

Operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$

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Received 7 April 2018; Revised 30 January 2019; Accepted 31 January 2019.

Communicated by Mohammad Sadegh Asgari

Abstract. Frames generalize orthonormal bases and allow representation of all the elements of the space. Frames play significant role in signal and image processing, which leads to many applications in informatics, engineering, medicine, and probability. In this paper, we introduce the concepts of operator frame for the space $End_{\mathcal{A}}^*(\mathcal{H})$ of all adjointable operators on a Hilbert \mathcal{A} -module \mathcal{H} and establish some results.

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Keywords: Frame, operator frame, C^* -algebra, Hilbert \mathcal{A} -modules.

2010 AMS Subject Classification: 42C15, 46L05.

1. Introduction and preliminaries

Frames were first introduced in 1952 by Duffin and Schaefer [6] in the study of nonharmonic Fourier series. Frames possess many nice properties which make them very useful in wavelet analysis, irregular sampling theory, signal processing and many other fields. The theory of frames has been generalized rapidly and various generalizations of frames in Hilbert spaces and Hilbert C^* -modules.

In this paper, we introduce the notion of operator frame for the space $End_{\mathcal{A}}^*(\mathcal{H})$ of all adjointable operators on a Hilbert \mathcal{A} -module \mathcal{H} , where is a generalization of operator frame for $B(\mathcal{H})$ [11].

Let \mathbb{J} be a finite or countable index subset of \mathbb{N} . In this section, we briefly recall the definitions and basic properties of C^* -algebra, Hilbert \mathcal{A} -modules, frame in Hilbert \mathcal{A} -modules. For information about frames in Hilbert spaces, we refer to [3]. Our reference

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for C^* -algebras is [4, 5]. For a C^* -algebra \mathcal{A} if $a \in \mathcal{A}$ is positive we write $a \geq 0$ and \mathcal{A}^+ denotes the set of positive elements of \mathcal{A} .

Definition 1.1 [7] Let \mathcal{A} be a unital C^* -algebra and \mathcal{H} be a left \mathcal{A} -module such that the linear structures of \mathcal{A} and \mathcal{H} are compatible. \mathcal{H} is a pre-Hilbert \mathcal{A} -module if \mathcal{H} is equipped with an \mathcal{A} -valued inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{A}$ such that is sesquilinear, positive definite and respects the module action. In the other words,

- (i) $\langle x, x \rangle \geq 0$ for all $x \in \mathcal{H}$ and $\langle x, x \rangle = 0$ if and only if $x = 0$;
- (ii) $\langle ax + y, z \rangle = a\langle x, y \rangle + \langle y, z \rangle$ for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$;
- (iii) $\langle x, y \rangle = \langle y, x \rangle^*$ for all $x, y \in \mathcal{H}$.

For $x \in \mathcal{H}$, we define $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$. If \mathcal{H} is complete with $\|\cdot\|$, it is called a Hilbert \mathcal{A} -module or a Hilbert C^* -module over \mathcal{A} . For every a in C^* -algebra \mathcal{A} , we have $|a| = (a^*a)^{\frac{1}{2}}$ and the \mathcal{A} -valued norm on \mathcal{H} is defined by $|x| = \langle x, x \rangle^{\frac{1}{2}}$ for $x \in \mathcal{H}$.

Let \mathcal{H} and \mathcal{K} be two Hilbert \mathcal{A} -modules, A map $T : \mathcal{H} \rightarrow \mathcal{K}$ is said to be adjointable if there exists a map $T^* : \mathcal{K} \rightarrow \mathcal{H}$ such that $\langle Tx, y \rangle_{\mathcal{A}} = \langle x, T^*y \rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$. We also reserve the notation $End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K})$ for the set of all adjointable operators from \mathcal{H} to \mathcal{K} and $End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H})$ is abbreviated to $End_{\mathcal{A}}^*(\mathcal{H})$.

Definition 1.2 [8] Let \mathcal{H} be a Hilbert \mathcal{A} -module over a unital C^* -algebra. A family $\{x_i\}_{i \in \mathbb{J}}$ of elements of \mathcal{H} is a frame for \mathcal{H} , if there exist two positive constants A and B such that for all $x \in \mathcal{H}$,

$$A\langle x, x \rangle \leq \sum_{i \in \mathbb{J}} \langle x, x_i \rangle \langle x_i, x \rangle \leq B\langle x, x \rangle. \tag{1}$$

The numbers A and B are called lower and upper bounds of the frame, respectively. If $A = B = \lambda$, the frame is λ -tight. If $A = B = 1$, it is called a normalized tight frame or a Parseval frame. If the sum in the middle of (1) is convergent in norm, the frame is called standard. If only upper inequality of (1) hold, then $\{x_i\}_{i \in I}$ is called a Bessel sequence for \mathcal{H} .

Definition 1.3 [9] Let \mathcal{A} be a unital C^* -algebra, \mathcal{H} be a Hilbert \mathcal{A} -module and $\{v_i : i \in \mathbb{J}\}$ be a family of weights in \mathcal{A} , i.e., each v_i is a positive invertible element from the center of the C^* -algebra \mathcal{A} . A sequence of closed submodules $\{M_i : i \in \mathbb{J}\}$ is a frame of submodules if every M_i is orthogonally complemented and there exist real constants $0 < C \leq D < \infty$ such that

$$C\langle x, x \rangle \leq \sum_{i \in \mathbb{J}} v_i^2 \langle \pi_{M_i} x, \pi_{M_i} x \rangle \leq D\langle x, x \rangle, \text{ for } x \in \mathcal{H}. \tag{2}$$

We call C and D the lower and upper bounds of the frame of submodules, and like (2) we call $\{(M_i, v_i) : i \in \mathbb{J}\}$ a fusion frame. As usual if $C = D = \lambda$, the family $\{M_i : i \in \mathbb{J}\}$ is called a λ -tight frame of submodules with respect to $\{v_i : i \in \mathbb{J}\}$ and if $C = D = 1$, it is called a Parseval or a normalized tight frame of submodules.

The following lemmas will be used to prove our mains results.

Lemma 1.4 [12] Let \mathcal{H} be a Hilbert \mathcal{A} -module. If $T \in End_{\mathcal{A}}^*(\mathcal{H})$, then

$$\langle Tx, Tx \rangle \leq \|T\|^2 \langle x, x \rangle, \forall x \in \mathcal{H}.$$

Lemma 1.5 [2] Let \mathcal{H} be a Hilbert \mathcal{A} -module over a C^* -algebra \mathcal{A} and $T \in End_{\mathcal{A}}^*(\mathcal{H})$

such that $T^* = T$. The following statements are equivalent.

- (i) T is surjective;
- (ii) There are $m, M > 0$ such that $m\|x\| \leq \|Tx\| \leq M\|x\|$, for all $x \in \mathcal{H}$;
- (iii) There are $m', M' > 0$ such that $m'\langle x, x \rangle \leq \langle Tx, Tx \rangle \leq M'\langle x, x \rangle$ for all $x \in \mathcal{H}$.

Lemma 1.6 [1] If $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is a $*$ -homomorphism between C^* -algebras, then φ is increasing; that is, if $a \leq b$, then $\varphi(a) \leq \varphi(b)$.

Lemma 1.7 [1] Let \mathcal{H} and \mathcal{K} two Hilbert \mathcal{A} -modules and $T \in \text{End}^*(\mathcal{H}, \mathcal{K})$. Then,

- (i) If T is injective and T has closed range, then the adjointable map T^*T is invertible and

$$\|(T^*T)^{-1}\|^{-1} \leq T^*T \leq \|T\|^2.$$

- (ii) If T is surjective, then the adjointable map TT^* is invertible and

$$\|(TT^*)^{-1}\|^{-1} \leq TT^* \leq \|T\|^2.$$

2. Operator Frame for $\text{End}^*_\mathcal{A}(\mathcal{H})$

Definition 2.1 A family of adjointable operators $\{T_i\}_{i \in \mathbb{J}}$ on a Hilbert \mathcal{A} -module \mathcal{H} over a unital C^* -algebra is said to be an operator frame for $\text{End}^*_\mathcal{A}(\mathcal{H})$, if there exists positive constants $A, B > 0$ such that

$$A\langle x, x \rangle \leq \sum_{i \in \mathbb{J}} \langle T_i x, T_i x \rangle \leq B\langle x, x \rangle, \forall x \in \mathcal{H}. \tag{3}$$

The numbers A and B are called lower and upper bounds of the operator frame, respectively. If $A = B = \lambda$, the operator frame is λ -tight. If $A = B = 1$, it is called a normalized tight operator frame or a Parseval operator frame. If only upper inequality of (3) hold, then $\{T_i\}_{i \in \mathbb{J}}$ is called an operator Bessel sequence for $\text{End}^*_\mathcal{A}(\mathcal{H})$. If the sum in the middle of (3) is convergent in norm, the operator frame is called standard. Throughout the paper, series like (3) are assumed to be convergent in the norm sense.

Example 2.2 Let \mathcal{A} be a Hilbert C^* -module over itself with the inner product $\langle a, b \rangle = ab^*$. Also, let $\{x_i\}_{i \in I}$ be a frame for \mathcal{A} with bounds A and B , respectively. For each $i \in I$, we define $T_i : \mathcal{A} \rightarrow \mathcal{A}$ by $T_i x = \langle x, x_i \rangle$ for all $x \in \mathcal{A}$. T_i is adjointable and $T_i^* a = ax_i$ for each $a \in \mathcal{A}$ and we have

$$A\langle x, x \rangle \leq \sum_{i \in I} \langle x, x_i \rangle \langle x_i, x \rangle \leq B\langle x, x \rangle, \forall x \in \mathcal{A}.$$

Then

$$A\langle x, x \rangle \leq \sum_{i \in I} \langle T_i x, T_i x \rangle \leq B\langle x, x \rangle, \forall x \in \mathcal{A}.$$

So, $\{T_i\}_{i \in I}$ is an operator frame in \mathcal{A} with bounds A and B , respectively.

Example 2.3 Let \mathcal{H} and \mathcal{K} be separable Hilbert spaces and $B(\mathcal{H}, \mathcal{K})$ be the set of all bounded linear operators from \mathcal{H} into \mathcal{K} . $B(\mathcal{H}, \mathcal{K})$ is a Hilbert $B(\mathcal{K})$ -module with a $B(\mathcal{K})$ -valued inner product $\langle S, T \rangle = ST^*$ for all $S, T \in B(\mathcal{H}, \mathcal{K})$, and with a linear operation of $B(\mathcal{K})$ on $B(\mathcal{H}, \mathcal{K})$ by composition of operators. Consider $\mathbb{J} = \mathbb{N}$ and fix

$(a_i)_{i \in \mathbb{N}} \in l^2(\mathbf{C})$. Define $T_i(X) = a_i X$ for all $X \in B(\mathcal{H}, \mathcal{K})$ and for all $i \in \mathbb{N}$. We have $\sum_{i \in \mathbb{N}} \langle T_i x, T_i x \rangle = \sum_{i \in \mathbb{N}} |a_i|^2 \langle X, X \rangle$ for all $X \in B(\mathcal{H}, \mathcal{K})$. Then $\{T_i\}_{i \in \mathbb{N}}$ is $\sum_{i \in \mathbb{N}} |a_i|^2$ -tight operator frame.

Example 2.4 Let $\{W_i\}_{i \in \mathbb{J}}$ be a frame of submodules with respect to $\{v_i\}_{i \in \mathbb{J}}$ for \mathcal{H} . Put $T_i = v_i \pi_{W_i}, \forall i \in \mathbb{J}$, then we get a sequence of operators $\{T_i\}_{i \in \mathbb{J}}$. Then there exist constants $A, B > 0$ such that

$$A \langle x, x \rangle \leq \sum_{i \in \mathbb{J}} v_i^2 \langle \pi_{W_i} x, \pi_{W_i} x \rangle \leq B \langle x, x \rangle, \forall x \in \mathcal{H}.$$

So, we have

$$A \langle x, x \rangle \leq \sum_{i \in \mathbb{J}} \langle T_i x, T_i x \rangle \leq B \langle x, x \rangle, \forall x \in \mathcal{H}.$$

Thus, the sequence $\{T_i\}_{i \in \mathbb{J}}$ becomes an operator frame for \mathcal{H} .

With this example a frame of submodules can be viewed as a special case of operator frames.

Let $\{T_i\}_{i \in \mathbb{J}}$ be an operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$. Define an operator $R : \mathcal{H} \rightarrow l^2(\mathcal{H})$ by $Rx = \{T_i x\}_{i \in \mathbb{J}}$ for all $x \in \mathcal{H}$. The operator R is called the analysis operator of the operator frame $\{T_i\}_{i \in \mathbb{J}}$. The adjoint of the analysis operator R ; that is, $R^* (\{x_i\}_{i \in \mathbb{J}}) : l^2(\mathcal{H}) \rightarrow \mathcal{H}$ is defined by

$$R^* (\{x_i\}_{i \in \mathbb{J}}) = \sum_{i \in \mathbb{J}} T_i^* x_i, \forall \{x_i\}_{i \in \mathbb{J}} \in l^2(\mathcal{H}).$$

The operator R^* is called the synthesis operator of the operator frame $\{T_i\}_{i \in \mathbb{J}}$. By composing R and R^* , the frame operator $S_T : \mathcal{H} \rightarrow \mathcal{H}$ for the operator frame is given by

$$S_T(x) = R^* Rx = \sum_{i \in \mathbb{J}} T_i^* T_i x.$$

Theorem 2.5 Let $\{T_i\}_{i \in \mathbb{J}}$ be a family of adjointable operators on a Hilbert \mathcal{A} -module \mathcal{H} . Suppose that $\sum_{i \in \mathbb{J}} \langle T_i x, T_i x \rangle$ converge in norm for all $x \in \mathcal{H}$. Then $\{T_i\}_{i \in \mathbb{J}}$ is an operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ if and only if there exist positive constants $A, B > 0$ such that

$$A \|x\|^2 \leq \left\| \sum_{i \in \mathbb{J}} \langle T_i x, T_i x \rangle \right\| \leq B \|x\|^2, \forall x \in \mathcal{H}. \quad (4)$$

Proof. Suppose that $\{T_i\}_{i \in \mathbb{J}}$ is an operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$. Since for any $x \in \mathcal{H}$, there is $\langle x, x \rangle \geq 0$, combined with the definition of an operator frame we know that (4) holds. Now, suppose that (4) holds. We know that the frame operator S_T is positive, self-adjoint and invertible. Hence,

$$\langle S_T^{\frac{1}{2}} x, S_T^{\frac{1}{2}} x \rangle = \langle S_T x, x \rangle = \sum_{i \in \mathbb{J}} \langle T_i x, T_i x \rangle.$$

So, we have $\sqrt{A} \|x\| \leq \|S_T^{\frac{1}{2}} x\| \leq \sqrt{B} \|x\|$ for any $x \in \mathcal{H}$. Using Lemma 1.5, there are constants $m, M > 0$ such that

$$m \langle x, x \rangle \leq \langle S_T^{\frac{1}{2}} x, S_T^{\frac{1}{2}} x \rangle = \sum_{i \in \mathbb{J}} \langle T_i x, T_i x \rangle \leq M \langle x, x \rangle,$$

which implies that $\{T_i\}_{i \in \mathbb{J}}$ is an operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$. ■

Theorem 2.6 Assume that S_T is the frame operator of an operator frame $T = \{T_i\}_{i \in \mathbb{J}}$ for $End_{\mathcal{A}}^*(\mathcal{H})$ with bounds A and B . Then S_T is positive, self-adjoint and invertible. Moreover, we have $AI \leq S_T \leq BI$ and the reconstruction formula

$$x = \sum_{i \in \mathbb{J}} S_T^{-1} T_i^* T_i x, \forall x \in \mathcal{H}. \tag{5}$$

Proof. It is clear that S_T is positive and self-adjoint. For any $x \in \mathcal{H}$, since $T = \{T_i\}_{i \in \mathbb{J}}$ is an operator frame with bounds A, B , we have

$$\langle Ax, x \rangle = A \langle x, x \rangle \leq \sum_{i \in \mathbb{J}} \langle T_i x, T_i x \rangle = \langle S_T x, x \rangle \leq B \langle x, x \rangle = \langle Bx, x \rangle.$$

This shows that $AI \leq S_T \leq BI$, which implies that S_T is invertible. Further, for any $x \in \mathcal{H}$, we have

$$x = S_T^{-1} S_T x = S_T^{-1} \sum_{i \in \mathbb{J}} T_i^* T_i x = \sum_{i \in \mathbb{J}} S_T^{-1} T_i^* T_i x.$$

■

Definition 2.7 Let $T = \{T_i\}_{i \in \mathbb{J}}$ be an operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$. A family of operators $\tilde{T} = \{\tilde{T}_i\}_{i \in \mathbb{J}}$ on \mathcal{H} is called a dual of operator frame $\{T_i\}_{i \in \mathbb{J}}$, if they satisfy

$$x = \sum_{i \in \mathbb{J}} T_i^* \tilde{T}_i x, \forall x \in \mathcal{H}. \tag{6}$$

Furthermore, we call $\{\tilde{T}_i\}_{i \in \mathbb{J}}$ a dual frame of the operator frame $\{T_i\}_{i \in \mathbb{J}}$, if $\{\tilde{T}_i\}_{i \in \mathbb{J}}$ is also an operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ and satisfies condition 6.

Theorem 2.8 Every operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ has a dual frame.

Proof. If $T = \{T_i\}_{i \in \mathbb{J}}$ is an operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ with bounds A and B , then the operator sequence $\tilde{T} = \{T_i S_T^{-1}\}_{i \in \mathbb{J}}$ is a dual frame of $T = \{T_i\}_{i \in \mathbb{J}}$. In fact, we have

$$x = S_T S_T^{-1} x = \sum_{i \in \mathbb{J}} T_i^* T_i S_T^{-1} x = \sum_{i \in \mathbb{J}} T_i^* \tilde{T}_i x, \forall x \in \mathcal{H}.$$

Also, $\tilde{T} = \{T_i S_T^{-1}\}_{i \in \mathbb{J}}$ satisfies

$$A \|S_T\|^{-2} \langle x, x \rangle \leq \sum_{i \in \mathbb{J}} \langle \tilde{T}_i x, \tilde{T}_i x \rangle = \sum_{i \in \mathbb{J}} \langle \tilde{T}_i S_T^{-1} x, \tilde{T}_i S_T^{-1} x \rangle \leq B \|S_T^{-1}\|^2 \langle x, x \rangle, \forall x \in \mathcal{H}.$$

Hence, $\{T_i S_T^{-1}\}_{i \in \mathbb{J}}$ is called the canonical dual frame of $\{T_i\}_{i \in \mathbb{J}}$. ■

Remark 1 Assume that $T = \{T_i\}_{i \in \mathbb{J}}$ is an operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ with analytic operator R_T and $\tilde{T} = \{\tilde{T}_i\}_{i \in \mathbb{J}}$ is a dual frame of T with analytic operator $R_{\tilde{T}}$. Then, for any x in \mathcal{H} , we have $x = \sum_{i \in \mathbb{J}} T_i^* \tilde{T}_i x = R_T^* R_{\tilde{T}} x$. This shows that every element of \mathcal{H} can be reconstructed with an operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ and its dual frame. Moreover, we have another fact that for any operator A on \mathcal{H} , $Ax = \sum_{i \in \mathbb{J}} T_i^* \tilde{T}_i Ax$; that is, an association of operator frame and its dual frame can reconstruct pointwisely every operator on \mathcal{H} . So, we can write $A = \sum_{i \in \mathbb{J}} T_i^* \tilde{T}_i A$.

For a frame of submodules $\{W_i\}_{i \in \mathbb{J}}$ with respect to the family of weights $\{v_i\}_{i \in \mathbb{J}}$ for \mathcal{H} with synthesis operator $T_{W,v}$, the sequence $\{u_i\}_{i \in \mathbb{J}} = \{S_{W,v}^{-1} W_i\}_{i \in \mathbb{J}}$ is called the dual frame of $\{W_i\}_{i \in \mathbb{J}}$, where the operator $S_{W,v} = T_{W,v} T_{W,v}^*$.

Theorem 2.9 For a frame of submodules $\{W_i\}_{i \in \mathbb{J}}$ with respect to the family of weights $\{v_i\}_{i \in \mathbb{J}}$ for \mathcal{H} , define $T_i = v_i S_{W,v} \pi_{W_i} S_{W,v}^{-1}$ and $Q_i = v_i \pi_{u_i} S_{W,v}^{-1}$. Then $Q = \{Q_i\}_{i \in \mathbb{J}}$ and $T = \{T_i\}_{i \in \mathbb{J}}$ are all operator frames for $End_{\mathcal{A}}^*(\mathcal{H})$, and Q is a dual frame of T .

Proof. Assume that $\{W_i\}_{i \in \mathbb{J}}$ has frame bounds A, B .

(1) $T = \{T_i\}_{i \in \mathbb{J}}$ is an operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$. For any $x \in \mathcal{H}$, we have

$$\begin{aligned} \sum_{i \in \mathbb{J}} \langle T_i x, T_i x \rangle &= \sum_{i \in \mathbb{J}} \langle v_i S_{W,v} \pi_{W_i} S_{W,v}^{-1} x, v_i S_{W,v} \pi_{W_i} S_{W,v}^{-1} x \rangle \\ &\leq \|S_{W,v}\|^2 B \langle S_{W,v}^{-1} x, S_{W,v}^{-1} x \rangle \leq B \|S_{W,v}\|^2 \|S_{W,v}^{-1}\|^2 \langle x, x \rangle. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{i \in \mathbb{J}} \langle T_i x, T_i x \rangle &= \sum_{i \in \mathbb{J}} \langle v_i S_{W,v} \pi_{W_i} S_{W,v}^{-1} x, v_i S_{W,v} \pi_{W_i} S_{W,v}^{-1} x \rangle \\ &\geq \|S_{W,v}^{-1}\|^{-2} A \langle S_{W,v}^{-1} x, S_{W,v}^{-1} x \rangle \geq A \|S_{W,v}^{-1}\|^{-2} \|S_{W,v}\|^{-2} \langle x, x \rangle. \end{aligned}$$

Thus, $T = \{T_i\}_{i \in \mathbb{J}}$ is an operator frame.

(2) $Q = \{Q_i\}_{i \in \mathbb{J}}$ is also an operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$. The proof is similar to (1).

(3) If $T_{U,v} = S_{W,v}^{-1} T_{W,v} S_{W,v}$, then $T_{U,v}^* = S_{W,v}^{-1} T_{W,v}^* S_{W,v}$, and $S_{U,v} = S_{W,v}$. It is easy to check that $\pi_{u_i} = S_{W,v}^{-1} \pi_{W_i} S_{W,v}$. Thus, $T_{U,v} = S_{W,v}^{-1} T_{W,v} S_{W,v}$, $T_{U,v}^* = S_{W,v}^{-1} T_{W,v}^* S_{W,v}$ and

$$\begin{aligned} S_{U,v} &= T_{U,v} T_{U,v}^* \\ &= S_{W,v}^{-1} T_{W,v} S_{W,v} S_{W,v}^{-1} T_{W,v}^* S_{W,v} \\ &= S_{W,v}^{-1} T_{W,v} T_{W,v}^* S_{W,v} \\ &= S_{W,v}^{-1} S_{W,v} S_{W,v} \\ &= S_{W,v} \end{aligned}$$

Hence, for any $x \in \mathcal{H}$, we compute

$$\begin{aligned} \sum_{i \in \mathbb{J}} T_i^* Q_i x &= \sum_{i \in \mathbb{J}} v_i S_{W,v}^{-1} \pi_{W_i} S_{W,v} v_i S_{W,v}^{-1} \pi_{W_i} S_{W,v} S_{W,v}^{-1} x \\ &= S_{W,v}^{-1} \left(\sum_{i \in \mathbb{J}} v_i^2 \pi_{W_i} x \right) \\ &= S_{W,v}^{-1} S_{W,v} x \\ &= x. \end{aligned}$$

This shows that $Q = \{Q_i\}_{i \in \mathbb{J}}$ is an dual operator frame of the operator frame $T = \{T_i\}_{i \in \mathbb{J}}$. ■

Theorem 2.10 Assume that $\{W_i\}_{i \in \mathbb{J}}$ is a Parseval frame of submodules for a Hilbert \mathcal{A} -module \mathcal{H} . Then $\{v_i \pi_{W_i}\}_{i \in \mathbb{J}}$ is an operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ and a dual frame of itself.

Proof. When $\{W_i\}_{i \in \mathbb{J}}$ is a Parseval frame of submodules, $S_{W,v} = I$. Clearly, this theorem is a consequence of the last theorem. ■

Theorem 2.11 Let $(\mathcal{H}, \mathcal{A}, \langle \cdot, \cdot \rangle_{\mathcal{A}})$ and $(\mathcal{H}, \mathcal{B}, \langle \cdot, \cdot \rangle_{\mathcal{B}})$ be two Hilbert C^* -modules, $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ be a $*$ -homomorphism and θ be an adjointable map on \mathcal{H} such that $\langle \theta x, \theta y \rangle_{\mathcal{B}} = \varphi(\langle x, y \rangle_{\mathcal{A}})$ for all $x, y \in \mathcal{H}$. Also, suppose that $\{\Lambda_i\}_i$ is an operator frame for $(\mathcal{H}, \mathcal{A}, \langle \cdot, \cdot \rangle_{\mathcal{A}})$ with frame operator $S_{\mathcal{A}}$ and lower and upper operator frame bounds A and B , respectively. If θ is invertible, then $\{\theta \Lambda_i\}_i$ is an operator frame for $(\mathcal{H}, \mathcal{B}, \langle \cdot, \cdot \rangle_{\mathcal{B}})$ with frame operator $S_{\mathcal{B}}$ and lower and upper operator frame bounds $\|(\theta^* \theta)^{-1}\|^{-1} A$ and $\|\theta\|^2 B$, respectively, and $\langle S_{\mathcal{B}} x, y \rangle_{\mathcal{B}} = \varphi(\langle S_{\mathcal{A}} x, y \rangle_{\mathcal{A}})$.

Proof. By the definition of operator frames, we have

$$A \langle x, x \rangle_{\mathcal{A}} \leq \sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle_{\mathcal{A}} \leq B \langle x, x \rangle_{\mathcal{A}}, \forall x \in \mathcal{H}.$$

By lemma 1.6, we have

$$\varphi(A \langle x, x \rangle_{\mathcal{A}}) \leq \varphi(\sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle_{\mathcal{A}}) \leq \varphi(B \langle x, x \rangle_{\mathcal{A}}), \forall x \in \mathcal{H}.$$

By the definition of $*$ -homomorphism, we have

$$A \varphi(\langle x, x \rangle_{\mathcal{A}}) \leq \sum_{i \in I} \varphi(\langle \Lambda_i x, \Lambda_i x \rangle_{\mathcal{A}}) \leq B \varphi(\langle x, x \rangle_{\mathcal{A}}), \forall x \in \mathcal{H}.$$

By the relation between θ and φ , we obtain

$$A \langle \theta x, \theta x \rangle_{\mathcal{B}} \leq \sum_{i \in I} \langle \theta \Lambda_i x, \theta \Lambda_i x \rangle_{\mathcal{B}} \leq B \langle \theta x, \theta x \rangle_{\mathcal{B}}, \forall x \in \mathcal{H}.$$

By lemma 1.7, we have $\|(\theta^* \theta)^{-1}\|^{-1} \langle x, x \rangle_{\mathcal{B}} \leq \langle \theta x, \theta x \rangle_{\mathcal{B}} \leq \|\theta\|^2 \langle x, x \rangle_{\mathcal{B}}$. Then

$$\|(\theta^* \theta)^{-1}\|^{-1} A \langle x, x \rangle_{\mathcal{B}} \leq \sum_{i \in I} \langle \theta \Lambda_i x, \theta \Lambda_i x \rangle_{\mathcal{B}} \leq \|\theta\|^2 B \langle x, x \rangle_{\mathcal{B}}, \forall x \in \mathcal{H}.$$

On the other hand, we have

$$\begin{aligned} \varphi(\langle S_{\mathcal{A}} x, y \rangle_{\mathcal{A}}) &= \varphi(\langle \sum_{i \in I} \Lambda_i^* \Lambda_i x, y \rangle_{\mathcal{A}}) \\ &= \sum_{i \in I} \varphi(\langle \Lambda_i x, \Lambda_i y \rangle_{\mathcal{A}}) \\ &= \sum_{i \in I} \langle \theta \Lambda_i x, \theta \Lambda_i y \rangle_{\mathcal{B}} \\ &= \sum_{i \in I} \langle (\theta \Lambda_i)^* (\theta \Lambda_i) x, y \rangle_{\mathcal{B}} \\ &= \langle \sum_{i \in I} (\theta \Lambda_i)^* (\theta \Lambda_i) x, y \rangle_{\mathcal{B}} \\ &= \langle S_{\mathcal{B}} x, y \rangle_{\mathcal{B}}, \end{aligned}$$

which completes the proof. ■

3. Tensor Product

Suppose that \mathcal{A} and \mathcal{B} are C^* -algebras and take $\mathcal{A} \otimes \mathcal{B}$ as the completion of $\mathcal{A} \otimes_{alg} \mathcal{B}$ with the spatial norm. $\mathcal{A} \otimes \mathcal{B}$ is the spatial tensor product of \mathcal{A} and \mathcal{B} . Also suppose that \mathcal{H} is a Hilbert \mathcal{A} -module and \mathcal{K} is a Hilbert \mathcal{B} -module. We want to define $\mathcal{H} \otimes \mathcal{K}$ as a Hilbert $(\mathcal{A} \otimes \mathcal{B})$ -module. Start by forming the algebraic tensor product $\mathcal{H} \otimes_{alg} \mathcal{K}$ of the vector spaces \mathcal{H}, \mathcal{K} (over \mathbb{C}). This is a left module over $(\mathcal{A} \otimes_{alg} \mathcal{B})$ (the module action being given by $(a \otimes b)(x \otimes y) = ax \otimes by$ ($a \in \mathcal{A}, b \in \mathcal{B}, x \in \mathcal{H}, y \in \mathcal{K}$)). For $(x_1, x_2 \in \mathcal{H}, y_1, y_2 \in \mathcal{K})$, we define $\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_1, x_2 \rangle \otimes \langle y_1, y_2 \rangle$. We also know that for $z = \sum_{i=1}^n x_i \otimes y_i$ in $\mathcal{H} \otimes_{alg} \mathcal{K}$ we have $\langle z, z \rangle = \sum_{i,j} \langle x_i, x_j \rangle \otimes \langle y_i, y_j \rangle \geq 0$ and $\langle z, z \rangle = 0$ iff $z = 0$. This extends by linearity to an $(\mathcal{A} \otimes_{alg} \mathcal{B})$ -valued sesquilinear form on $\mathcal{H} \otimes_{alg} \mathcal{K}$, which makes $\mathcal{H} \otimes_{alg} \mathcal{K}$ into a semi-inner-product module over the pre- C^* -algebra $(\mathcal{A} \otimes_{alg} \mathcal{B})$. The semi-inner-product on $\mathcal{H} \otimes_{alg} \mathcal{K}$ is actually an inner product, see [10]. Then $\mathcal{H} \otimes_{alg} \mathcal{K}$ is an inner-product module over the pre- C^* -algebra $(\mathcal{A} \otimes_{alg} \mathcal{B})$, and we can perform the double completion discussed in chapter 1 of [10] to conclude that the completion $\mathcal{H} \otimes \mathcal{K}$ of $\mathcal{H} \otimes_{alg} \mathcal{K}$ is a Hilbert $(\mathcal{A} \otimes \mathcal{B})$ -module. We call $\mathcal{H} \otimes \mathcal{K}$ the exterior tensor product of \mathcal{H} and \mathcal{K} . With \mathcal{H}, \mathcal{K} as above, we wish to investigate the adjointable operators on $\mathcal{H} \otimes \mathcal{K}$. Suppose that $S \in L(\mathcal{H})$ and $T \in L(\mathcal{K})$. We define a linear operator $S \otimes T$ on $\mathcal{H} \otimes \mathcal{K}$ by $S \otimes T(x \otimes y) = Sx \otimes Ty$ ($x \in \mathcal{H}, y \in \mathcal{K}$). It is a routine verification that $S^* \otimes T^*$ is the adjoint of $S \otimes T$. In fact, $S \otimes T \in L(\mathcal{H} \otimes \mathcal{K})$. For more details, see [5, 10]. We note that if $a \in \mathcal{A}^+$ and $b \in \mathcal{B}^+$, then $a \otimes b \in (\mathcal{A} \otimes \mathcal{B})^+$. Plainly if a and b are Hermitian elements of \mathcal{A} and $a \geq b$, then we have $a \otimes x \geq b \otimes x$ for every positive element x of \mathcal{B} .

Theorem 3.1 Let \mathcal{H} and \mathcal{K} be two Hilbert C^* -modules over unitary C^* -algebras \mathcal{A} and \mathcal{B} , respectively. Also, let $\{\Lambda_i\}_{i \in I}$ and $\{\Gamma_j\}_{j \in J}$ be two operator frame for \mathcal{H} and \mathcal{K} with frame operator S_Λ and S_Γ and operator frame bounds (A, B) and (C, D) , respectively. Then $\{\Lambda_i \otimes \Gamma_j\}_{i \in I, j \in J}$ is an operator frame for the Hilbert $\mathcal{A} \otimes \mathcal{B}$ -module, $\mathcal{H} \otimes \mathcal{K}$ with frame operator $S_\Lambda \otimes S_\Gamma$ and lower and upper operator frame bounds AC and BD , respectively.

Proof. By the definition of operator frame $\{\Lambda_i\}_{i \in I}$ and $\{\Gamma_j\}_{j \in J}$, we have

$$\begin{aligned} A\langle x, x \rangle &\leq \sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle \leq B\langle x, x \rangle, \forall x \in \mathcal{H}, \\ C\langle y, y \rangle &\leq \sum_{j \in J} \langle \Gamma_j y, \Gamma_j y \rangle \leq D\langle y, y \rangle, \forall y \in \mathcal{K}. \end{aligned}$$

Therefore,

$$\begin{aligned} (A\langle x, x \rangle) \otimes (C\langle y, y \rangle) &\leq \sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle \otimes \sum_{j \in J} \langle \Gamma_j y, \Gamma_j y \rangle \\ &\leq (B\langle x, x \rangle) \otimes (D\langle y, y \rangle), \forall x \in \mathcal{H}, \forall y \in \mathcal{K}. \end{aligned}$$

Then, we have

$$\begin{aligned} (AC)(\langle x, x \rangle \otimes \langle y, y \rangle) &\leq \sum_{i \in I, j \in J} \langle \Lambda_i x, \Lambda_i x \rangle \otimes \langle \Gamma_j y, \Gamma_j y \rangle \\ &\leq (BD)(\langle x, x \rangle \otimes \langle y, y \rangle), \forall x \in \mathcal{H}, \forall y \in \mathcal{K}. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} (AC)\langle x \otimes y, x \otimes y \rangle &\leq \sum_{i \in I, j \in J} \langle \Lambda_i x \otimes \Gamma_j y, \Lambda_i x \otimes \Gamma_j y \rangle \\ &\leq (BD)\langle x \otimes y, x \otimes y \rangle, \forall x \in \mathcal{H}, \forall y \in \mathcal{K}. \end{aligned}$$

Then, for all $x \otimes y \in \mathcal{H} \otimes \mathcal{K}$, we have

$$\begin{aligned} (AC)\langle x \otimes y, x \otimes y \rangle &\leq \sum_{i \in I, j \in J} \langle (\Lambda_i \otimes \Gamma_j)(x \otimes y), (\Lambda_i \otimes \Gamma_j)(x \otimes y) \rangle \\ &\leq (BD)\langle x \otimes y, x \otimes y \rangle. \end{aligned}$$

The last inequality is satisfied for every finite sum of elements in $\mathcal{H} \otimes_{alg} \mathcal{K}$ and then it is satisfied for all $z \in \mathcal{H} \otimes \mathcal{K}$. It shows that $\{\Lambda_i \otimes \Gamma_j\}_{i \in I, j \in J}$ is an operator frame for Hilbert $\mathcal{A} \otimes \mathcal{B}$ -module $\mathcal{H} \otimes \mathcal{K}$ with lower and upper operator frame bounds AC and BD , respectively. By the definition of frame operator S_Λ and S_Γ , we have

$$S_\Lambda x = \sum_{i \in I} \Lambda_i^* \Lambda_i x, \forall x \in \mathcal{H} \text{ and } S_\Gamma y = \sum_{j \in J} \Gamma_j^* \Gamma_j y, \forall y \in \mathcal{K}.$$

Therefore,

$$\begin{aligned} (S_\Lambda \otimes S_\Gamma)(x \otimes y) &= S_\Lambda x \otimes S_\Gamma y \\ &= \sum_{i \in I} \Lambda_i^* \Lambda_i x \otimes \sum_{j \in J} \Gamma_j^* \Gamma_j y \\ &= \sum_{i \in I, j \in J} \Lambda_i^* \Lambda_i x \otimes \Gamma_j^* \Gamma_j y \\ &= \sum_{i \in I, j \in J} (\Lambda_i^* \otimes \Gamma_j^*)(\Lambda_i x \otimes \Gamma_j y) \\ &= \sum_{i \in I, j \in J} (\Lambda_i^* \otimes \Gamma_j^*)(\Lambda_i \otimes \Gamma_j)(x \otimes y) \\ &= \sum_{i \in I, j \in J} (\Lambda_i \otimes \Gamma_j)^*(\Lambda_i \otimes \Gamma_j)(x \otimes y). \end{aligned}$$

Now, by the uniqueness of frame operator, the last expression is equal to $S_{\Lambda \otimes \Gamma}(x \otimes y)$. Consequently, we have $(S_\Lambda \otimes S_\Gamma)(x \otimes y) = S_{\Lambda \otimes \Gamma}(x \otimes y)$. The last equality is satisfied for every finite sum of elements in $\mathcal{H} \otimes_{alg} \mathcal{K}$ and then it is satisfied for all $z \in \mathcal{H} \otimes \mathcal{K}$. It shows that $(S_\Lambda \otimes S_\Gamma)(z) = S_{\Lambda \otimes \Gamma}(z)$. So $S_{\Lambda \otimes \Gamma} = S_\Lambda \otimes S_\Gamma$. ■

Theorem 3.2 If $Q \in L(\mathcal{H})$ is an invertible \mathcal{A} -linear map and $\{\Lambda_i\}_{i \in I}$ is an operator frame for $\mathcal{H} \otimes \mathcal{K}$ with lower and upper operator frame bounds A and B respectively and frame operator S , then $\{\Lambda_i(Q^* \otimes I)\}_{i \in I}$ is an operator frame for $\mathcal{H} \otimes \mathcal{K}$ with lower and upper operator frame bounds $\|Q^{*-1}\|^{-2}A$ and $\|Q\|^2B$ respectively and frame operator $(Q \otimes I)S(Q^* \otimes I)$.

Proof. Since $Q \in L(\mathcal{H})$, $Q \otimes I \in L(\mathcal{H} \otimes \mathcal{K})$ with inverse $Q^{-1} \otimes I$. It is obvious that $Q \otimes I$ is $\mathcal{A} \otimes \mathcal{B}$ -linear and adjointable with adjoint $Q^* \otimes I$. An easy calculation shows

that for every elementary tensor $x \otimes y$,

$$\begin{aligned} \|(Q \otimes I)(x \otimes y)\|^2 &= \|Q(x) \otimes y\|^2 \\ &= \|Q(x)\|^2 \|y\|^2 \\ &\leq \|Q\|^2 \|x\|^2 \|y\|^2 \\ &= \|Q\|^2 \|x \otimes y\|^2. \end{aligned}$$

So, $Q \otimes I$ is bounded and it can be extended to $\mathcal{H} \otimes \mathcal{K}$. Similarly for $Q^* \otimes I$. Hence $Q \otimes I$ is $\mathcal{A} \otimes \mathcal{B}$ -linear, adjointable with adjoint $Q^* \otimes I$. Hence, for every $z \in \mathcal{H} \otimes \mathcal{K}$, we have

$$\|Q^{*-1}\|^{-1} \cdot |z| \leq |(Q^* \otimes I)z| \leq \|Q\| \cdot |z|.$$

By the definition of operator frames, we have

$$A\langle z, z \rangle \leq \sum_{i \in I} \langle \Lambda_i z, \Lambda_i z \rangle \leq B\langle z, z \rangle.$$

Then,

$$\begin{aligned} A\langle (Q^* \otimes I)z, (Q^* \otimes I)z \rangle &\leq \sum_{i \in I} \langle \Lambda_i (Q^* \otimes I)z, \Lambda_i (Q^* \otimes I)z \rangle \\ &\leq B\langle (Q^* \otimes I)z, (Q^* \otimes I)z \rangle. \end{aligned}$$

So,

$$\|Q^{*-1}\|^{-2} A\langle z, z \rangle \leq \sum_{i \in I} \langle \Lambda_i (Q^* \otimes I)z, \Lambda_i (Q^* \otimes I)z \rangle \leq \|Q\|^2 B\langle z, z \rangle.$$

Now,

$$\begin{aligned} (Q \otimes I)S(Q^* \otimes I) &= (Q \otimes I)\left(\sum_{i \in I} \Lambda_i^* \Lambda_i\right)(Q^* \otimes I) \\ &= \sum_{i \in I} (Q \otimes I)\Lambda_i^* \Lambda_i(Q^* \otimes I) \\ &= \sum_{i \in I} (\Lambda_i(Q^* \otimes I))^* \Lambda_i(Q^* \otimes I), \end{aligned}$$

which completes the proof. ■

Theorem 3.3 If $Q \in L(\mathcal{K})$ is an invertible \mathcal{B} -linear map and $\{\Lambda_i\}_{i \in I}$ is an operator frame for $\mathcal{H} \otimes \mathcal{K}$ with lower and upper operator frame bounds A and B respectively and frame operator S , then $\{\Lambda_i(I \otimes Q^*)\}_{i \in I}$ is an operator frame for $\mathcal{H} \otimes \mathcal{K}$ with lower and upper operator frame bounds $\|Q^{*-1}\|^{-2}A$ and $\|Q\|^2B$ respectively and frame operator $(I \otimes Q)S(I \otimes Q^*)$.

Acknowledgments

The authors would like to thank the anonymous reviewers for careful reading of the manuscript and giving useful comments, which will help to improve the paper.

References

- [1] A. Alijani, M. A. Dehghan, $*$ -frames in Hilbert C^* -modules, U.P.B. Sci. Bull. (Ser. A). 73 (4) (2011), 89-106.
- [2] Lj. Arambaić, On frames for countably generated Hilbert C^* -modules, Proc. Amer. Math. Soc. 135 (2007), 469-478.
- [3] O. Christensen, An Introduction to Frames and Riesz Bases, Birkhäuser, 2016.
- [4] J. B. Conway, A Course in Operator Theory, American Mathematical Society, 2000.
- [5] F. R. Davidson, C^* -algebra by example, Fields Ins. Monog, 1996.
- [6] R. J. Duffin, A. C. Schaeffer, A class of nonharmonic fourier series, Trans. Amer. Math. Soc. 72 (1952), 341-366.
- [7] I. Kaplansky, Modules over operator algebras, Amer. J. Math. 75 (1953), 839-858.
- [8] A. Khosravi, B. Khosravi, Frames and bases in tensor products of Hilbert spaces and Hilbert C^* -modules, Proc. Indian Acad. Sci. Math. Sci. 117 (2007), 1-12.
- [9] A. Khosravi, B. Khosravi, Fusion frames and g-frames in Hilbert C^* -modules, Int. J. Wavelets Multiresolut. Inf. Process. 6 (2008), 433-446.
- [10] E. C. Lance, Hilbert C^* -modules: A Toolkit for Operator Algebraists, London Math. Soc. Lecture Series, 1995.
- [11] C. Y. Li, H. X. Cao, Operator Frames for $B(\mathcal{H})$, Wavelet Analysis and Applications, Applications of Numerical Harmonic Analysis, Springer, 2006.
- [12] W. Paschke, Inner product modules over B^* -algebras, Trans. Amer. Math. Soc. 182 (1973), 443-468.