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On some properties of the hyperspace $\theta(X)$ and the study of the space $\downarrow \theta C(X)$

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Abstract. The aim of the paper is to first investigate some properties of the hyperspace $\theta(X)$, and then in the next part of the paper to deal with a detailed study of a special type of subspace $\downarrow \theta C(X)$ of the space $\theta(X \times \mathbb{I})$.

Keywords: θ-closed set, H-closed space, H-set, locally θ-H space.

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1. Introduction

The study of hyperspace topology started with Hausdorff [6], where he topologized a collection of all nonempty closed subsets of a bounded metric space X by defining a metric on that collection. After that, Vietoris introduced a new topology on the collection of all nonempty closed subsets of a topological space (X, σ), which is known as "*Vietoris Topology*" or "*Finite Topology*". Michael also in his paper [7] dealt with different types of subsets for construction of topologies. Subsequently, Fell in his paper [3] constructed a compact, Hausdorff topology for the collection of all closed subsets of a topological space (X, σ).

In [5], we have introduced a new topology on the collection of all nonempty θ-closed subsets of a topological space (X, σ). In Section 3, we continue our study of the space $\theta(X)$ endowed with the above defined topology described in [5]. There, a necessary and sufficient condition has been established for a space X to be locally θ-H. Also the local connectedness of an H-closed, Urysohn space X is studied in terms of that of $\theta(X)$.

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Section 4 deals with the hyperspace $\downarrow \theta C(X)$. Here $\theta C(X)$ denotes the set of all θ-continuous maps from a topological space X to $[0,1](\equiv \mathbb{I})$, endowed with the subspace topology of the real line. For each $f \in \theta C(X)$, we define the hypograph of f by $\downarrow f$. By identifying each $f \in \theta C(X)$ with $\downarrow f \in \theta(X \times \mathbb{I})$, we can regard $\theta C(X)$ as the subset $\downarrow \theta C(X) \subset \theta(X \times \mathbb{I})$. So any topology on $\theta(X \times \mathbb{I})$ will induce a topology on $\downarrow \theta C(X)$. In this section, we investigate some properties of $\downarrow \theta C(X)$ endowed with the above defined topology. At first investigations are made how the first countability and local θ-H-ness of a space X are related. Then we have obtained that first countability of $\downarrow \theta C(X)$ always implies the separability of $\downarrow \theta C(X)$. Finally it has been proved that for an H-closed space X, the second countability of $\downarrow \theta C(X)$ always implies the second countability of X.

Recall that H-closedness of the space $(\mathbb{K}(X), \vee)$ of all nonempty compact subsets of a space X endowed with the Vietoris topology $\vee$ was considered in [2].

2. Preliminaries

Throughout the paper all spaces are assumed to be Tychonoff. Let us first recall the following.

Definition 2.1 [8] A point $x \in X$ is said to be a θ-contact point (also called a θ-cluster point or a θ-adherent point) of a set $A \subseteq X$ if for every neighborhood U of x, we get $cl_X U \cap A \neq \phi$. The set of all θ-contact points of a set A is called the θ-closure of A and we denote this set by $\overline{A}^{\theta}$ (or, $cl_\theta A$). A set A is called θ-closed if $A = \overline{A}^{\theta}$. A set A is called θ-open if $X \setminus A$ is θ-closed.

Remark 1 *The collection of all θ-open sets in X forms a topology. By $\theta(X)$ we mean*

$$\theta(X) = \{A \subseteq X : A \neq \phi \text{ and } A \text{ is } \theta\text{-closed}\}.$$

Definition 2.2 A T_2-space X is called H-closed if any open cover of X has a finite proximate subcover, i.e. a finite collection whose union is dense in X. A set $A \subseteq X$ is called an H-set if any open cover $\{U_\alpha : \alpha \in \Lambda\}$ of A by open sets in X has a finite subfamily $\{U_{\alpha_i} : i = 1, 2, .., n\}$ such that $A \subseteq \bigcup_{i=1}^{n} cl_X U_{\alpha_i}$.

Definition 2.3 [5] On $\theta(X)$ we define a topology as follows. For each $W \subseteq X$, let $W^+ = \{A \in \theta(X) : A \subseteq W\}$ and $W^- = \{A \in \theta(X) : A \cap W \neq \phi\}$. Consider

$$S_\theta = \{W^- : W \text{ is open in } X\} \cup \{W^+ : W \text{ is } \theta\text{-open in } X \text{ with } X \setminus W \text{ an } H\text{-set}\}.$$

Then S_θ forms a subbase for some topology on $\theta(X)$ which we denote by τ.

Remark 2 *[5] Any basic open set in the above defined topology is of the form $V_1^- \cap V_2^- \cap ... \cap V_n^- \cap V_0^+$, where $V_i \subseteq V_0$ for each $i = 1, 2, ..., n$ and $V_1, V_2, ..., V_n$ are open sets, V_0 is a θ-open set with $X \setminus V_0$ an H-set.*

Definition 2.4 [5] A space X is locally θ-H if X contains a base $\mathcal{B}$ for its topology such that for each $B \in \mathcal{B}$, $cl_X B$ is an H-set which is θ-closed also.

Proposition 2.5 [5] If X is H-closed and Urysohn, then X is locally θ-H.

Corollary 2.6 [8] Any θ-closed set in an H-closed space is an H-set.

Corollary 2.7 [1] In an H-closed Urysohn space, every H-set is θ-closed and every θ-closed set is an H-set.

3. The hyperspace $\theta(X)$

In this section, we investigate the properties of $\theta(X)$ endowed with the topology τ as defined above.

Definition 3.1 Let (X, σ) be a topological space. A map $f : (X, \sigma) \to \mathbb{R}$ is said to be θ-lower semicontinuous if for any $t \in \mathbb{R}$, $f^{-1}[t, \infty)$ is θ-closed in X.

Definition 3.2 For an extended real-valued function $f : X \to [-\infty, \infty]$, the epigraph of f is denoted by $epi(f)$ and is defined by $epi(f) = \{(x, t) \in X \times \mathbb{R} : f(x) \leqslant t\}$.

Remark 3 *It should be observed that f is θ-lower semicontinuous if and only if $epi(f)$ is θ-closed in $X \times \mathbb{R}$.*

Consider $\theta L(X) = \{f : X \to [-\infty, \infty] : f$ is θ-lower semi continuous$\}$. By identifying each f with $epi(f)$, we can consider $\theta L(X)$ as a subspace of $\theta(X \times \mathbb{R})$.

Theorem 3.3 A Urysohn space X is locally θ-H if and only if $\theta L(X)$ is closed in $\theta(X \times \mathbb{R})$.

Proof. First let X be locally θ-H. Then for each $A \in \theta(X \times \mathbb{R}) \setminus \theta L(X)$, there exist $x \in X$ and $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$ such that $(x, r_1) \in A$ but $(x, r_2) \notin A$. Since X is locally θ-H, there exist an open neighbourhood V of x and a $\delta > 0$ such that $cl\ V$ is a θ-closed, H-set and $cl\ V \times (r_2 - \delta, r_2 + \delta) \subset X \times \mathbb{R} \setminus A$. Put $K = cl\ V \times [r_2 - \delta, r_2 + \delta]$ and $U = V \times (-\infty, r_2 - \delta)$. Then K is an H-set in $X \times \mathbb{R}$, U is an open set in $X \times \mathbb{R}$ such that $A \in U^- \cap ((X \times \mathbb{R}) \setminus K)^+ \subset \theta(X \times \mathbb{R}) \setminus \theta L(X)$. Hence $\theta L(X)$ is closed in $\theta(X \times \mathbb{R})$.

Conversely, let X be not locally θ-H. Then there exists $x_0 \in X$ which has no θ-closed, H-set neighbourhood in X. Consider

$$A = (X \times [1, \infty)) \cup \{(x_0, 0)\} \in \theta(X \times \mathbb{R}) \setminus \theta L(X).$$

For each neighbourhood W of A in $\theta(X \times \mathbb{R})$, choose open sets $U_1, ..., U_n \subset X \times \mathbb{R}$ and an H-set $K \subset X \times \mathbb{R}$ such that $(x_0, 0) \in U_1$ and $A \in U_1^- \cap ... \cap U_n^- \cap ((X \times \mathbb{R}) \setminus K)^+ \subset W$. If we denote the projection map $p_1 : X \times \mathbb{R} \to X$, then as $p_1(K)$ is an H-set, $p_1(K)$ is not a neighbourhood of $x_0 \in X$, i.e. $p_1(U_1) \not\subset p_1(K)$. Choose $x_1 \in p_1(U_1) \setminus p_1(K)$. Now define $g \in \theta L(X)$ by

$$g(x) = \begin{cases} 0, & x = x_1 \\ 1, & x \neq x_1 \end{cases}$$

Then by identifying g with its epigraph, we can write $g = (X \times [1, \infty)) \cup (\{x_1\} \times [0, \infty))$. Now, $g \in U_1^- \cap ... \cap U_n^- \cap ((X \times \mathbb{R}) \setminus K)^+ \subset W$, which implies that $W \cap \theta L(X) \neq \phi$, i.e. $A \in cl\ \theta L(X)$. Thus $\theta L(X)$ is not closed. ■

Proposition 3.4 For an H-closed, Urysohn space X, if $\theta(X)$ is locally connected, then so is X.

Proof. Since, by Proposition 2.5 X is locally θ-H, there exists an open neighbourhood U of $x_0 \in X$ such that $cl\ U$ is a θ-closed, H-set. As $\theta(X)$ is locally connected and $U^- \cap (X \setminus bd\ U)^+$ is a neighbourhood of $\{x_0\}$ in $\theta(X)$, there exists a connected neighbourhood $\mathcal{W}$ of $\{x_0\}$ in $\theta(X)$ such that $\mathcal{W} \subset U^- \cap (X \setminus bd\ U)^+$. Hence for each $A \in \mathcal{W}$, $A \cap U \neq \phi$

and $A \cap bd\ U = \phi$. As $\phi : X \to \theta(X), x \to \{x\}$ is an embedding, $\{x \in X : \{x\} \in \mathcal{W}\}$ is a neighbourhood of x_0 in X, thus $V = U \cap \cup\mathcal{W}$ is also a neighbourhood of x in X. We claim that V is connected. If not, then there exist two nonempty, disjoint open sets V_0 and V_1 in X such that $V \subset V_0 \cup V_1 \subset U$, $x_0 \in V_0$ and $V \cap V_1 \neq \phi$, i.e. $V \cap cl\ V_1 = V \cap V_1$, $V \cap cl\ V_0 = V \cap V_0$. Now, for each $A \in \mathcal{W}$, $A \cap U \neq \phi$ and $A \cap cl\ U = A \cap U \subset V \subset V_0 \cup V_1$, so that $\mathcal{W}$ is being covered by the following pairwise, disjoint open sets $V_0^- \cap (X \setminus cl\ V_1)^+, V_1^- \cap (X \setminus cl\ V_0)^+, V_0^- \cap V_1^-$. Clearly, $\{x_0\} \in \mathcal{W} \cap V_0^- \cap (X \setminus cl\ V_1)^+$. As $V \cap V_1 \neq \phi$, $A \in \mathcal{W}$ such that $A \cap V_1 \neq \phi$, whence $A \in V_1^- \cap (X \setminus cl\ V_0)^+$ or $A \in V_0^- \cap V_1^-$. Thus $\mathcal{W}$ meets one of $V_1^- \cap (X \setminus cl\ V_0)^+$ or $V_0^- \cap V_1^-$, which contradicts the fact that $\mathcal{W}$ is connected. ■

Proposition 3.5 For an H-closed, Urysohn space X, if $\theta(X)$ is connected, then any non-empty open set in X is not an H-set.

Proof. If possible, let X has a non-empty open set U that is an H-set. Then U^- and $(X \setminus U)^+$ are disjoint non-empty open sets in $\theta(X)$ such that $\theta(X) = U^- \cup (X \setminus U)^+$, hence $\theta(X)$ is disconnected. ■

4. The hyperspace $\downarrow \theta C(X)$

In this section we investigate the properties of the hyperspace $\downarrow\ \theta C(X)$. We first recollect the following:

Definition 4.1 [4] A function $f : (X, \sigma) \to (Y, \gamma)$ is said to be θ-continuous at a point $x \in X$ if for each open neighbourhood V of $f(x)$, there exists an open neighbouhood U of x such that $f(cl\ U) \subseteq cl\ V$. The function f is said to be θ-continuous on X if it is θ-continuous at each point x of X.

The family of all θ-continuous functions from a topological space (X, σ) to $\mathbb{I} = [0, 1]$ with the subspace topology of the reals will be denoted by $\theta C(X)$.

Definition 4.2 For every $f \in \theta C(X)$, the hypograph of f is defined by $\downarrow f = \{(x, y) \in X \times \mathbb{I} : y \leqslant f(x)\}$.

Remark 4 *It is to be noted that for each $f \in \theta C(X)$, $\downarrow f \in \theta(X \times \mathbb{I})$. So by identifying each $f \in \theta C(X)$ with $\downarrow f \in \theta(X \times \mathbb{I})$, we can regard $\theta C(X)$ as the subset $\downarrow \theta C(X) = \{\downarrow f : f \in \theta C(X)\} \subset \theta(X \times \mathbb{I})$. So any topology on $\theta(X \times \mathbb{I})$ will give rise to a subspace topology on $\downarrow \theta C(X)$. Thus the above defined topology will induce a topology τ' on $\downarrow \theta C(X)$ which is being generated by*

$$\{\bigcap_{i=1}^{n} V_i^- \cap V_0^+ \cap \downarrow \theta C(X) : V_1, ..., V_n \textit{ are open in } X \times (0,1],\ V_0 \textit{ is } \theta\textit{-open in } X \times (0,1] \textit{ with its complement an } H\textit{-set}\}.$$

Notation 4.3 For a closed set F in a topological space (X, σ),

$$F^* = (X \setminus F)^+ = \{A \in \theta(X) : A \cap F = \phi\}.$$

Theorem 4.4 $(\downarrow \theta C(X), \tau')$ is always T_1.

Proof. Let $f, g \in \theta C(X)$ be such that $f \neq g$. Then there exists $x_0 \in X$ such that $f(x_0) \neq g(x_0)$. Let $f(x_0) < g(x_0)$. As f, g are θ-continuous, there exists an open neighbourhood W of x_0 such that $f(x) \leqslant a < b \leqslant g(x)$, for all $x \in cl\ W$, where $a = f(x_0)$ and $b = g(x_0)$. Then $\downarrow f \in (\{x_0\} \times [b, 1])^*$ and $\downarrow g \in (W \times (a, 1])^-$, but $\downarrow g \notin (\{x_0\} \times [b, 1])^*$ and

$\downarrow f \notin (W \times (a,1])^{-}$. Hence $(\downarrow \theta C(X), \tau')$ is T_1. ■

Theorem 4.5 For an H-closed, Urysohn space X, $\downarrow \theta C(X)$ is T_2 if and only if there exists a dense open subset U of X which is locally θ-H.

Proof. Take $f, g \in \theta C(X)$, $x_0 \in cl\ W$ and $a, b \in \mathbb{I}$ as in the proof of the above theorem. Since $f, g \in \theta C(X)$, we assume that $x_0 \in U$. As U is locally θ-H, there exists an open set V in X such that $x_0 \in V \subseteq cl\ V \subseteq cl(U \cap W)$ and $cl\ V$ is a θ-closed, H-set. As for all $x \in cl\ V$, $f(x) \leqslant a < b \leqslant g(x)$, $(cl\ V \times [c,1])^{*} \cap \downarrow \theta C(X)$ and $(V \times (c,1])^{-} \cap \downarrow \theta C(X)$ are disjoint neighbourhoods of $\downarrow f$ and $\downarrow g$ respectively, where $c = \frac{a+b}{2}$.

Conversely, let us define $U = \cup\{int\ K : K$ is an H-set in $X\}$. Then U is open, so that $cl\ U = cl_{\theta} U$. As X is H-closed, $cl\ U$ becomes θ-closed and hence an H-set. Thus U is locally θ-H. If possible, let U be not dense in X. Then there exists a nonempty open set V in X such that interior of every H-set of V is empty. As X is Tychonoff, there exists $f \in \theta C(X)$ such that $f(X \setminus V) = \{1\}$ and $f(x_0) = 0$ for some $x_0 \in V$. As $\downarrow \theta C(X)$ is T_2, there exist disjoint open neighbourhoods $\mathcal{U}$ and $\mathcal{V}$ in $\downarrow \theta C(X)$ such that $\downarrow \underline{1} \in \mathcal{U}$ and $\downarrow f \in \mathcal{V}$. Then there exist open sets $G_1, ..., G_n, ..., G_m \subset X \times (0,1]$ and an H-set $K \subset X \times (0,1]$ such that

$$\downarrow \underline{1} \in G_1^{-} \cap ... \cap G_n^{-} \cap \downarrow \theta C(X) \subset \mathcal{U} \text{ and } \downarrow f \in G_{n+1}^{-} \cap ... \cap G_m^{-} \cap K^{*} \cap \downarrow \theta C(X) \subset \mathcal{V}.$$

As $f(X \setminus V) = \{1\}$, $p_1(K) \subset V$, so that $int\ p_1(K) = \phi$. For every $i \leqslant m$, $p_1(G_i) \setminus p_1(K) \neq \phi$, since $p_1(G_i)$ is a non-empty open set in X. Take $x_i \in p_1(G_i) \setminus p_1(K)$. As X is Tychonoff, there exists an $h \in \theta C(X)$ such that $h(x_i) = 1$, for $i \leqslant m$ and $h(p_1(K)) = \{0\}$. Then $\downarrow h \in \mathcal{U} \cap \mathcal{V}$, a contradiction. ■

Theorem 4.6 For an H-closed, Urysohn space X, if $\downarrow \theta C(X)$ is first countable, then there exist H-sets $H_1 \subset H_2 \subset ...$ in X such that every H-set in X in contained in some H_n. In particular, $X = \bigcup_{n=1}^{\infty} H_n$.

Proof. Since $\downarrow \theta C(X)$ is first countable, there exist H-sets $K_1, K_2, ...$ in $X \times (0,1]$ such that $\{K_n^{*} \cap \downarrow \theta C(X) : n \in \mathbb{N}\}$ is a neighbourhood base of $\downarrow \underline{0}$ in $\downarrow \theta C(X)$. Then $p_1(K_n) = H_n, n \in \mathbb{N}$ are H-sets in X. We have to show that every H-set H_0 in X is a subset of some H_n. If not, choose $x_n \in H_0 \setminus H_n$ and define $f_n \in \theta C(X)$ by $f_n(x_n) = 1$, $f_n(H_n) = \{0\}$. Then $\downarrow f_n \in K_n^{*}$, for all $n \in \mathbb{N}$ and hence $\downarrow f_n \to \downarrow \underline{0}$ in $\downarrow \theta C(X)$, whereas $\downarrow f_n \not\subset (H_0 \times \{1\})^{*}$ which is a neighbourhood of $\downarrow \underline{0}$, a contradiction. ■

Theorem 4.7 If X and $\downarrow \theta C(X)$ are both first countable, then X is locally θ-H.

Proof. If possible, let there exists $x_0 \in X$ which has no H-set neighbourhood. As X is first countable, there exists a decreasing sequence of open neighbourhood base $\{U_n : n \in \mathbb{N}\}$ at x_0. Also, as $\downarrow \theta C(X)$ is first countable, there exist H-sets $K_1 \subset K_2 \subset ...$ in $X \times (0,1]$ such that $\{K_n^{*} \cap \downarrow \theta C(X) : n \in \mathbb{N}\}$ is a neighbourhood base of $\downarrow \underline{0}$ in $\downarrow \theta C(X)$. As for all $n \in \mathbb{N}$, $U_n \not\subset p_1(K_n)$, choose $x_n \in U_n \setminus p_1(K_n)$. Then $x_n \to x_0$ in X. Since X is Tychonoff, there exists $f_n \in \theta C(X)$ such that $f_n(x_n) = 1$ and $f_n(p_1(K_n) \cup (X \setminus U_n)) = \{0\}$. So, $\downarrow f_n \in K_n^{*}$ and hence $\downarrow f_n \to \downarrow \underline{0}$. But, $(\{x_n : n \in \mathbb{N}\} \times \{1\})^{*} \cap \downarrow \theta C(X)$ is a neighbourhood of $\downarrow \underline{0}$ in $\downarrow \theta C(X)$ containing no $\downarrow f_n$, a contradiction. ■

Theorem 4.8 Consider the following statements :
(a) $\downarrow \theta C(X)$ is first countable.
(b) There exists a countable family $\mathcal{U}$ of non-empty open sets in X such that every non-empty open set in X includes an element of $\mathcal{U}$.
(c) $\downarrow \theta C(X)$ is separable.

Then $(a) \Rightarrow (b) \Rightarrow (c)$ hold in general.

In addition, if X is H-closed, $(b) \Rightarrow (a)$ also holds.

Proof. (**a**) $\Rightarrow$ (**b**): As $\downarrow \theta C(X)$ is first countable, let

$$\{(G_1^n)^- \cap ... \cap (G_{k(n)}^n)^- \cap \downarrow \theta C(X) : n \in \mathbb{N}\}$$

be a countable neighbourhood base at $\downarrow \underline{1}$ in $\downarrow \theta C(X)$. Consider $\mathcal{U} = \{p_1(G_i^n) : i = 1, 2, ..., k(n), n \in \mathbb{N}\}$. Then $\mathcal{U}$ is a countable family of non-empty open sets in X. It remains to show that every non-empty open set U in X includes an element of $\mathcal{U}$. Take $f \in \theta C(X)$ such that $f(X \setminus U) = \{1\}$ and $f(x_0) = 0$ for some point $x_0 \in U$. As $\downarrow \theta C(X)$ is T_1, $\downarrow f \notin \bigcap_{i=1}^{k(n)} (G_i^n)^-$, for $n \in \mathbb{N}$ and hence $\downarrow f \notin (G_i^n)^-$, for some $i = 1, 2, ..., k(n)$. Then $\downarrow f \cap G_i^n = \phi$. As $f(X \setminus U) = \{1\}$, we have $p_1(G_i^n) \subset U$.

(**b**) $\Rightarrow$ (**c**): Let $\mathcal{U}$ be a countable family of non-empty open sets in X satisfying condition (b). For every $U \in \mathcal{U}$, $r \in \mathbb{Q} \cap (0, 1]$ and $x \in U$, there exists θ-continuous $f_{U,r} : X \to [0, r]$ such that $f_{U,r}(X \setminus U) = \{0\}$ and $f_{U,r}(x) = r$. Let

$$D = \{\max\{f_{U,r} : U \in \mathcal{F}, r \in F\} : \mathcal{F} \text{ and } F \text{ are finite subsets of } \mathcal{U} \text{ and } \mathbb{Q} \cap (0,1] \text{ respectively}\}.$$

Then $\downarrow D = \{\downarrow f : f \in D\}$ is a countable subset of $\downarrow \theta C(X)$. We show that $\downarrow D$ is dense in $\downarrow \theta C(X)$. Let $f \in \theta C(X)$, K be an H-set in $X \times (0, 1]$ and $G_1, G_2, ..., G_k$ be open in $X \times (0, 1]$ such that $\downarrow f \in G_1^- \cap ... \cap G_k^- \cap K^* \cap \downarrow \theta C(X)$. We have $x_1, ..., x_k \in X$ such that $\{x_i\} \times [0, f(x_i)] \cap G_i \neq \phi$ for each $i \leqslant k$. As $\{x_i\} \times [0, f(x_i)] \cap K = \phi$, there exist an open neighbourhood W_i of x_i in X and $s_i < t_i$ such that $W_i \times (s_i, t_i) \subset G_i$ and $W_i \times [0, t_i] \cap K = \phi$. Choose $r_i \in \mathbb{Q} \cap (s_i, t_i)$ and $U_i \in \mathcal{U}$ such that $U_i \subset W_i$. Then $\downarrow f_{U_i, r_i} \in G_i^- \cap K^*$ and thus $\downarrow \max\{f_{U_i, r_i} : i \leqslant k\} \in \downarrow D \cap G_1^- \cap ... \cap G_k^- \cap K^*$.

Next, let X be H-closed.

(**b**) $\Rightarrow$ (**a**): Let $\mathcal{U}$ be a countable family of non-empty open sets in X satisfying condition (b). Then $\mathcal{G} = \{U \times (s, t) : U \in \mathcal{U}, s < t \in \mathbb{Q} \cap (0, 1)\}$ is a countable family of non-empty open sets in $X \times \mathbb{I}$ satisfying condition (b). For every $f \in \theta C(X)$ and $n \in \mathbb{N}$, let

$$\mathcal{G}(f) = \{G \in \mathcal{G} : \downarrow f \in G^-\} \text{ and } K_n(f) = \{(x, t) \in X \times \mathbb{I} : t \geqslant f(x) + \tfrac{1}{n}\}.$$

For every open set H in $X \times (0, 1]$ with $\downarrow f \in H^-$, there exists $x_0 \in X$ such that $\{x_0\} \times [0, f(x_0)] \cap H \neq \phi$. As $f(x_0) > 0$, there exist an open neighbourhood V of x_0 in X and $s < t \in \mathbb{Q} \times (0, 1)$ such that $s < f(x_0), V \times (s, t) \subset H$ and $s < f(x)$ for every $x \in V$. Then there exists $U \in \mathcal{U}$ such that $U \subset V$. Thus $U \times (s, t) \in \mathcal{G}$ and $\downarrow f \in G^- \subset H^-$. Again, for every H-set K in $X \times \mathbb{I}$ with $\downarrow f \in K^*$, by H-closedness of X, there exists $n \in \mathbb{N}$ such that $K \subset K_n(f)$ and thus $\downarrow f \in K_n(f)^* \subset K^*$. Thus

$$\{G_1^- \cap ... \cap G_k^- \cap K_n(f)^* \cap \downarrow \theta C(X) : G_i \in \mathcal{G}(f), i \leqslant k; k, n \in \mathbb{N}\}$$

is a countable neighbourhood base at $\downarrow f$ in $\downarrow \theta C(X)$. ■

Notation 4.9 If X is H-closed, then every θ-closed subset of an H-closed space is an H-set and thus in this case the topology τ' on $\downarrow \theta C(X)$ is generated by

$$\{\bigcap_{i=1}^{n} V_i^- \cap V_0^+ \cap \downarrow \theta C(X) : V_1, ..., V_n \text{ are open in } X \times (0,1], V_0 \text{ is } \theta\text{-open in } X \times (0,1]\}.$$

Theorem 4.10 For an H-closed space X, if $\downarrow \theta C(X)$ is second countable, then X is also a second countable space.

Proof. Let

$$\{U_1^{n^-} \cap ... \cap U_{m(n)}^{n^-} \cap (\bigcup_{i=1}^{m(n)} U_i^n)^+ \cap \downarrow \theta C(X) : n \in \mathbb{N}\}$$

be a countable base for $\downarrow \theta C(X)$ and $\mathcal{B}$ be a countable base for $\mathbb{I}$. For $n \in \mathbb{N}$, $i \leqslant m(n)$ and $B \in \mathcal{B}$, let

$$V(n, i, B) = \{x \in X : H \times B \subset U_i^n, \text{ for some open set } H \text{ containing } x \text{ in } X\}.$$

Then $V(i, n, B)$ is open in X and $V(i, n, B) \times B \subset U_i^n$. Let $\mathcal{C}$ be the family of all finite intersections of sets of the form $V(i, n, B)$. Then $\mathcal{C}$ is a countable open base for X, in fact, for any open set V in X and $x \in V$, there exists $f \in \theta C(X)$ such that $f(x) = 0$ and $f(X \setminus V) = \{1\}$. Let $U_1 = X \times [0, \frac{1}{2})$ and $U_2 = V \times [0, 1]$. Then

$$\downarrow f \in U_1^- \cap U_2^- \cap (U_1 \cup U_2)^+ \cap \downarrow \theta C(X).$$

Then there exists $n \in \mathbb{N}$ such that

$$\downarrow f \in U_1^{n^-} \cap ... \cap U_{m(n)}^{n^-} \cap (\bigcup_{i=1}^{m(n)} U_i^n)^+ \subset U_1^- \cap U_2^- \cap (U_1 \cup U_2)^+ \cap \downarrow \theta C(X).$$

Then for every $t \in \mathbb{I}$ there exists $i(t) \leqslant m(n)$ such that $(x, t) \in U_{i(t)}^n$. Hence there exist $B_t \in \mathcal{B}$ and an open set H in X such that $(x, t) \in H \times B_t \subset U_{i(t)}^n$. Then $(x, t) \in V(n, i(t), B_t) \times B_t \subset U_{i(t)}^n$. Choose a finite subcover $\{B_{t_j} : j = 1, 2, ..., l\}$ of the open cover $\{B_t : t \in \mathbb{I}\}$ of $\mathbb{I}$ and let $G = \bigcap_{j=1}^{l} V(n, i(t_j), B_{t_j})$. Then $x \in G \in \mathcal{C}$. It now suffices to show that $G \subset V$. Otherwise, choose $y \in G \setminus V$ and $g \in \theta C(X)$ such that $g(y) = 1$ and $g(X \setminus G) = \{0\}$. Let $h = f \vee g \in \theta C(X)$. Then $\downarrow h \notin < U_1, U_2 > (\equiv U_1^- \cap U_2^- \cap (U_1 \cup U_2)^+)$. Again,

$$G \times \mathbb{I} = \bigcap_{j=1}^{l} V(n, i(t_j), B_{t_j}) \times (\bigcup_{j=1}^{l} B_{t_j}) \subset \bigcup_{j=1}^{l} U_{i(t_j)}^n \subset \bigcup_{i=1}^{m(n)} U_i^n$$

$$\Rightarrow \downarrow h = \downarrow f \cup \downarrow g \subset \downarrow f \cup (G \times \mathbb{I}) \subset \bigcup_{i=1}^{m(n)} U_i^n.$$

Thus, $\downarrow h \in < U_1^n \cap ... \cap U_{m(n)}^n > \cap \downarrow \theta C(X)$. Since $\downarrow h \supset \downarrow f$ and $\downarrow f \cap U_i^n \neq \phi$ for every $i \leqslant m(n)$, a contradiction. ■

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