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The Mackey topology in linearly topologized modules

D. P. Pombo Jr.[a], P. C. G. Mauro[b]

[a] *Instituto de Matemática e Estatística, Universidade Federal Fluminense, Niterói, RJ Brasil.*
[b] *Centro Interdisciplinar de Ciências da Natureza, Universidade Federal da Integração Latino-Americana, Foz do Iguaçu, PR Brasil.*

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Abstract. The Mackey topology in the context of Hausdorff linearly topologized modules over a complete discrete valuation ring is introduced and characterizations of this concept are established. Moreover the interplay between the concept of Mackey topology and two special classes of linearly topologized modules is discussed.

Keywords: Complete discrete valuation rings, linearly topologized modules, Mackey topology.

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1. Introduction

The notion of linearly topologized module over a discrete valuation ring plays an important role in Algebraic Geometry [2], Commutative Algebra [1], Number Theory [10, 11] and Topological Algebra [12]. The purpose of this paper is to introduce the Mackey topology in the context of Hausdorff linearly topologized modules over an arbitrary complete discrete valuation ring and to obtain characterizations of this notion by means of universal properties. Additionally it is proved that certain Hausdorff linearly topologized modules, characterized by the validity of the Banach-Steinhaus property, are endowed with the Mackey topology.

Let us recall that a principal ring R is a discrete valuation ring ([10], Chap. I) if the set M of all non-invertible elements of R is a non-trivial ideal of R (in this case, M is a maximal ideal of R). If π is a generator of M, then $\{\pi^n R : n = 1, 2, \ldots\}$ constitutes a

*Corresponding author.
E-mail address: dpombojr@gmail.com (D. P. Pombo Jr.); patricia.mauro@unila.edu.br (P. C. G. Mauro).

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fundamental system of neighborhoods of 0 in R formed by ideals of R such that $\bigcap_{n\geqslant 1} \pi^n R = \{0\}$. The following important examples of discrete valuation rings may be mentioned: the ring $\mathbb{Z}_p$ of p-adic integers (p a prime natural number), M being $p\mathbb{Z}_p$; the ring $\mathbb{K}\,[[X]]$ of formal power series with coefficients in an arbitrary field $\mathbb{K}$, M being $X\mathbb{K}\,[[X]]$; for each $z_0 \in \mathbb{C}$, the ring $\mathcal{H}_{z_0}$ of complex analytic mappings on an open ball (in $\mathbb{C}$) with center at z_0, M being $(z - z_0)\mathcal{H}_{z_0}$. E is a linearly topologized R-module ([12], Chap. V) if E is a topological R-module whose origin admits a fundamental system of neighborhoods consisting of submodules of E.

Let R be a complete discrete valuation ring and K the field of fractions of R. Then K is endowed with a discrete valuation under which it is complete. Since the residual field R/M is finite, Proposition 1, p. 37 of [10] implies that R is compact. Thus R is a linearly compact ring and Theorem 31.12 of [12] ensures that K is a linearly compact R-module. Now, let R_0 be the R-module K/R endowed with the discrete topology, under which R_0 is a linearly topologized R-module [5]. Since the canonical R-linear mapping $\pi : \lambda \in K \longmapsto \lambda + R \in R_0$ is continuous ($\mathrm{Ker}(\pi) = R$), it follows from Theorem 31.6 of [12] that R_0 is linearly compact. Consequently, by Theorem 31.10 of [12], $(R_0)^I$ (endowed with the product topology) is a linearly compact R-module if I is an arbitrary non-empty set.

2. The Mackey topology

In this work R will denote an arbitrary complete discrete valuation ring.

Let (E, τ) be an arbitrary Hausdorff linearly topologized R-module, E^* the R-module of all continuous R-linear mappings from (E, τ) into R_0, $\sigma(E, E^*)$ the weak topology on E (one has $(E, \sigma(E, E^*))^* = E^*$) and $\sigma(E^*, E)$ the weak topology on E^* [3, 5]. Then

$$\mathcal{B} = \left\{ L^\perp : L \text{ is a } \sigma(E^*, E)\text{-bounded and } \sigma(E^*, E)\text{-linearly compact submodule of } E^* \right\}$$

is a filter base on E. In fact, if L_1, L_2 are $\sigma(E^*, E)$-bounded and $\sigma(E^*, E)$-linearly compact submodules of E^*, $L_1 + L_2$ is a $\sigma(E^*, E)$-bounded ([12], Theorem 15.2(4)) and $\sigma(E^*, E)$-linearly compact ([12], Theorem 31.6) submodule of E^* such that $(L_1 + L_2)^\perp \subset L_1^\perp \cap L_2^\perp$. Moreover, conditions (ATG 1), (ATG 2), (TMN 1) and (TMN 3) of Theorem 12.3 of [12] are obviously satisfied and the validity of condition (TMN 2) follows from the fact that each $L^\perp \in \mathcal{B}$ comes from an L which is $\sigma(E^*, E)$-bounded. Thus the just mentioned Theorem 12.3 guarantees the existence of a unique R-module topology $\tau(E, E^*)$ on E for which $\mathcal{B}$ is a fundamental system of neighborhoods of 0. And, since each element of $\mathcal{B}$ is a submodule of E, $\tau(E, E^*)$ is a linear topology on E.

Definition 2.1 $\tau(E, E^*)$ is said to be the Mackey topology on E.

Proposition 2.2 If (E, τ) is an arbitrary Hausdorff linearly topologized R-module, then

$$\sigma(E, E^*) \leqslant \tau \leqslant \tau(E, E^*).$$

Proof. Since $\sigma(E, E^*) \leqslant \tau$, it remains to show that $\tau \leqslant \tau(E, E^*)$. Indeed, if U is a τ-neighborhood of 0 in E which is a submodule of E, Proposition 4.12 of [5] furnishes $U = U^{\perp\perp} = (U^\perp)^\perp$, $U^\perp$ being an equicontinuous (hence $\sigma(E^*, E)$-bounded) submodule of E^* which is $\sigma(E^*, E)$-linearly compact in view of Theorem 1 of [7]. Therefore $U \in \mathcal{B}$, and $\tau \leqslant \tau(E, E^*)$. ■

Proposition 2.3 If (E, τ) is an arbitrary Hausdorff linearly topologized R-module, then $(E, \tau(E, E^*))^* = E^*$.

Proof. By Proposition 2.2, it is enough to prove that $(E, \tau(E, E^*))^* \subset E^*$. In fact, if $\varphi \in (E, \tau(E, E^*))^*$, there is a $\sigma(E^*, E)$-bounded and $\sigma(E^*, E)$-linearly compact submodule L of E^* so that $\varphi(x) = 0$ for all $x \in L^\perp$. Now let us consider the dual system (E, E'), where E' is the R-module of all R-linear mappings from E into R_0. Since the inclusion mapping $\xi \in (E^*, \sigma(E^*, E)) \longmapsto \xi \in (E', \sigma(E', E))$ is continuous, L is $\sigma(E', E)$-linearly compact, hence $\sigma(E', E)$-closed by Theorem 31.9(1) of [12]. On the other hand, by Proposition 4.13 of [5], $\varphi \in L^{\perp\perp} = \overline{L^{\sigma(E',E)}} = L \subset E^*$, which concludes the proof. ■

The next result furnishes necessary and sufficient conditions for a Hausdorff linear R-module topology to coincide with the Mackey topology. More precisely, we shall establish the following:

Theorem 2.4 For an arbitrary Hausdorff linearly topologized R-module (E, τ), the following conditions are equivalent:
(a) for every Hausdorff linearly topologized R-module (F, τ'), each R-linear mapping $u : (E, \tau) \to (F, \tau')$ such that $\Psi \circ u \in E^*$ for all $\Psi \in F^*$ is continuous (where $F^* = ((F, \tau'))^*$);
(b) every Hausdorff linear R-module topology θ on E such that $((E, \theta))^* \subset E^*$ is coarser than τ (in particular, if $((E, \theta))^* = E^*$, then θ is coarser than τ);
(c) $\tau = \tau(E, E^*)$.

Proof. $(a) \Rightarrow (b)$: It suffices to take, in (a), $(F, \tau') = (E, \theta)$ and $u : (E, \tau) \to (F, \tau')$ the identity mapping.
$(b) \Rightarrow (c)$: Since τ is coarser than $\tau(E, E^*)$ by Proposition 2.2, it remains to show that $\tau(E, E^*)$ is coarser than τ. But, by Proposition 2.3, $(E, \tau(E, E^*))^* = E^*$, and consequently $\tau(E, E^*)$ is coarser than τ by hypothesis.
$(c) \Rightarrow (a)$: Let (F, τ'), F^* and u be as in (a). Then the R-linear mapping

$$u^t : \Psi \in (F^*, \sigma(F^*, F)) \longmapsto \Psi \circ u \in (E^*, \sigma(E^*, E))$$

is continuous. Let V be a neighborhood of 0 in (F, τ') which is a submodule of F. Since $V^\perp$ is a $\sigma(F^*, F)$-bounded and $\sigma(F^*, F)$-linearly compact submodule of F^*, $u^t(V^\perp)$ is a $\sigma(E^*, E)$-bounded and $\sigma(E^*, E)$-linearly compact submodule of E^* by Theorems 15.2(2) and 31.6 of [12]. Hence, by hypothesis, $[u^t(V^\perp)]^\perp$ is a neighborhood of 0 in (E, τ). Moreover, the relations

$$x \in [u^t(V^\perp)]^\perp, \Psi \in V^\perp$$

imply $\Psi(u(x)) = 0$, and so

$$u(x) \in (V^\perp)^\perp = V^{\perp\perp} = V \text{ for all } x \in [u^t(V^\perp)]^\perp.$$

Thus $u : (E, \tau) \to (F, \tau')$ is continuous, as was to be shown. ■

It is known that every continuous linear mapping between two Hausdorff linearly topologized R-modules is weakly continuous. The last result of this section ensures that, conversely, every weakly continuous linear mapping is continuous with respect to the Mackey topology.

Proposition 2.5 Let E and F be Hausdorff linearly topologized R-modules. If

$$u : (E, \sigma(E, E^*)) \longrightarrow (F, \sigma(F, F^*))$$

is a continuous R-linear mapping, then

$$u : (E, \tau(E, E^*)) \longrightarrow (F, \tau(F, F^*))$$

is continuous. Consequently, by Proposition 2.2, $u : (E, \tau(E, E^*)) \longrightarrow F$ is continuous.

Proof. Let M be an arbitrary $\sigma(F^*, F)$-bounded and $\sigma(F^*, F)$-linearly compact submodule of F^*. By the continuity of the R-linear mapping

$$u^t : \Psi \in (F^*, \sigma(F^*, F)) \longmapsto \Psi \circ u \in (E^*, \sigma(E^*, E)),$$

the submodule $u^t(M)$ of E^* is $\sigma(E^*, E)$-bounded and $\sigma(E^*, E)$-linearly compact. Hence $\left[u^t(M)\right]^\perp$ is a $\tau(E, E^*)$-neighborhood of 0 in E such that $u\left(\left[u^t(M)\right]^\perp\right) \subset M^\perp$, which concludes the proof. ■

3. Linearly topologized R-modules with the Mackey topology

A linearly topologized R-module E is bornological [8] if every bornivorous in E is a neighborhood of 0 in E. As we have observed in Examples 2.5 and 2.9 of the same paper, the Hausdorff linearly topologized R-modules R_0, R^n $(n = 1, 2, \ldots)$ and (the topological direct sum) $R^{(\mathbb{N})}$ are bornological.

Our first result, motivated by Proposition 27 of [6], furnishes a sufficient condition for a Hausdorff linearly topologized R-module to be bornological.

Proposition 3.1 If (E, τ) is a Hausdorff linearly topologized R-module such that $\tau = \tau(E, E^*)$ and every R-linear mapping from E into R_0 which transforms τ-bounded subsets of E into bounded subsets of R_0 belongs to E^*, then (E, τ) is bornological.

Proof. Let $\mathcal{B}$ be the filter base on E consisting of all τ-bornivorous subsets of E ($B \in \mathcal{B}$ if B is a submodule of E such that for each τ-bounded subset D of E there is an integer $m \geqslant 1$ for which $\pi^m D \subset B$). Since conditions (ATG 1), (ATG 2), (TMN 1), (TMN 2) and (TMN 3) of Theorem 12.3 of [12] are easily verified, one may guarantee the existence of a unique linear R-module topology $\tau_{\mathcal{B}}$ on E for which $\mathcal{B}$ is a fundamental system of $\tau_{\mathcal{B}}$-neighborhoods of 0 ($\tau_{\mathcal{B}}$ is a Hausdorff topology because τ is coarser than $\tau_{\mathcal{B}}$). If $\varphi \in ((E, \tau_{\mathcal{B}}))^*$ is arbitrary, there is a $B \in \mathcal{B}$ so that $\varphi(B) = \{0\}$. If D is an arbitrary τ-bounded subset of E, there is an integer $k \geqslant 1$ such that $\pi^k D \subset B$, which implies $\pi^k \varphi(D) \subset \{0\}$ and shows that $\varphi(D)$ is a bounded subset of R_0. Hence, by hypothesis, $\varphi \in E^*$. Finally, by condition (b) of Theorem 2.4, $\tau_{\mathcal{B}}$ is coarser than τ. Therefore $\tau = \tau_{\mathcal{B}}$, and (E, τ) is bornological. ■

A linearly topologized R-module E is barrelled [4] if every barrel in E is a neighborhood of 0 in E. As we have observed in Examples 2.7, 2.6 and 2.10 of the same paper, the Hausdorff linearly topologized R-modules R_0, R^n $(n = 1, 2, \ldots)$ and (the topological direct sum) $R^{(\mathbb{N})}$ are barrelled.

Our last result, motivated by Proposition 26 of [6] and Proposition 14.4 of [9], furnishes a necessary and sufficient condition for a Hausdorff linearly topologized R-module to be barrelled.

Proposition 3.2 For a Hausdorff linearly topologized R-module (E, τ), the following conditions are equivalent:
(a) (E, τ) is barrelled;
(b) $\tau = \tau(E, E^*)$ and every $\sigma(E^*, E)$-closed and $\sigma(E^*, E)$-bounded submodule of E^* is $\sigma(E^*, E)$-linearly compact.

Proof. $(a) \Rightarrow (b)$: We shall prove the validity of condition (a) of Theorem 2.4. Indeed, let (F, τ') be a Hausdorff linearly topologized R-module and $u : E \to F$ an R-linear mapping satisfying the just mentioned condition (a). We claim that u is continuous. In fact, if V is a τ'-neighborhood of 0 in F which is a submodule of F, then $N = V^{\perp} \circ u \subset E^*$ and is $\sigma(E^*, E)$-bounded (note that $V^{\perp}$ is equicontinuous, hence $\sigma(F^*, F)$-bounded, where $F^* = ((F, \tau'))^*$). Thus, by Theorem 3.4 of [4], N is equicontinuous. Consequently, there is a τ-neighborhood U of 0 in E such that $\Psi(u(x)) = 0$ for all $x \in U$ and $\Psi \in V^{\perp}$, and so $u(x) \in (V^{\perp})^{\perp} = V^{\perp\perp} = V$ for all $x \in U$. Therefore, u is continuous, and Theorem 2.4 gives $\tau = \tau(E, E^*)$. Finally, if M is a $\sigma(E^*, E)$-closed and $\sigma(E^*, E)$-bounded submodule of E^*, M is equicontinuous. Thus, by Theorem 1 of [7], M is $\sigma(E^*, E)$-linearly compact.
$(b) \Rightarrow (a)$: If T is a barrel in E, the $\sigma(E^*, E)$-closed submodule $T^{\perp}$ of E^* is $\sigma(E^*, E)$-bounded by Proposition 3.1 of [4]; hence, by hypothesis, $T^{\perp}$ is $\sigma(E^*, E)$-linearly compact. Moreover, since $\tau = \tau(E, E^*)$ and $T = T^{\perp\perp} = (T^{\perp})^{\perp}$, it follows that T is a τ-neighborhood of 0 in E, proving that (E, τ) is barrelled. ■

In conclusion one may guarantee that, for a given Hausdorff linearly topologized R-module (E, τ), conditions (a) and (b) of Proposition 3.2 are also equivalent to (the equivalent) conditions:
(α) for every Hausdorff linearly topologized R-module (F, τ'), each τ_s-bounded subset of the R-module $\mathcal{L}(E; F)$ of all continuous R-linear mappings from (E, τ) into (F, τ') is equicontinuous (where τ_s denotes the linear R-module topology of simple convergence on $\mathcal{L}(E; F)$),
(β) each $\sigma(E^*, E)$-bounded subset of E^* is equicontinuous, which occur in Proposition 3.5 of [4].

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