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An efficient method for the numerical solution of functional integral equations

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Abstract. We propose an efficient mesh-less method for functional integral equations. Its convergence analysis has been provided. It is tested via a few numerical experiments which show the efficiency and applicability of the proposed method. Attractive numerical results have been obtained.

Keywords: Functional integral equations, mesh-less method, radial basis functions, collocation method.

2010 AMS Subject Classification: 45G10.

1. Introduction

In this paper, we focus on the following functional integral equation:

$$u(t) = g(t, u(t)) + \int_0^t G(t, s, u(s))ds, \qquad t \in (0, T), \tag{1}$$

where g and G are Lipschitz continuous functions with respect to their second and third variables, respectively. For the list of applications of Eq. (1) and some consolidated arguments we refer to [8, 14].

In recent years, mesh-less methods have been widely applied in a number of fields such as multivariate function interpolation and approximation, neural networks and solution of differential and integral equations. The meshless methods are based upon the

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scattered data approximations that estimate a function without any mesh generation on the domain. Among meshless methods, the radial basis functions (RBFs) method has become known as a powerful tool for the scattered data interpolation problem. The main advantage of radial basis functions is that they involve a single independent variable regardless of the dimension of the problem [7]. RBFs have been used to approximate the solution of one-dimensional integral equations [1]. A numerical technique based on the spectral method is presented for the numerical solution of one-dimensional Volterra-Fredholm-Hammerstein integral equations using RBFs in [10]. To cite a few more references considering such techniques we refer the reader to [6, 8, 9, 13].

In [11] the existence of at least one solution for Eq. (1) has been proved by using fixed-point methods and measure of non-compactness theory. But no numerical procedures have been suggested therein. In this paper, we propose and analyze an efficient mesh-less method for numerical solution of Eq. (1). The rest of this paper is organized as follows: In Section 2, we bring a brief introduction into RBF approximation. In Section 3, the proposed method is described for functional integral equations. Error analysis of the proposed method is provided in Section 4. Numerical experiments are carried out in Section 5.

2. RBF approximation

A mesh-free method does not require a mesh to discretize the domain of the problem. So the approximate solution is constructed entirely based on a set of scattered nodes. Radial basis functions are one of the most developed mesh-less methods that have attracted attention in recent years. They form a primary tool for multivariate function interpolation [2, 4, 5, 7, 14]. Hence they are receiving increasing attention for solving PDEs on irregular domains. In the sequel, we need the following definitions [12].

Definition 2.1 A function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is said to be radial if there exists a function $\phi : [0, \infty) \to \mathbb{R}$ such that $\Phi(\mathbf{x}) = \phi(\| \mathbf{x} \|_2)$ for all $\mathbf{x} \in \mathbb{R}^d$.

Some of the most popular functions that generate RBFs are listed in Table 1. Suppose

Table 1. Some well-known functions that generate RBFs

Name of function	Definition
Gaussian (GA)	$\phi(r) = \exp(-cr^2), \quad c > 0$
Multiquadrics (MQ)	$\phi(r) = (r^2 + c^2)^{\frac{1}{2}}$
Inverse multiquadrics (IMQ)	$\phi(r) = (r^2 + c^2)^{\frac{\beta}{2}}, \quad \beta < 0$
Thin plate splines	$\phi(r) = (-1)^{k+1} r^{2k} \log(r), \ k \in \mathbb{N}$

that $E \subset \mathbb{R}^d$ and $X = \{x_1, ..., x_N\}$ is a given set of distinct nodal points in E. In order to interpolate the scattered data using RBFs, we usually consider the approximation of a function f in the form:

$$\mathcal{P}f = \sum_{i=1}^{N} \lambda_i \phi(\| x - x_i \|), \quad x \in E. \tag{2}$$

The interpolation problem is to find $\lambda_i, \ i = 1, ..., N$ such that the interpoant $\mathcal{P}f$ satisfies

$$\mathcal{P}f(x_i) = f(x_i), \ i = 1, ..., N. \tag{3}$$

One can write this system in matrix form as

$$A\Lambda = \mathbf{f}, \tag{4}$$

in which $A_{ij} = \phi(\| x_i - x_j \|)$, $i, j = 1, ..., N$, $\mathbf{\Lambda} = [\lambda_1, ..., \lambda_N]^T$ and $\mathbf{f} = [f(x_1), ..., f(x_N)]^T$.

Definition 2.2 [12] A radial basis function ϕ on $[0, \infty)$ is positive definite, if for all choices of sets $X = \{x_1, ..., x_N\}$ of finitely many points $x_1, ..., x_N$ and arbitrary N the $N \times N$ symmetric matrices A of (4) are positive definite.

If ϕ satisfies the above definition, the solvability of (4) is guaranteed. This holds for several standard radial basis functions such as Gaussians and Inverse multiquadrics. We are interested in how well the interpolant $\mathcal{P}f$ approximates the function f on E as the set of data X becomes denser in E. This can be measured by fill distance who's definition follows:

Definition 2.3 [12] The fill distance of a given set $X = \{x_1, ..., x_N\}$ consisting of pairwise distinct points in E is defined as $h_{X,E} = \sup_{x \in E} \min_{x_j \in X} \| x - x_j \|$, which indicates how well the data in the set X fill out the domain E.

Definition 2.4 [12] A set E is said to satisfy an interior cone condition if there exists an angle $\theta \in (0, \frac{\pi}{2})$ and a radius $r > 0$ such that for every given $x \in E$ a unit vector $\eta(x)$ exists such that the cone

$$C(x, \eta(x), \theta, r) = \{x + \lambda y : y \in \mathbb{R}^d, \| y \|_2 = 1, y^T \eta(x) \geqslant \cos(\theta), \lambda \in [0, r]\}$$

is contained in E.

Definition 2.5 [4] The native space of ϕ is defined as

$$\mathcal{N}_\phi = \left\{ f \in L_2(\mathbb{R}^s) \cap C(\mathbb{R}^s) : \frac{\hat{f}}{\sqrt{\hat{\phi}}} \in L_2(\mathbb{R}^s) \right\},$$

where $\hat{\phi}$ is the Fourier transform of ϕ. This is the unique Hilbert space on which ϕ introduces a natural inner product.

3. Description of the method

In this section, we provide a simple method for solving Eq. (1). Let

$$\bar{u}(t) = \sum_{i=1}^{n} \mu_i B_i(t),$$

in which $B_i(t) = \phi(|t - t_i|)$ and ϕ is an RBF basis. Eq. (1) implies

$$\sum_{i=1}^{n} \mu_i B_i(t) = g\left(t, \sum_{i=1}^{n} \mu_i B_i(t)\right) + \int_0^t G\left(t, s, \sum_{i=1}^{n} \mu_i B_i(s)\right) ds.$$

By change of variables $s = \frac{t}{2}(z+1)$ we obtain

$$\sum_{i=1}^{n} \mu_i B_i(t) = g\left(t, \sum_{i=1}^{n} \mu_i B_i(t)\right) + \int_{-1}^{1} \hat{G}\left(t, \frac{t}{2}(z+1), \sum_{i=1}^{n} \mu_i B_i(\frac{t}{2}(z+1))\right) dz,$$

where $\hat{G} = \frac{t}{2}G$. Collocating at $t_j : j = 1, ..., n$ yields

$$\sum_{i=1}^{n} \mu_i B_i(t_j) = g\left(t_j, \sum_{i=1}^{n} \mu_i B_i(t_j)\right) + \int_{-1}^{1} \hat{G}\left(t_j, \frac{t_j}{2}(z+1), \sum_{i=1}^{n} \mu_i B_i(\frac{t_j}{2}(z+1))\right) dz$$

for $j = 1, ..., n$. Using n-point Gaussian quadrature with abscissa z_l and weights w_l we obtain

$$\sum_{i=1}^{n} \hat{\mu}_i B_i(t_j) = g\left(t_j, \sum_{i=1}^{n} \hat{\mu}_i B_i(t_j)\right) + \sum_{l=1}^{n} w_l \hat{G}\left(t_j, \frac{t_j}{2}(z_l+1), \sum_{i=1}^{n} \hat{\mu}_i B_i(\frac{t_j}{2}(z_l+1))\right)$$

for $j = 1, ..., n$ in which $\hat{\mu}_i$ are approximations for μ_i while using numerical quadrature formula. This is a nonlinear system of algebraic equations for $\hat{\mu}_i$ which can be solved by any nonlinear solver. We have used fsolve command of Maple. Finally, we obtain $\tilde{u}(t) = \sum_{i=1}^{n} \hat{\mu}_i B_i(t)$ as approximation for the solution.

4. Error analysis

In this section, the error analysis of the proposed method is presented. In the sequel, we need the following:

Theorem 4.1 [12] Let $E \subseteq \mathbb{R}^d$ be open and bounded, satisfying an interior cone condition. Suppose that ϕ is positive definite with infinitely smooth. Suppose the interpolant of $u \in \mathcal{N}_\phi$ is based on set $X = \{x_1, ..., x_N\}$ and it is denoted by $\mathcal{P}_N u$. Then for every $l \in \mathbb{N}$ there exist positive constants $h_0(l)$ and C_l such that

$$\| u - \mathcal{P}_N u \| \leqslant C_l h_{X,E} \| u \|_{\mathcal{N}_\phi(E)}, \tag{5}$$

provided $h_{X,E} \leqslant h_0(l)$.

As a conclusion of Theorem 4.1, for inverse multiquadrics we can write

$$\| u - \mathcal{P}_N u \| \leqslant \exp\left(-\frac{C}{h_{X,E}}\right) \| u \|_{\mathcal{N}_\phi(E)},$$

provided $h_{X,E}$ is sufficiently small and $u \in \mathcal{N}_\phi(E)$. Also for Gaussians we get

$$\| u - \mathcal{P}_N u \| \leqslant \exp\left(-\frac{C|\log h_{X,E}|}{h_{X,E}}\right) \| u \|_{\mathcal{N}_\phi(E)} .$$

Lemma 4.2 [3] Let $f \in H_m(-1,1)$ with $m \geqslant 1$, x_j, $0 \leqslant j \leqslant n$, be the Gauss, or the Gauss-Radau, or the Gauss-Lobatto points relative to the Legendre weight $w(x) \equiv 1$ and

$I_n f$ denote the polynomial of degree n that interpolates f at one of these sets of points, namely $I_n f = \sum_{j=0}^{n} f(x_j)L_j(x)$, where L_j is the j-th Lagrange basis function. Then

$$\| f - I_n f \|_{L_2(-1,1)} \leqslant Cn^{-m}|f|_{H^{m,n}(-1,1)},$$

with

$$|f|_{H^{m,n}(-1,1)} = \left(\sum_{k=\min\{m,n+1\}}^{m} \| \frac{d^k f}{dx^k} \|^2_{L_2(-1,1)} \right)^{\frac{1}{2}}.$$

We consider the integral operator $\mathcal{F}[u](t)$ in the Banach space C^p as

$$\mathcal{F}[u](t) = g(t, u(t)) + \int_0^t G(t, s, u(s))ds.$$

The following theorem ensures the convergence of the proposed method:

Theorem 4.3 Suppose g is Lipschitz continuous with respect to its second variable with constant M_1 and G is Lipschitz continuous with respect to its third variable with constant M_2. Assume the assumptions of Lemma 4.2 are fulfilled. Let $\xi = M_1 + M_2 T$. Then we have

$$||u - \tilde{u}|| \leqslant \xi(1 + cn^{-m}) \exp\left(-\frac{C|\log h_{X,E}|}{h_{X,E}}\right)||u||_{\mathcal{N}_\phi(0,T)} + \xi cn^{-m}|u|_{H(0,T)}.$$

Proof. The triangle inequality yields

$$||\mathcal{F}[u] - \mathcal{F}[\tilde{u}]|| \leqslant ||\mathcal{F}[u] - \mathcal{F}[\bar{u}]|| + ||\mathcal{F}[\bar{u}] - \mathcal{F}[\tilde{u}]||.$$

On the one hand,

$$\begin{aligned} ||\mathcal{F}[u] - \mathcal{F}[\bar{u}]|| &\leqslant ||g(t, u(t)) - g(t, \bar{u}(t))|| + || \int_0^t G(t, s, u(s)) - G(t, s, \bar{u}(s))ds|| \\ &\leqslant M_1||u - \bar{u}|| + M_2 T||u - \bar{u}|| \\ &\leqslant \xi \exp\left(-\frac{C|\log h_{X,E}|}{h_{X,E}}\right)||u||_{\mathcal{N}_\phi}. \end{aligned}$$

On the other hand,

$$\begin{aligned} ||\mathcal{F}[\bar{u}] - \mathcal{F}[\tilde{u}]|| &\leqslant ||g(t, \bar{u}(t)) - g(t, I_n\bar{u}(t))|| + || \int_0^T G(t, s, \bar{u}(s)) - G(t, s, I_n\bar{u}(s))ds|| \\ &\leqslant M_1||\bar{u} - I_n\bar{u}|| + M_2 T||\bar{u} - I_n\bar{u}|| \\ &\leqslant \xi cn^{-m}|\bar{u}|_{H^{m,n}(0,T)} \\ &\leqslant \xi cn^{-m}(|u - \bar{u}|_H + |u|_H) \\ &\leqslant \xi cn^{-m}\left(\exp\left(-\frac{C|\log h_{X,E}|}{h_{X,E}}\right)||u||_{\mathcal{N}_\phi} + |u|_H\right). \end{aligned}$$

Thus, by fixed point property of $\mathcal{F}$, we have

$$||u-\tilde{u}|| \leqslant \xi(1+cn^{-m})\exp\left(-\frac{C|\log h_{X,E}|}{h_{X,E}}\right)||u||_{\mathcal{N}_\phi}+\xi cn^{-m}|u|_H.$$

The proof is complete. ■

5. Numerical experiments

In order to validate our theoretical findings, in this section we bring a few numerical examples. We take n zeros of shifted Chebyshev polynomials as RBF centers. At the same time, n-point Gaussian quadrature has been employed. IMQs have been used as RBF basis. In all examples we take $c = 10$, $\beta = -1$ and $T = 1$. Calculations are done with Maple 16 using 200 digits of accuracy.

***Example* 5.1** Consider the equation (1) with $g(t, v) = t-1-\frac{3}{2}t^2+\cos(v)$ and $G(t, s, v) = t+s+\sin(v)$. The exact solution is $u(t) = t$. For $n = 6$, absolute error has been depicted in Fig. 1.

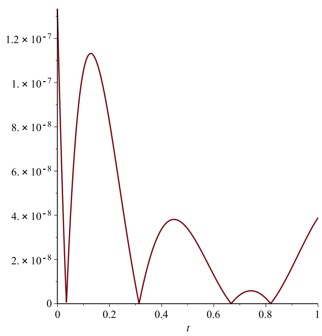

Figure 1. Absolute error profile for $n = 6$ for Example 5.1

***Example* 5.2** Consider the equation (1) with $g(t, v) = v - \frac{1}{2}t^3 - \sin(t)$ and $G(t, s, v) = ts + v$. The exact solution is $u(t) = \cos(t)$. For $n = 50$, absolute error has been depicted in Fig. 2.

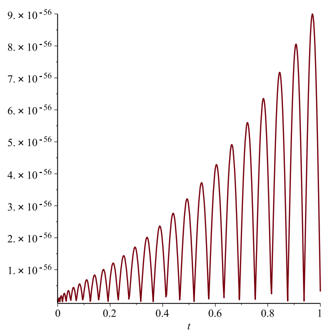

Figure 2. Absolute error profile for $n = 50$ for Example 5.2

***Example* 5.3** Consider the equation (1) with $g(t, v) = v - \frac{1}{4}t^5$ and $G(t, s, v) = tsv$. The exact solution is $u(t) = t^2$. For $n = 50$, absolute error has been depicted in Fig. 3.

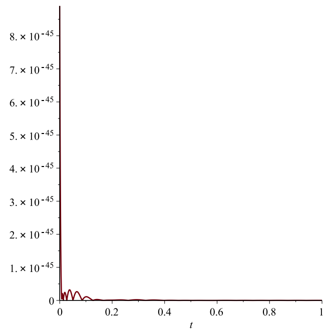

Figure 3. Absolute error profile for $n = 50$ for Example 5.3

6. Conclusion

We have applied the RBF collocation method for solving functional integral equations. Numerical results suggest using zeros of shifted Chebyshev polynomials as centers of RBFs and n-point Gaussian quadrature. IMQs are the most efficient RBF bases for this kind of problems. The provided error analysis suggests that the proposed method converges to the exact solution of the problem by a reasonable cost of computation. The results of numerical experiments approve the efficiency of the proposed method.

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References

[1] A. Alipanah, M. Dehghan, Numerical solution of the nonlinear Fredholm integral equations by positive definite functions, Appl. Math. Comput. 190 (2007), 1754-1761.
[2] I. Babuska, U. Banerjee, J.E. Osborn, Q. Zhang, Effect of numerical integration on meshless methods, Comput. Methods Appl. Mech. Engrg. 198 (2009), 2886-2897.
[3] C. Canuto, M. Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods, Fundamentals in Single Domains, Springer-Verlag, 2006.
[4] G.E. Fasshauer, Meshfree Approximation Methods with MATLAB, World Scientific Publishing. Co. Pte. Ltd., Hackensack, NJ, 2007.
[5] R. L. Hardy, Theory and applications of the multiquadric-biharmonic method. 20 years of discovery 1968-1988, Comput. Math. Appl. 19 (8-9) (1990), 163-208.
[6] T. Ismaeelpour, A. Askari Hemmat, H. Saeedi, B-spline operational matrix of fractional integration, Optik. 130 (2017), 291-305.
[7] E. J. Kansa, Multiquadrics-A scattered data approximation scheme with applications to computational fluid-dynamics. II. Solutions to parabolic, hyperbolic and elliptic partial differential equations, Comput. Math. Appl. 19 (8-9) (1990), 147-161.
[8] K. Parand, M. Delkhosh, Operational matrices to solve nonlinear Volterra-Fredholm integro-differential equations of multi-arbitrary order, Gazi J. of Sci. 29 (2016), 895-907.
[9] K. Parand, M. Delkhosh, Solving Volterra's population growth model of arbitrary order using the generalized fractional order of the Chebyshev functions, Ricerche mat. 65 (2016), 307-328.
[10] K. Parand, J. A. Rad, Numerical solution of nonlinear Volterra-Fredholm-Hammerstein integral equations via collocation method based on radial basis functions, Appl. Math. Comput. 218 (2012), 5292-5309.
[11] C. Vetro, F. Vetro, On the existence of at least a solution for functional integral equations via measure of noncompactness, Banach J. Math. Anal. 11 (3) (2017), 497-512.
[12] H. Wendland, Scattered Data Approximation, Cambridge University Press, 2005.
[13] S. Yousefi, A. Lotfi, M. Dehghan, The use of a Legendre multiwavelet collocation method for solving the fractional optimal control problems. J. Vibration and Control. 17 (13) (2011), 2059-2065.
[14] M. Zerroukat, H. Power, C. S. Chen, A numerical method for heat transfer problem using collocation and radial basis functions, Int. J. Numer. Meth. Eng. 42 (1992), 1263-1278.