$$\delta(Tx_n, Tx_m) > 0 \Rightarrow f(\delta(Tx_n, Tx_m)) \leqslant f\big(\sum_{i=n}^{m-1} \delta(Tx_i, Tx_{i+1})\big) + B < f(\epsilon),$$

$$\delta(Ty_n, Ty_m) > 0 \Rightarrow f(\delta(Ty_n, Ty_m)) \leqslant f\big(\sum_{i=n}^{m-1} \delta(Ty_i, Ty_{i+1})\big) + B < f(\epsilon).$$

Hence, $\{Tx_n\}$ and $\{Ty_n\}$ are cauchy sequence. By completeness of X, there exist $Z_{x_0}, Z_{y_0} \in X$ such that

$$\lim_{n\to\infty} TF^n(x_0, y_0) = Z_{x_0} \text{ and } \lim_{n\to\infty} TF^n(y_0, x_0) = Z_{y_0}.$$

Now if T is subsequentially convergent, then $F^n(x_0, y_0)$ and $F^n(y_0, x_0)$ have convergent subsequences. Thus, there exist $W_{x_0}, W_{y_0} \in X$ and two sequences $\{x_{n_i}\}$ and $\{y_{n_i}\}$ so that

$$\lim_{i\to\infty} F^{n_i}(x_0, y_0) = W_{x_0} \text{ and } \lim_{i\to\infty} F^{n_i}(y_0, x_0) = W_{y_0}.$$

Because of the continuity of T, we obtain

$$\lim_{n\to\infty} TF^{n_i}(x_0, y_0) = TW_{x_0} \text{ and } \lim_{n\to\infty} TF^{n_i}(y_0, x_0) = TW_{y_0}.$$

So we have $TW_{x_0} = Z_{x_0}$ and $TW_{y_0} = Z_{y_0}$. Therefore,

$$\begin{aligned}
\delta\big(TF(W_{x_0}, W_{y_0}), TW_{x_0}\big) &= \lim_{i\to\infty} [\delta\big(TF(W_{x_0}, W_{y_0}), TF^{n_i}(x_0, y_0)\big)] \\
&= \lim_{i\to\infty} [\delta\big(TF(W_{x_0}, W_{y_0}), TF(x_{n_{i-1}}, y_{n_{i-1}})\big)] \\
&\leqslant \lim_{i\to\infty} [\alpha\delta\big(TF(W_{x_0}, W_{y_0}), TW_{x_0}\big) \\
&\quad + \beta\delta\big(TF(x_{n_{i-1}}, y_{n_{i-1}}), Tx_{n_{i-1}}\big)],
\end{aligned}$$

which induces that $\delta\big(TF(W_{x_0}, W_{y_0}), TW_{x_0}\big) = 0$. Thus, $TF(W_{x_0}, W_{y_0}) = TW_{x_0}$. Since T is one to one, then $F(W_{x_0}, W_{y_0}) = W_{x_0}$. Similarly, we can obtain $F(W_{y_0}, W_{x_0}) = W_{y_0}$. Thus, (W_{x_0}, W_{y_0}) is a coupled fixed point of F. Now, we show that W_{x_0} and W_{y_0} are unique. Suppose that (U_{x_0}, U_{y_0}) is another coupled fixed point. Then, from (1), we get

$$\begin{aligned}
\delta(TW_{x_0}, TU_{x_0}) &= \delta\big(TF(W_{x_0}, W_{y_0}), TF(U_{x_0}, U_{y_0})\big) \\
&\leqslant \alpha\delta\big(TF(W_{x_0}, W_{y_0}), TW_{x_0}\big) + \beta\big(TF(U_{x_0}, U_{y_0}), TU_{x_0}\big),
\end{aligned}$$

which implies that $\delta(TW_{x_0}, TU_{x_0}) = 0$. Thus, $TW_{x_0} = TU_{x_0}$. Since T is one to one, then $W_{x_0} = U_{x_0}$. Similarly, we can obtain $W_{y_0} = U_{y_0}$. Thus, $(W_{x_0}, W_{y_0}) = (U_{x_0}, U_{y_0})$. Finally if T is sequentially convergent, then we can replace n by n_i. Thus, we have

$$\lim_{n\to\infty} TF^n(x_0, y_0) = W_{x_0} \text{ and } \lim_{n\to\infty} TF^n(y_0, x_0) = W_{y_0}.$$

■

***Example* 2.5** Let$X = \mathbb{R}^+$. Define $\delta : X \times X \to [0,\infty)$ by $\delta(x,y) = |x-y|$ for all $x, y \in X$. Then δ is an $\mathcal{F}$-metric with $f(t) = \ln t$ and $B = 0$. Consider $F : X \times X \to X$ and $T : X \to X$

$$F(x,y) = \frac{x+y}{4} \text{ and } T(x) = \frac{x}{2}.$$

Then

$$\begin{aligned}\delta\big(TF(x,y), TF(x^*,y^*)\big) &= \delta\big(T(\frac{x+y}{4}), T(\frac{x^*+y^*}{4})\big)\\ &= |\frac{x+y}{8} - \frac{x^*+y^*}{8}|\\ &= |\frac{x-x^*}{8} + \frac{y-y^*}{8}|\\ &\leqslant \frac{1}{4}[\delta(Tx,Tx^*) + \delta(Ty,Ty^*)].\end{aligned}$$

Hence, F satisfies the contractive condition (1) by $\alpha = \beta = \frac{1}{4}$. Thus, by Theorem 2.3, F has an unique coupled fixed point. Obviously, $(0,0)$ is a coupled fixed point of F.

3. An Application

Consider the following system of integral equations:

$$\begin{cases} x(t) = \int_a^b M(t,s)K(s,x(s),y(s))ds, \\ y(t) = \int_a^b M(t,s)K(s,y(s),x(s))ds, \end{cases} \tag{12}$$

for all $t \in I = [a,b]$, where $b > a$, $M \in C(I \times I, [0,\infty))$ and $K \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

Let $C(I,\mathbb{R})$ be the Banach space of all real continuous functions defined on I with the sup norm and $C(I \times I \times C(I,\mathbb{R}), \mathbb{R})$ be the space of all continuous functions defined on $I \times I \times C(I,\mathbb{R})$ endowed with the $\mathcal{F}$-metric $\delta(u,v) = \sup\big|u(t) - v(t)\big|$ for all $u, v \in X$ and $t \in I$.

Theorem 3.1 Let $(C(I,\mathbb{R}),\delta)$ be a complete $\mathcal{F}$-metric space and $f : C(I,\mathbb{R}) \times C(I,\mathbb{R}) \to C(I,\mathbb{R})$ be a operator defined by $f(x,y)t = \int_a^b M(t,s)K(s,x(s),y(s))ds$, where $M \in C(I \times I, [0,\infty))$ and $K \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ are two operators satisfying the following conditions:

(i) $||K||_\infty = \sup\limits_{s \in I,\, x,y \in C(I,\mathbb{R})} |K(s,x(s),y(s))| < \infty$;

(ii) for all $x, y \in C(I,\mathbb{R})$ and for each $t, s \in I$, we have

$$\Big|K\big(t,x(t),y(t)\big) - K\big(t,u(t),v(t)\big)\Big| \leqslant \alpha\Big|x(t) - u(t)\Big| + \beta\Big|y(t) - v(t)\Big|;$$

(iii) $\sup\limits_{t \in I} \int_a^b M(t,s)ds < 1$.

Then the system of integral equations (12) has a solution in $C(I,\mathbb{R}) \times C(I,\mathbb{R})$.

Proof. It is easy to show that (x,y) is a solution of the system (12) if and only if (x,y) is a coupled fixed point of f. To establish the existence of such a point, assume T is the identity mapping and apply Theorem 2.3. Then, for $x,y,x^*,y^* \in X$ and for all $t \in I$, we have

$$\begin{aligned}
\delta\Big(Tf(x,y),Tf(x^*,y^*)\Big) &= \sup_{t\in I}\Big|f(x,y)(t)-f(x^*,y^*)(t)\Big| \\
&= \sup_{t\in I}\Big|\int_a^b M(t,s)K\big(s,x(s),y(s)\big)ds \\
&\quad -\int_a^b M(t,s)K\big(s,x^*(s),y^*(s)\big)ds\Big| \\
&= \sup_{t\in I}\Big|\int_a^b M(t,s)\ [K\big(s,x(s),y(s)\big)-K\big(s,x^*(s),y^*(s)\big)]ds\Big| \\
&\leqslant \int_a^b \big|M(t,s)\big|\ \big|K\big(s,x(s),y(s)\big)-K\big(s,x^*(s),y^*(s)\big)\big|ds.
\end{aligned}$$

By (ii), we obtain

$$\begin{aligned}
\Big|f(x,y)(t)-f(x^*,y^*)(t)\Big| &\leqslant \int_a^b \big|M(t,s)\big|\big[\alpha\big|x(s)-x^*(s)\big|+\beta\big|y(s)-y^*(s)\big|\big]ds \\
&\leqslant \Big[\alpha\delta(Tx,Tx^*)+\beta\delta(Ty,Ty^*)\Big]\Big(\int_a^b \big|M(t,s)\big|ds\Big).
\end{aligned}$$

Using condition (iii), we obtain

$$\Big|f(x,y)(t)-f(x^*,y^*)(t)\Big| \leqslant \alpha\delta(Tx,Tx^*)+\beta\delta(Ty,Ty^*).$$

Hence, we have

$$\delta\big(Tf(x,y),Tf(x^*,y^*)\big) \leqslant \alpha\delta(Tx,Tx^*)+\beta\delta(Ty,Ty^*)$$

for all $x,y,x^*,y^* \in X$. Thus the contractive condition of Theorem 2.3 is satisfied. Thus, f has an unique coupled fixed point in $C\big([a,b],\mathbb{R}\big)$. Consequently, the system of integral equations (12) has a solution in $C(I,\mathbb{R})\times C(I,\mathbb{R})$. ■

References

[1] M. Abbas, M. Alikhan, S. Radenović, Common coupled fixed point theorems in cone metric space for w-compatiable mapping, Appl. Math. Comput. 217 (2010), 195-202.

[2] O. Alghatani, E. Karapinar, P. Shahi, Common fixed-point results in function weighted metric space, J. Inequal Appl. 2019, 2019:164.

[3] H. Aydi, E. Karapinar, Tripled fixed point in ordered metric spaces, Bull. Math. Anal. Appl. 4 (1) (2012), 197-207.

[4] H. Aydi, E. Karapinar, Z.D. Mitrović, T. Rashid, A remark on "Existence and uniqueness for a neutral differential problem with unbounded delay via fixed point results $\mathcal{F}$-metric spaces", RACSAM. 113 (4) (2019), 3197-3206.

[5] H. Aydi, B. Samet, C. Vetro, Coupled fixed-point results in cone metric space for $\bar{w}$-compatible mappings, Fixed point Theory Appl. 2011, 2011:27.

[6] A. Bera, H. Garai, B. Damjanović, A. Chanda, Some interesting results on $\mathcal{F}$-metric spaces, Filomat. 33 (10) (2019), 3257-3268.
[7] T. Bhaskar, V. lakshmikantham, Fixed point theorems in partially ordered metric space and applications, Nonlinear Anal. 65 (2006), 1379-1393.
[8] M. Filipović, L. Paunović, S. Radenović, M. Rajović, Remarks on "Cone metric spaces and fixed point theorems of T-Kannan and T-Chatterjea contractive mappings", Math. Comput. Modelling. 54 (2011), 1467-1472.
[9] M. Jleli, B. Samet, On a new generalization of metric spaces, J. Fixed Point Theory Appl. 20, 2018:128.
[10] V. Lakshmikantham, L. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70 (12) (2009), 4341-4349.
[11] S. Moradi, Kannan fixed-point theorem on complete metric spaces and generalized metric spaces depended on another function, Int. J. Math. Anal. 5 (47) (2011), 2313-2320.
[12] J. R. Morales, E. Rojas, Some results on T-zamfirescu operators, Revista Notas de Matematica. 5 (1) (2009), 64-71.
[13] H. Rahimi, S. Radenović, G. Soleimani Rad, P. Kumam, Quadrupled fixed point results in abstract metric spaces, Comp. Appl. Math. 33 (2014), 671-685.
[14] H. Rahimi, P. Vetro, G.S. Rad, Coupled fixed-point results for T-contractions on cone metric spaces with applications, Math Notes. 98 (2015), 158-167.
[15] F. Sabetghadam, H.P. Masiha, A.H. Sanatpour, Some coupled fixed point theorems in cone metric space, Fixed Point Theory Appl. 2009, 2009:125426.
[16] G. Soleimani Rad, S. Shukla, H. Rahimi, Some relations between n-tuple fixed point and fixed point results, RACSAM. 109 (2) (2015), 471-481.