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# Conjectures on the anti-automorphism of $Z$-basis of the Steenrod algebra

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**Abstract.** In this paper, we compute the images of some of the $Z$-basis elements under the anti-automorphism map $\chi$ of the mod 2 Steenrod algebra $\mathcal{A}_2$ and propose some conjectures based on our computations.

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## 1. Introduction

The mod 2 Steenrod algebra, $\mathcal{A}_2$, is the algebra generated by all stable primary cohomology operations, called the Steenrod operations, which play a crucial role in the solution of many problems, such as calculating homotopy groups of $n$-sphere, Hopf invariant problem, and characteristic classes of vector bundles in algebraic topology. Its Hopf algebraic structure allows us to define an anti-automorphism on it. Milnor constructed a base system on the Steenrod algebra and gave a formula for the images of the Milnor basis elements under the anti-automorphism map [7]. Through this formula, Davis [4] and Silverman [9] compute the images of some certain Steenrod operations, Barrat and Miller [3] obtain new identities related to the anti-automorphism. For prime $p > 2$, the mod $p$ Steenrod algebra has also been studied in the literature [10] and we refer to [5, 6, 11] for the computations of the images of some certain Steenrod operations under the anti-automorphism map.

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There is no general rule for the computation of the elements in $\mathcal{A}_2$ under the anti-automorphism, that means each computation is important on its own. In this paper, we consider some certain finite subalgebras of the mod 2 Steenrod algebra and compute the images of some of their basis elements under the anti-automorphism map and propose conjectures based on our computations.

## 2. Preliminaries

The Steenrod squares, $\mathrm{Sq}^i$ for $i \geqslant 0$, are group homomorphisms

$$\mathrm{Sq}^i : H^n(X;\mathbb{Z}_2) \to H^{n+i}(X;\mathbb{Z}_2)$$

on the cohomology of the topological space $X$, where $\mathrm{Sq}^0$ is the identity and $\mathrm{Sq}^i$ is zero whenever $n < i$ [10]. These squares satisfy the Adem relations

$$\mathrm{Sq}^i\mathrm{Sq}^j = \sum_{k=0}^{[\frac{i}{2}]} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k}\mathrm{Sq}^k \tag{1}$$

for $0 < i < 2j$, where $[\frac{i}{2}]$ denotes the greatest integer less than or equal to $\frac{i}{2}$ and the binomial coefficients are taken modulo 2. The grading of the $\mathrm{Sq}^i$ is $i$, and for the composition of the Steenrod squares $\mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2}\ldots\mathrm{Sq}^{i_n}$, the grading is $i_1+i_2+\cdots+i_n$. The length of the monomial $\mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2}\ldots\mathrm{Sq}^{i_n}$ is the number of Steenrod squares, $n$. Note that the Adem relations preserve the grading and do not extend the length of the composition when it is applied on $\mathrm{Sq}^i\mathrm{Sq}^j$ for $0 < i < 2j$.

These operations form a free graded associative $\mathbb{Z}_2$-algebra, called the mod 2 Steenrod algebra $\mathcal{A}_2$ subject to the Adem relations. Note that $\mathcal{A}_2$ is generated algebraically by $\mathrm{Sq}^{2^k}$ for every non-negative integer $k$ [8]. For details of the Steenrod algebra, we refer to [1, 10, 12, 13].

$\mathcal{A}_2$ has a Hopf algebraic structure so it admits a unique anti-automorphism $\chi$ on itself $\chi : \mathcal{A}_2 \to \mathcal{A}_2$ defined on the Steenrod squares by

$$\chi(\mathrm{Sq}^k) = \sum_{1\leqslant i\leqslant k} \mathrm{Sq}^i\chi(\mathrm{Sq}^{k-i})$$

such that $\chi(\mathrm{Sq}^0) = \mathrm{Sq}^0$ and $\chi^2 = 1$ [7]. We will pay attention to the finite subalgebras $\mathcal{A}_2(n)$ of $\mathcal{A}_2$ which are algebraically generated by $\mathrm{Sq}^{2^i}$ for $0 \leqslant i \leqslant n$ [7]. Note that each subalgebra $\mathcal{A}_2(n)$ is a vector subspace of $\mathcal{A}_2$ over $\mathbb{Z}_2$ with dimension $2^{(n+1)(n+2)/2}$. Wood [14] defines $Z$-basis on $\mathcal{A}_2$ that can be also extended to the whole algebra $\mathcal{A}_2$. Consider the monomial $X_n = \mathrm{Sq}^{1.2^n}\mathrm{Sq}^{3.2^{n-1}}\mathrm{Sq}^{7.2^{n-2}}\cdots\mathrm{Sq}^{2^{n+1}-1}$ for an integer $n \geqslant 0$ and define $Z_n = X_nX_{n-1}\ldots X_1X_0$.

For instance, $Z_0 = \mathrm{Sq}^1$, $Z_1 = \mathrm{Sq}^2\mathrm{Sq}^3\mathrm{Sq}^1$ and $Z_2 = \mathrm{Sq}^4\mathrm{Sq}^6\mathrm{Sq}^7\mathrm{Sq}^2\mathrm{Sq}^3\mathrm{Sq}^1$.

Wood [14] shows that $Z_n$ is the top element, the monomial with a maximum length, of $\mathcal{A}_2(n)$ and the set of $2^{(n+1)(n+2)/2}$ monomials obtained by selecting all subsets of $Z_n$ in the given order together with $\mathrm{Sq}^0$ is an additive basis for $\mathcal{A}_2(n)$. For instance, the basis elements for $\mathcal{A}_2(2)$ are $\mathrm{Sq}^2\mathrm{Sq}^3\mathrm{Sq}^1$, $\mathrm{Sq}^2\mathrm{Sq}^3$, $\mathrm{Sq}^2\mathrm{Sq}^1$, $\mathrm{Sq}^3\mathrm{Sq}^1$, $\mathrm{Sq}^2$, $\mathrm{Sq}^3$, $\mathrm{Sq}^1$, $\mathrm{Sq}^0$.

Now we compute the images of some of the basis elements under the anti-automorphism map $\chi$. So, we will use the stripping technique which is practical to derive a new relation from old relation considering the Adem relations. It is based on the action of the dual Steenrod algebra to the original algebra $\mathcal{A}_2$. We refer to [15] for the details.

The basic idea of the stripping technique is as follows [15]. We call a finite non-negative sequence $(\omega_1, \omega_2, \ldots, \omega_k)$ an allowable vector if all $\omega_i$'s are some power of 2 and $\omega_i \geqslant \omega_{i+1}$ for all $i = 1, 2, \ldots, k-1$. For example, the vectors $(16, 4, 2)$, $(4, 1)$, $(1)$, and $(8, 4, 2, 1)$ are allowable while $(16, 0, 4)$ and $(2, 8)$ are not allowable. To obtain a new relation from old relation, we use these types of vectors and apply them to each of the monomial in the relation. Let $\mathrm{Sq}^{n_1}\mathrm{Sq}^{n_2} \ldots \mathrm{Sq}^{n_m}$ be a monomial in the relation and $(\omega_1, \omega_2, \ldots, \omega_k)$ be the allowable vector. Stripping the relation with the vector is to strip each monomial in the relation and this is done by subtracting $(\omega_1, \omega_2, \ldots, \omega_k)$ componentwise from the vector of the gradings $(n_1, n_2, \ldots, n_m)$ of the monomial. For the case $n \neq k$, we will add 0's to make the length of the sequence equal, and for the case $n_s - \omega_s < 0$, we do not substract. Then the new monomial derived from the original will be a monomial in the new relation. For instance, if the relation

$$\mathrm{Sq}^{15}\mathrm{Sq}^{8} = 0 \tag{2}$$

is stripped by the allowable vector $(4, 2)$, then we get the new relation $\mathrm{Sq}^{11}\mathrm{Sq}^{6} = 0$. To strip the relation given in (2) with the allowable vector $(4)$, first we add 0 to the sequence $(4)$ to make the length equal so we have two possible vectors $(4, 0)$ and $(0, 4)$ and then we strip the relation with both to get $\mathrm{Sq}^{11}\mathrm{Sq}^{8} + \mathrm{Sq}^{15}\mathrm{Sq}^{4} = 0$.

The following theorem obtained from the stripping technique gives us a new relation in $\mathcal{A}_2$.

**Theorem 2.1** For $n > 1$, $\mathrm{Sq}^{2^n-1}\mathrm{Sq}^{2^n}\mathrm{Sq}^{2^{n+1}} = \mathrm{Sq}^{2^{n+2}-1}$.

**Proof.** By the Adem relations, we have

$$\mathrm{Sq}^{2k-1}\mathrm{Sq}^{k} = 0 \tag{3}$$

for $k > 1$ [15]. If we take $k = 2^n$, then we get

$$\mathrm{Sq}^{2^{n+1}-1}\mathrm{Sq}^{2^n} = 0. \tag{4}$$

If we strip the relation (4) with the allowable vector $(2^n)$, we get

$$\mathrm{Sq}^{2^n-1}\mathrm{Sq}^{2^n} + \mathrm{Sq}^{2^{n+1}-1} = 0. \tag{5}$$

Next, we take $k = 2^{n+1}$ in the relation (3), then the relation transforms into

$$\mathrm{Sq}^{2^{n+2}-1}\mathrm{Sq}^{2^{n+1}} = 0.$$

Then we strip (6) with the allowable vector $(2^{n+1})$ to get

$$\mathrm{Sq}^{2^{n+1}-1}\mathrm{Sq}^{2^{n+1}} + \mathrm{Sq}^{2^{n+2}-1} = 0. \tag{6}$$

The above relation together with the relation (5) gives the desired relation

$$\mathrm{Sq}^{2^n-1}\mathrm{Sq}^{2^n}\mathrm{Sq}^{2^{n+1}} = \mathrm{Sq}^{2^{n+2}-1}.$$

■

## 3. Computations

The anti-automorphism $\chi$ is compatible with the subalgebras $\mathcal{A}_2(n)$ for $n \geqslant 0$. Since $\chi$ does not change the degree of the monomial, we will derive conjectures for the images of the top element $Z_n$ in $\mathcal{A}_2(n)$ under $\chi$. Since $Z_n = X_n Z_{n-1}$, we start with computing $\chi(X_n)$ for $n = 0, 1, 2, 3$. The computations in the following examples are obtained from the definition of $\chi$ and the Adem relations.

***Example* 3.1** It is trivial that $\chi(X_0) = \mathrm{Sq}^1$ and

$$\chi(X_1) = \chi(\mathrm{Sq}^2\mathrm{Sq}^3) = \chi(\mathrm{Sq}^3)\chi(\mathrm{Sq}^2) = (\mathrm{Sq}^2\mathrm{Sq}^1)\mathrm{Sq}^2 = \mathrm{Sq}^5 = \mathrm{Sq}^2\mathrm{Sq}^3.$$

***Example* 3.2** For the monomial $X_2 = \mathrm{Sq}^4\mathrm{Sq}^6\mathrm{Sq}^7$, the image of $X_2$ under $\chi$ can be obtained as follows:

$$\begin{aligned}
\chi(X_2) &= \chi(\mathrm{Sq}^4\mathrm{Sq}^6\mathrm{Sq}^7) = \chi(\mathrm{Sq}^7)\chi(\mathrm{Sq}^6)\chi(\mathrm{Sq}^4) \\
&= (\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1)(\mathrm{Sq}^4\mathrm{Sq}^2)(\mathrm{Sq}^3\mathrm{Sq}^1 + \mathrm{Sq}^4) \\
&= \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^3\mathrm{Sq}^1 + \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^4 \\
&= 0 + \mathrm{Sq}^{12}\mathrm{Sq}^4\mathrm{Sq}^1 + \mathrm{Sq}^{14}\mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^{15}\mathrm{Sq}^2 + \mathrm{Sq}^{16}\mathrm{Sq}^1 + \mathrm{Sq}^{17} + \mathrm{Sq}^{12}\mathrm{Sq}^5 \\
&= \mathrm{Sq}^4\mathrm{Sq}^6\mathrm{Sq}^7.
\end{aligned}$$

***Example* 3.3** Similar to Example 3.2, the image of the monomial $X_3 = \mathrm{Sq}^8\mathrm{Sq}^{12}\mathrm{Sq}^{14}\mathrm{Sq}^{15}$ under $\chi$ is obtained as follows:

$$\begin{aligned}
\chi(X_3) &= \chi(\mathrm{Sq}^8\mathrm{Sq}^{12}\mathrm{Sq}^{14}\mathrm{Sq}^{15}) = \chi(\mathrm{Sq}^{15})\chi(\mathrm{Sq}^{14})\chi(\mathrm{Sq}^{12})\chi(\mathrm{Sq}^8) \\
&= (\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1)(\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^2)(\mathrm{Sq}^8\mathrm{Sq}^3\mathrm{Sq}^1 + \mathrm{Sq}^8\mathrm{Sq}^4) \\
&\quad + (\mathrm{Sq}^7\mathrm{Sq}^1 + \mathrm{Sq}^5\mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^6\mathrm{Sq}^2 + \mathrm{Sq}^8) \\
&= \mathrm{Sq}^{32}\mathrm{Sq}^{12}\mathrm{Sq}^4\mathrm{Sq}^1 + \mathrm{Sq}^{36}\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^1 + \mathrm{Sq}^{38}\mathrm{Sq}^7\mathrm{Sq}^3\mathrm{Sq}^1 \\
&\quad + \mathrm{Sq}^{32}\mathrm{Sq}^{16}\mathrm{Sq}^1 + \mathrm{Sq}^{38}\mathrm{Sq}^8\mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^{40}\mathrm{Sq}^6\mathrm{Sq}^2\mathrm{Sq}^1 \\
&\quad + \mathrm{Sq}^{42}\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^{32}\mathrm{Sq}^{14}\mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^{33}\mathrm{Sq}^{16} \\
&\quad + \mathrm{Sq}^{32}\mathrm{Sq}^{15}\mathrm{Sq}^2 + \mathrm{Sq}^{32}\mathrm{Sq}^{12}\mathrm{Sq}^5 + \mathrm{Sq}^{36}\mathrm{Sq}^{12}\mathrm{Sq}^1 \\
&\quad + \mathrm{Sq}^{38}\mathrm{Sq}^{10}\mathrm{Sq}^1 + \mathrm{Sq}^{40}\mathrm{Sq}^8\mathrm{Sq}^1 + \mathrm{Sq}^{44}\mathrm{Sq}^4\mathrm{Sq}^1 + \mathrm{Sq}^{46}\mathrm{Sq}^2\mathrm{Sq}^1 \\
&\quad + \mathrm{Sq}^{39}\mathrm{Sq}^{10} + \mathrm{Sq}^{47}\mathrm{Sq}^2 + \mathrm{Sq}^{48}\mathrm{Sq}^1 + \mathrm{Sq}^{49} \\
&\quad + \mathrm{Sq}^{39}\mathrm{Sq}^8\mathrm{Sq}^2 + \mathrm{Sq}^{43}\mathrm{Sq}^4\mathrm{Sq}^2 + \mathrm{Sq}^{45}\mathrm{Sq}^3\mathrm{Sq}^1 \\
&\quad + \mathrm{Sq}^{36}\mathrm{Sq}^{13} + \mathrm{Sq}^{40}\mathrm{Sq}^9 + \mathrm{Sq}^{45}\mathrm{Sq}^4 + \mathrm{Sq}^{38}\mathrm{Sq}^8\mathrm{Sq}^3 \\
&\quad + \mathrm{Sq}^{38}\mathrm{Sq}^9\mathrm{Sq}^2 + \mathrm{Sq}^{40}\mathrm{Sq}^7\mathrm{Sq}^2 + \mathrm{Sq}^{36}\mathrm{Sq}^9\mathrm{Sq}^4 \\
&= \mathrm{Sq}^8\mathrm{Sq}^{12}\mathrm{Sq}^{14}\mathrm{Sq}^{15}.
\end{aligned}$$

The computations of $\chi(X_n)$ for $n = 1, 2, 3$ and the stripping technique will be used for computing the images $\chi(Z_n)$ for $n = 1, 2, 3$.

**Theorem 3.4** For the top element $Z_2$ considering the $Z$-basis in $\mathcal{A}_2(2)$, $\chi(Z_2) = \mathrm{Sq}^{17}\mathrm{Sq}^5\mathrm{Sq}^1$.

**Proof.** By Example 3.2 and

$$\chi(Z_1) = \chi(\mathrm{Sq}^2\mathrm{Sq}^3\mathrm{Sq}^1) = \chi(\mathrm{Sq}^1)\chi(\mathrm{Sq}^3)\chi(\mathrm{Sq}^2) = (\mathrm{Sq}^1)(\mathrm{Sq}^2\mathrm{Sq}^1)(\mathrm{Sq}^2) = \mathrm{Sq}^5\mathrm{Sq}^1,$$

we have

$$\chi(Z_2) = \chi(Z_1)\chi(X_2) = (\mathrm{Sq}^5\mathrm{Sq}^1)(\mathrm{Sq}^4\mathrm{Sq}^6\mathrm{Sq}^7) = Z_2.$$

We apply $\mathrm{Sq}^4$, $\mathrm{Sq}^6$, and $\mathrm{Sq}^7$ respectively to the monomial $\mathrm{Sq}^5\mathrm{Sq}^1$ from the right. By the relation (3) we have

$$\mathrm{Sq}^9\mathrm{Sq}^5 = 0. \tag{7}$$

If the monomial $\mathrm{Sq}^9\mathrm{Sq}^5$ is stripped by the allowable vector (4), then the following holds:

$$\mathrm{Sq}^5\mathrm{Sq}^5 = \mathrm{Sq}^9\mathrm{Sq}^1.$$

Since $\mathrm{Sq}^1\mathrm{Sq}^4 = \mathrm{Sq}^5$ we obtain

$$\mathrm{Sq}^5\mathrm{Sq}^1\mathrm{Sq}^4 = \mathrm{Sq}^9\mathrm{Sq}^1. \tag{8}$$

Next, we apply $\mathrm{Sq}^6$ from right to the above relation

$$\mathrm{Sq}^5\mathrm{Sq}^1\mathrm{Sq}^4\mathrm{Sq}^6 = \mathrm{Sq}^9\mathrm{Sq}^1\mathrm{Sq}^6.$$

Note that we have the following relation

$$\mathrm{Sq}^9\mathrm{Sq}^1\mathrm{Sq}^6 = \mathrm{Sq}^{13}\mathrm{Sq}^3 \tag{9}$$

since the relation (8) is obtained from stripping it by the allowable vector $(4, 2)$. Hence, we obtain $\mathrm{Sq}^5\mathrm{Sq}^1\mathrm{Sq}^4\mathrm{Sq}^6 = \mathrm{Sq}^{13}\mathrm{Sq}^3$. Finally, we apply $\mathrm{Sq}^7$ to the above relation from right to have

$$\mathrm{Sq}^5\mathrm{Sq}^1\mathrm{Sq}^4\mathrm{Sq}^6\mathrm{Sq}^7 = \mathrm{Sq}^{13}\mathrm{Sq}^3\mathrm{Sq}^7.$$

Again the following relation $\mathrm{Sq}^{13}\mathrm{Sq}^3\mathrm{Sq}^7 = \mathrm{Sq}^{17}\mathrm{Sq}^5\mathrm{Sq}^1$ holds since (9) is obtained from stripping it by the allowable vector $(4, 2, 1)$. This follows that

$$\chi(Z_2) = \mathrm{Sq}^5\mathrm{Sq}^1\mathrm{Sq}^4\mathrm{Sq}^6\mathrm{Sq}^7 = \mathrm{Sq}^{17}\mathrm{Sq}^5\mathrm{Sq}^1 = Z_2.$$

■

**Theorem 3.5** For the top element $Z_3$ considering the $Z$-basis in $\mathcal{A}_2(3)$, $\chi(Z_3) = \mathrm{Sq}^{49}\mathrm{Sq}^{17}\mathrm{Sq}^5\mathrm{Sq}^1 = Z_3$.

**Proof.** We will compute $\chi(Z_3)$ by using the results in Examples 3.3 and Theorem 3.4. We have that

$$\chi(Z_3) = \chi(Z_2)\chi(X_3) = (\mathrm{Sq}^{17}\mathrm{Sq}^5\mathrm{Sq}^1)(\mathrm{Sq}^8\mathrm{Sq}^{12}\mathrm{Sq}^{14}\mathrm{Sq}^{15}).$$

We apply $\mathrm{Sq}^8$, $\mathrm{Sq}^{12}$, $\mathrm{Sq}^{14}$, and $\mathrm{Sq}^{15}$ respectively to the monomial $\mathrm{Sq}^{17}\mathrm{Sq}^5\mathrm{Sq}^1$ from the right. Note that the relation

$$\mathrm{Sq}^{17}\mathrm{Sq}^5\mathrm{Sq}^1\mathrm{Sq}^8 = \mathrm{Sq}^{25}\mathrm{Sq}^5\mathrm{Sq}^1 \tag{10}$$

holds since if it is stripped by the allowable vector $(8)$, we have

$$\mathrm{Sq}^9\mathrm{Sq}^5\mathrm{Sq}^1\mathrm{Sq}^8 + \mathrm{Sq}^{17}\mathrm{Sq}^5\mathrm{Sq}^1 = \mathrm{Sq}^{17}\mathrm{Sq}^5\mathrm{Sq}^1$$

so $\mathrm{Sq}^9\mathrm{Sq}^5\mathrm{Sq}^1\mathrm{Sq}^8 = 0$ by the relation (7). Then apply $\mathrm{Sq}^{12}$ from right to the relation (10), we have now that

$$\mathrm{Sq}^{17}\mathrm{Sq}^5\mathrm{Sq}^1\mathrm{Sq}^8\mathrm{Sq}^{12} = \mathrm{Sq}^{25}\mathrm{Sq}^5\mathrm{Sq}^1\mathrm{Sq}^{12}.$$

Again we have the relation

$$\mathrm{Sq}^{25}\mathrm{Sq}^5\mathrm{Sq}^1\mathrm{Sq}^{12} = \mathrm{Sq}^{33}\mathrm{Sq}^9\mathrm{Sq}^1. \tag{11}$$

If it is stripped by the allowable vector $(8,4)$, we obtain

$$\mathrm{Sq}^{17}\mathrm{Sq}^1\mathrm{Sq}^1\mathrm{Sq}^{12} + \mathrm{Sq}^{17}\mathrm{Sq}^5\mathrm{Sq}^1\mathrm{Sq}^8 = \mathrm{Sq}^{25}\mathrm{Sq}^5\mathrm{Sq}^1$$

so it turns into the relation (10) since $\mathrm{Sq}^1\mathrm{Sq}^1 = 0$. Hence, we have

$$\mathrm{Sq}^{17}\mathrm{Sq}^5\mathrm{Sq}^1\mathrm{Sq}^8\mathrm{Sq}^{12} = \mathrm{Sq}^{33}\mathrm{Sq}^9\mathrm{Sq}^1.$$

Next, we apply $\mathrm{Sq}^{14}$ to the above relation from right to obtain the new relation

$$\mathrm{Sq}^{17}\mathrm{Sq}^5\mathrm{Sq}^1\mathrm{Sq}^8\mathrm{Sq}^{12}\mathrm{Sq}^{14} = \mathrm{Sq}^{33}\mathrm{Sq}^9\mathrm{Sq}^1\mathrm{Sq}^{14}.$$

Note that the relation

$$\mathrm{Sq}^{33}\mathrm{Sq}^9\mathrm{Sq}^1\mathrm{Sq}^{14} = \mathrm{Sq}^{41}\mathrm{Sq}^{13}\mathrm{Sq}^3 \tag{12}$$

holds since the relation (11) can be obtained from it by stripping with the allowable vector $(8,4,2)$. In that case,

$$\mathrm{Sq}^{17}\mathrm{Sq}^5\mathrm{Sq}^1\mathrm{Sq}^8\mathrm{Sq}^{12}\mathrm{Sq}^{14} = \mathrm{Sq}^{41}\mathrm{Sq}^{13}\mathrm{Sq}^3. \tag{13}$$

Finally, we apply $\mathrm{Sq}^{15}$ to the above relation from right and get

$$\mathrm{Sq}^{17}\mathrm{Sq}^5\mathrm{Sq}^1\mathrm{Sq}^8\mathrm{Sq}^{12}\mathrm{Sq}^{14}\mathrm{Sq}^{15} = \mathrm{Sq}^{41}\mathrm{Sq}^{13}\mathrm{Sq}^3\mathrm{Sq}^{15}. \tag{14}$$

Then we have the relation $\mathrm{Sq}^{41}\mathrm{Sq}^{13}\mathrm{Sq}^3\mathrm{Sq}^{15} = \mathrm{Sq}^{49}\mathrm{Sq}^{17}\mathrm{Sq}^5\mathrm{Sq}^1$, since the relation (12) is obtained from it by stripping with the allowable vector $(8,4,2,1)$. Therefore, we have

$$\chi(Z_3) = \mathrm{Sq}^{17}\mathrm{Sq}^5\mathrm{Sq}^1\mathrm{Sq}^8\mathrm{Sq}^{12}\mathrm{Sq}^{14}\mathrm{Sq}^{15} = \mathrm{Sq}^{49}\mathrm{Sq}^{17}\mathrm{Sq}^5\mathrm{Sq}^1 = Z_3.$$

■

## 4. Conjectures

According to the computations and results given in the previous section, we propose the following conjectures for $\chi(X_n)$ and $\chi(Z_n)$. Similar to $Z_2$ and $Z_3$, the authors think that applying the certain stripping relations to the images of the top elements $Z_n$ under $\chi$ for $n > 4$ is also a way to prove the following conjectures.

**Conjecture 4.1** $\chi(Z_0) = \mathrm{Sq}^1$ and $\chi(Z_n) = \mathrm{Sq}^{n2^{n+1}+1}\chi(Z_{n-1}) = Z_n$ for $n \geqslant 1$. The extended version of the recursive formula above is

$$\chi(Z_n) = \mathrm{Sq}^{n2^{n+1}+1}\mathrm{Sq}^{(n-1)2^n+1}\mathrm{Sq}^{(n-2)2^{n-1}+1}\dots\mathrm{Sq}^5\mathrm{Sq}^1$$

for $n \geqslant 0$.

**Conjecture 4.2** For $n \geqslant 1$, $\chi(X_n) = X_n$.

Next, consider the monomial

$$Q_k^n = \mathrm{Sq}^{2^n}\mathrm{Sq}^{2^{n-1}}\dots\mathrm{Sq}^{2^k} \tag{15}$$

for $n \geqslant k \geqslant 0$ [2]. Then we have the following conjectures.

**Conjecture 4.3** For $n \geqslant 0$,

$$Q_0^n Q_1^{n+1}\mathrm{Sq}^{2^{n+2}} = \mathrm{Sq}^{7.2^n}\mathrm{Sq}^{2^{n-1}+2^n}\mathrm{Sq}^{2^{n-2}+2^{n-1}}\dots\mathrm{Sq}^{2^0+2^1}.$$

**Conjecture 4.4** For $n \geqslant 0$,

$$Q_0^n\mathrm{Sq}^{2^{n+1}} = \mathrm{Sq}^{2^n+2^{n+1}}Q_0^{n-1}. \tag{16}$$

For instance $\mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^4 = \mathrm{Sq}^6\mathrm{Sq}^1$ for $n = 1$ and $\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^{16} = \mathrm{Sq}^{24}\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1$ for $n = 3$.

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