

C-class functions on common fixed point theorems for weak contraction mapping of integral type in modular spaces

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Abstract. In this paper, we use the concept of *C*-class functions introduced by Ansari [4] to prove the existence and uniqueness of common fixed point for self-mappings in modular spaces of integral inequality. Our results extended and generalized previous known results in this direction.

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1. Introduction and preliminaries

Let (X, d) be a complete metric space and $f : X \rightarrow X$ be a self-mapping on X . Suppose that $F_f = \{x \in X : F(x) = x\}$ is the set of fixed points of f . The classical Banach's fixed point theorem is established in Banach [6] by using the following contractive definition: there exists $k \in [0, 1)$ such that

$$d(fx, fy) \leq kd(x, y). \quad (1)$$

for all $x, y \in X$. Rhoades [23] proved the following theorem:

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Theorem 1.1 Let T be a mapping from a complete metric space (X, d) into itself satisfying

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)) \quad (2)$$

for all $x, y \in X$, where $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous and nondecreasing such that ϕ is positive on $\mathbb{R}^+ \setminus \{0\}$, $\phi(0) = 0$ and $\lim_{t \rightarrow \infty} \phi(t) = \infty$. Then T has a unique fixed point in X .

We note that (1) is a special case of (2) by taking $\phi(t) = (1 - k)t$ for $0 \leq k < 1$.

Branciari [9] and Rhoades [24] proved the following theorems for contraction mapping and weakly contractive mapping of integral type, respectively, which are generalization of the Banach fixed point theorem.

Theorem 1.2 [9] Let T be a mapping from complete metric space (X, d) into itself satisfying

$$\int_0^{d(Tx, Ty)} \varphi(t) dt \leq k \int_0^{d(x, y)} \varphi(t) dt$$

for all $x, y \in X$, where $k \in [0, 1)$ is a constant and $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a Lebesgue-integrable mapping which is summable, nonnegative, and for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t) dt > 0$. Then T has a unique fixed point $z \in X$ such that $\lim_{n \rightarrow \infty} T^n x = z$ for all $x \in X$.

Theorem 1.3 [24] Let T be a mapping from complete metric space (X, d) into itself satisfying

$$\int_0^{d(Tx, Ty)} \varphi(t) dt \leq k \int_0^{m(x, y)} \varphi(t) dt$$

for all $x, y \in X$, where

$$m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}$$

and

$$\int_0^{d(Tx, Ty)} \varphi(t) dt \leq k \int_0^{M(x, y)} \varphi(t) dt, \quad \forall x, y \in X,$$

with $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$, where $k \in [0, 1)$ and $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ in both cases is as defined in Theorem 1.1. Then T has a unique fixed point $z \in X$ such that $\lim_{n \rightarrow \infty} T^n x = z$ for each $x \in X$.

Afterward, many authors extended this work to more general contractive conditions. The works noted in [1, 3, 12, 20, 25]. The following definition is taken from Berinde [7].

Definition 1.4 A single-valued mapping f on a metric space X is called a weak contraction or (δ, L) -weak contraction if and only if there exists two constants $L \geq 0$ and

$\delta \in [0, 1)$ such that

$$d(fx, fy) \leq \delta d(x, y) + L(d(y, fx))$$

for all $x, y \in X$

We shall employ the following definitions in the sequel to obtain our results.

Definition 1.5 [1] A function $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is called a comparison function if it satisfies the following conditions

- (i) Ψ is monotone increasing, $\Psi(t) < t$ for some $t > 0$,
- (ii) $\Psi(0) = 0$,
- (iii) $\lim_{n \rightarrow \infty} \Psi^n(t) = 0$ for all $t \geq 0$.

Definition 1.6 [5, 15] The function $\psi : [0, \infty) \rightarrow [0, \infty)$ is called an altering distance function if and only if the following properties are satisfied

1. ψ is continuous and non-decreasing.
2. $\psi(t) = 0 \Leftrightarrow t = 0$.

Afterwards, the authors in [16, 17] continued the study of the existence of fixed points and common fixed points for several contractive mappings of integral type in complete metric spaces. In 2010, Olatinwo [17] generalized the result of Branciari and established the following fixed point theorems.

Theorem 1.7 [17] Let (X, d) be a complete metric space and $f : X \rightarrow X$ satisfies a (L, ψ) -weak contraction of integral type

$$\int_0^{d(fx, fy)} \varphi(t)dv(t) \leq L \left(\int_0^{d(x, fx)} \varphi(t)dv(t) \right)^r \left(\int_0^{d(y, fx)} \varphi(t)dv(t) \right) + \psi \left(\int_0^{d(x, y)} \varphi(t)dv(t) \right)$$

for all $x, y \in X$, where $L, r \geq 0$. Suppose that

- (i) $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous comparison function and $v : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a monotone increasing functions,
- (ii) $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a Lebesgue-Stieltjes integrable mapping which is summable, non-negative and for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)dv(t) > 0$. Then f has a unique fixed point $x^* \in X$ such that $\lim_{n \rightarrow \infty} f^n x = x^*$ for each $x \in X$.

Theorem 1.8 [17] Let (X, d) be a complete metric space and $f : X \rightarrow X$ satisfies a (ϕ, ψ) -weak contraction of integral type

$$\int_0^{d(fx, fy)} \varphi(t)dv(t) \leq \phi \left(\int_0^{d(x, fx)} \varphi(t)dv(t) \right) \left(\int_0^{d(y, fx)} \varphi(t)dv(t) \right) + \psi \left(\int_0^{d(x, y)} \varphi(t)dv(t) \right)$$

for all $x, y \in X$. Suppose that

- (i) $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous comparison function,
- (ii) $\phi, v : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are monotone increasing functions such that ϕ is continuous and $\Phi(0) = 0$,
- (iii) $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a Lebesgue-Stieltjes integrable mapping which is summable, non-

negative and for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)dv(t) > 0$ and $v : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is also an increasing function. Then f has a unique fixed point $x^* \in X$ such that $\lim_{n \rightarrow \infty} f^n x = x^*$ for each $x \in X$.

In 2012, Aydi [5] presented the following definition and fixed point theorem for contractive condition of integral type involving altering distances as the following.

Definition 1.9 [5] $u : [0, +\infty) \rightarrow [0, +\infty)$ is subadditive on each $[a, b] \subset [0, +\infty)$ if

$$\int_0^{a+b} u(t)dt \leq \int_0^a u(t)dt + \int_0^b u(t)dt.$$

Theorem 1.10 [5] Let (X, d) be a complete metric space and $f : X \rightarrow X$ such that

$$\psi\left(\int_0^{d(fx, fy)} u(t)dt\right) \leq \psi(\theta(x, y)) - \Phi(\theta(x, y))$$

for each $x, y \in X$ with non-negative real numbers α, β, γ such that $2\alpha + \beta + 2\gamma < 1$, where ψ and Φ are altering distances, and

$$\theta(x, y) = \alpha \int_0^{d(x, fx) + d(y, fy)} u(t)dt + \beta \int_0^{d(x, y)} u(t)dt + \gamma \int_0^{\max\{d(x, fy), d(y, fx)\}} u(t)dt,$$

where $u(t) : [0, +\infty) \rightarrow [0, +\infty)$ be a Lebesgue-integrable mapping which is summable, subadditive on each subset of \mathbb{R}^+ , non-negative such that for each $\epsilon > 0$, $\int_0^\epsilon u(t)dt > 0$.

Then f has a unique fixed point in X .

The notion of modular space, as a generalization of a metric space, was introduced by Nakano [14] in 1950 and redefined and generalized by Musielak and Orlicz [13] in 1959. In the existence of fixed point theory and Banach contraction principle occupies a prominent place in the study of metric spaces, it become a most popular tool in solving problems in mathematical analysis and construct methods in mathematics to solve problems in applied mathematics and sciences.

In this article we study and prove the existence of fixed point theorems for mappings in modular spaces. Now, we begin with a brief recollection of basic concepts and facts in modular spaces and modular metric spaces (see [2, 8, 10, 11, 21, 22]).

Definition 1.11 [22] Let X be an arbitrary vector space over $K = \mathbb{R}$ or \mathbb{C} .

a) A functional $\rho : X \rightarrow [0, \infty]$ is called modular if:

- (i) $\rho(x) = 0$ iff $x = 0$;
- (ii) $\rho(\alpha x) = \rho(x)$ for $\alpha \in K$ with $|\alpha| = 1$ for all $x \in X$;
- (iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha, \beta \geq 0$, $\alpha + \beta = 1$ for all $x, y \in X$.

If (iii) is replaced by:

- (iv) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ for $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ for all $x, y \in X$, then the modular is called convex modular.

b) If ρ a modular in X , then the set $X_\rho = \{x \in X : \rho(\alpha x) \rightarrow 0 \text{ as } \alpha \rightarrow 0\}$ is called a modular space.

Remark 1 [22] Note that ρ is an increasing function as the following, suppose $0 < a < b$, then, property (iii) with $y = 0$ shows that $\rho(ax) = \rho(\frac{a}{b}(bx)) \leq \rho(bx)$.

Definition 1.12 [22] Let X_ρ be a modular space.

- a) A sequence $(x_n)_{n \in \mathbb{N}}$ in X_ρ is said to be:
 - (i) ρ -convergent to x if $\rho(x_n - x) \rightarrow 0$ as $n \rightarrow \infty$.
 - (ii) ρ -Cauchy if $\rho(x_n - x_m) \rightarrow 0$ as $n, m \rightarrow \infty$.
- b) X_ρ is ρ -complete if every ρ -Cauchy sequence is ρ -convergent.
- c) A subset $B \subset X_\rho$ is said to be ρ -closed if for any sequence $(x_n)_{n \in \mathbb{N}} \subset B$ and $x_n \rightarrow x$ we have $x \in B$.
- d) A subset $B \subset X_\rho$ is called ρ -bounded if $\delta_\rho(B) = \sup \rho(x - y) < \infty$ for all $x, y \in B$, where $\delta_\rho(B)$ is called the ρ -diameter of B .
- e) ρ has the Fatou property, if $\rho(x - y) \leq \liminf \rho(x_n - y_n)$ whenever $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$.
- f) ρ is said to satisfy the Δ_2 -condition if $\rho(2x_n) \rightarrow 0$, whenever $(x_n) \rightarrow 0$ as $n \rightarrow \infty$.

Remark 2 [8] Note that since ρ does not satisfy a priori the triangle inequality, we cannot expect that if $\{x_n\}$ and $\{y_n\}$ are ρ -convergent, respectively, to x and y then $\{x_n + y_n\}$ is ρ -convergent to $\{x + y\}$, neither that a ρ -convergent sequence is ρ -Cauchy.

Definition 1.13 [22] Let X_ρ be a modular space, where ρ satisfies the Δ_2 -condition. Two self-mappings T and h of X are called ρ -compatible if $\rho(Thx_n - hTx_n) \rightarrow 0$, whenever $(x_n)_{n \in \mathbb{N}}$ is a sequence in X_ρ such that $hx_n \rightarrow z$ and $Tx_n \rightarrow z$ for some point $z \in X_\rho$.

In 2014, the concept of C -class functions was introduced by Ansari [4]. By using this concept, some authors generalize many fixed point theorems in the literature (for example, see [18, 19]).

Definition 1.14 [4] Let $F : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ be a continuous mapping, it is called a C -class function if it satisfies the following conditions:

- (F₁) $F(s, t) \leq s$, for all $(s, t) \in \mathbb{R}_+^2$;
- (F₂) $F(s, t) = s$ implies that $s = 0$, or $t = 0$, for all $(s, t) \in \mathbb{R}_+^2$.

Note that for some F we have $F(0, 0) = 0$. We denote C -class functions as \mathcal{C} .

Example 1.15 [4] The following functions $F : [0, \infty)^2 \rightarrow \mathbb{R}$ are elements of \mathcal{C} , for all $s, t \in [0, \infty)$:

- (1) $F(s, t) = s - t, F(s, t) = s \Rightarrow t = 0$;
- (2) $F(s, t) = ms, 0 < m < 1, F(s, t) = s \Rightarrow s = 0$;
- (3) $F(s, t) = \frac{s}{(1+t)^r}, r \in (0, \infty), F(s, t) = s \Rightarrow s = 0$ or $t = 0$;
- (4) $F(s, t) = \log(t + a^s)/(1 + t), a > 1, F(s, t) = s \Rightarrow s = 0$ or $t = 0$;
- (5) $F(s, t) = \ln(1 + a^s)/2, a > e, F(s, 1) = s \Rightarrow s = 0$;
- (6) $F(s, t) = (s + l)^{1/(1+t)^r} - l, l > 1, r \in (0, \infty), F(s, t) = s \Rightarrow t = 0$;
- (7) $F(s, t) = s \log_{t+a} a, a > 1, F(s, t) = s \Rightarrow s = 0$ or $t = 0$;
- (8) $F(s, t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t}), F(s, t) = s \Rightarrow t = 0$;
- (9) $F(s, t) = s\beta(s), \beta : [0, \infty) \rightarrow [0, 1)$, and is continuous, $F(s, t) = s \Rightarrow s = 0$;
- (10) $F(s, t) = s - \frac{t}{k+t}, F(s, t) = s \Rightarrow t = 0$;
- (11) $F(s, t) = s - \varphi(s), F(s, t) = s \Rightarrow s = 0$, here $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;

(12) $F(s, t) = sh(s, t), F(s, t) = s \Rightarrow s = 0$, here $h : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is a continuous function such that $h(t, s) < 1$ for all $t, s > 0$;

(13) $F(s, t) = s - \frac{(2+t)}{1+t}t, F(s, t) = s \Rightarrow t = 0$;

(14) $F(s, t) = \sqrt[n]{\ln(1+s^n)}, F(s, t) = s \Rightarrow s = 0$;

(15) $F(s, t) = \phi(s), F(s, t) = s \Rightarrow s = 0$, where $\phi : [0, \infty) \rightarrow [0, \infty)$ is a upper semicontinuous function such that $\phi(0) = 0$, and $\phi(t) < t$ for $t > 0$,

(16) $F(s, t) = \frac{s}{(1+s)^r}; r \in (0, \infty), F(s, t) = s \Rightarrow s = 0$;

(17) $F(s, t) = \vartheta(s); \vartheta : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is a generalized Mizoguchi-Takahashi type function $F(s, t) = s \Rightarrow s = 0$;

(18) $F(s, t) = \frac{s}{\Gamma(1/2)} \int_0^\infty \frac{e^{-x}}{\sqrt{x+t}} dx$, where Γ is the Euler Gamma function.

Let Φ_u denote the set of all functions $\phi : [0, +\infty) \rightarrow [0, +\infty)$, that satisfy the following conditions:

- (1) ϕ is lower semi-continuous on $[0, +\infty)$;
- (2) $\phi(0) \geq 0$;
- (3) $\phi(s) > 0$ for each $s > 0$.

Let Ψ_c denote the set of all functions $\psi : [0, \infty) \rightarrow [0, \infty)$, that satisfy the following conditions:

- (1) ψ is continuous and strictly increasing;
- (2) $\psi(t) = 0$ iff $t = 0$.

2. Main results

Now, we study the existence of a common fixed point for ρ -compatible mappings satisfying an $F(\phi, \psi)$ -weak contraction of integral type in modular spaces

Theorem 2.1 Let X_ρ be a ρ -complete modular space, where ρ satisfies the Δ_2 -condition. Suppose $\psi \in \Phi_u$, $F \in \mathcal{C}$, and $T, h : X_\rho \rightarrow X_\rho$ are two ρ -compatible mappings such that $T(X_\rho) \subseteq h(X_\rho)$ and

$$\int_0^{\rho(Tx-Ty)} \varphi(t) d\nu(t) \leq F \left(\int_0^{\rho(hx-hy)} \varphi(t) d\nu(t), \psi \left(\int_0^{\rho(hx-hy)} \varphi(t) d\nu(t) \right) \right) + \phi \left(\int_0^{\rho(hx-Tx)} \varphi(t) d\nu(t) \right) \left(\int_0^{\rho(hy-Tx)} \varphi(t) d\nu(t) \right) \quad (3)$$

for all $x, y \in X_\rho$, and $\nu, \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are monotone increasing functions such that $\phi(0) = 0$. Let $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a Lebesgue–Stieltjes integrable mapping which is summable and nonnegative such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t) d\nu(t) > 0$. If one of h or T is continuous, then there exists a unique common fixed point of h and T .

Proof. Let x_0 be an arbitrary point of X_ρ and generate inductively the sequence

$(Tx_n)_{n \in \mathbb{N}}$ as $Tx_n = hx_{n+1}$. For each integer $n \geq 1$, (3) shows that

$$\begin{aligned} \int_0^{\rho(Tx_{n-1}-Tx_n)} \varphi(t) d\nu(t) &\leq F \left(\int_0^{\rho(hx_{n-1}-hx_n)} \varphi(t) d\nu(t), \psi \left(\int_0^{\rho(hx_{n-1}-hx_n)} \varphi(t) d\nu(t) \right) \right) \\ &\quad + \phi \left(\int_0^{\rho(hx_{n-1}-Tx_{n-1})} \varphi(t) d\nu(t) \right) \left(\int_0^{\rho(hx_n-Tx_{n-1})} \varphi(t) d\nu(t) \right) \\ &\leq F \left(\int_0^{\rho(Tx_{n-2}-Tx_{n-1})} \varphi(t) d\nu(t), \psi \left(\int_0^{\rho(Tx_{n-2}-Tx_{n-1})} \varphi(t) d\nu(t) \right) \right) \\ &\quad + \phi \left(\int_0^{\rho(Tx_{n-2}-Tx_{n-1})} \varphi(t) d\nu(t) \right) \left(\int_0^{\rho(Tx_{n-1}-Tx_{n-1})} \varphi(t) d\nu(t) \right) \\ &\leq F \left(\int_0^{\rho(Tx_{n-2}-Tx_{n-1})} \varphi(t) d\nu(t), \psi \left(\int_0^{\rho(Tx_{n-2}-Tx_{n-1})} \varphi(t) d\nu(t) \right) \right) \\ &\leq \int_0^{\rho(Tx_{n-2}-Tx_{n-1})} \varphi(t) d\nu(t). \end{aligned}$$

This leads to

$$\int_0^{\rho(Tx_{n-1}-Tx_n)} \varphi(t) d\nu(t) \rightarrow r \geq 0. \tag{4}$$

Taking the limit in (4) as $n \rightarrow \infty$ yields $r \leq F(r, \psi(r))$. Thus, $\psi(r) = 0$ or $r = 0$. Hence,

$$\lim_{n \rightarrow \infty} \int_0^{\rho(Tx_{n-1}-Tx_n)} \varphi(t) d\nu(t) = 0.$$

Therefore,

$$\lim_{n \rightarrow \infty} \rho(Tx_{n-1} - Tx_n) = 0. \tag{5}$$

Now, we show that $(Tx_n)_{n \in \mathbb{N}}$ is ρ -Cauchy. If not, then there exists an $\epsilon > 0$ and two sequences of integers $\{n(s)\}, \{m(s)\}$ with $n(s) > m(s) \geq s$ such that

$$\rho(Tx_{n(s)} - Tx_{m(s)}) \geq \epsilon \tag{6}$$

for $s = 1, 2, \dots$. We can assume that

$$\rho(Tx_{n(s)-1} - Tx_{m(s)}) < \epsilon. \quad (7)$$

To prove inequality (7), we can see [18]. Again, from (3), we obtain

$$\begin{aligned} \int_0^{\rho(Tx_{m(s)} - Tx_{n(s)})} \varphi(t) d\nu(t) &\leq F \left(\int_0^{\rho(hx_{m(s)} - hx_{n(s)})} \varphi(t) d\nu(t), \psi \left(\int_0^{\rho(hx_{m(s)} - hx_{n(s)})} \varphi(t) d\nu(t) \right) \right) \\ &\quad + \phi \left(\int_0^{\rho(hx_{m(s)} - Tx_{m(s)})} \varphi(t) d\nu(t) \right) \left(\int_0^{\rho(hx_{n(s)} - Tx_{m(s)})} \varphi(t) d\nu(t) \right) \\ &= F \left(\int_0^{\rho(Tx_{m(s)-1} - Tx_{n(s)-1})} \varphi(t) d\nu(t), \psi \left(\int_0^{\rho(Tx_{m(s)-1} - Tx_{n(s)-1})} \varphi(t) d\nu(t) \right) \right) \\ &\quad + \phi \left(\int_0^{\rho(Tx_{m(s)-1} - Tx_{m(s)})} \varphi(t) d\nu(t) \right) \left(\int_0^{\rho(Tx_{n(s)-1} - Tx_{m(s)})} \varphi(t) d\nu(t) \right). \end{aligned} \quad (8)$$

Moreover,

$$\begin{aligned} \rho(Tx_{m(s)-1} - Tx_{n(s)-1}) &= \rho\left(\frac{2}{2}(Tx_{m(s)-1} - Tx_{m(s)}) + \rho\left(\frac{2}{2}(Tx_{m(s)} - Tx_{n(s)-1})\right)\right) \\ &\leq \rho(2(Tx_{m(s)-1} - Tx_{m(s)})) + \rho(2(Tx_{m(s)} - Tx_{n(s)-1})). \end{aligned} \quad (9)$$

Take the limit as $s \rightarrow \infty$ in (9) and using (5), (7) and using Δ_2 -condition, we have

$$\int_0^{\rho(Tx_{m(s)-1} - Tx_{n(s)-1})} \varphi(t) d\nu(t) \leq \int_0^{\epsilon} \varphi(t) d\nu(t). \quad (10)$$

In (8), taking the limit as $s \rightarrow \infty$ and using (6), (7) and (10), we have

$$\begin{aligned} \int_0^{\epsilon} \varphi(t) d\nu(t) &\leq \int_0^{\rho(Tx_{n(s)} - Tx_{m(s)})} \varphi(t) d\nu(t) \\ &\leq F \left(\int_0^{\rho(Tx_{n(s)-1} - Tx_{m(s)-1})} \varphi(t) d\nu(t), \psi \left(\int_0^{\rho(Tx_{n(s)-1} - Tx_{m(s)-1})} \varphi(t) d\nu(t) \right) \right) \\ &\leq F \left(\int_0^{\epsilon} \varphi(t) d\nu(t), \psi \left(\int_0^{\epsilon} \varphi(t) d\nu(t) \right) \right) < \int_0^{\epsilon} \varphi(t) d\nu(t). \end{aligned}$$

So, $\psi(\int_0^\epsilon \varphi(t)d\nu(t)) = 0$ or $\int_0^\epsilon \varphi(t)d\nu(t) = 0$. Thus $\int_0^\epsilon \varphi(t)d\nu(t) = 0$; that is, $\epsilon = 0$, which is a contradiction. Therefore, by Δ_2 -condition, $(Tx_n)_{n \in \mathbb{N}}$ is ρ -Cauchy. Since X_ρ is ρ -complete, then there exists $z \in X_\rho$ such that $\rho(Tx_n - z) \rightarrow 0$ as $n \rightarrow \infty$. If T is continuous, then $T^2x_n \rightarrow Tz$ and $Thx_n \rightarrow Tz$. Since $\rho(hTx_n - Thx_n) \rightarrow 0$, then $hTx_n \rightarrow Tz$ by ρ -compatibility. We now prove that z is a fixed point of T . If not, we have from (3) that

$$\int_0^{\rho(T^2x_n - Tx_n)} \varphi(t)d\nu(t) \leq F \left(\int_0^{\rho(hTx_n - hx_n)} \varphi(t)d\nu(t), \psi \left(\int_0^{\rho(hTx_n - hx_n)} \varphi(t)d\nu(t) \right) \right) + \phi \left(\int_0^{\rho(hTx_n - T^2x_n)} \varphi(t)d\nu(t) \right) \left(\int_0^{\rho(hx_n - T^2x_n)} \varphi(t)d\nu(t) \right).$$

Take the limit as $n \rightarrow \infty$. We obtain

$$\int_0^{\rho(Tz - z)} \varphi(t)d\nu(t) \leq F \left(\int_0^{\rho(Tz - z)} \varphi(t)d\nu(t), \psi \left(\int_0^{\rho(Tz - z)} \varphi(t)d\nu(t) \right) \right).$$

So, $\psi(\int_0^{\rho(Tz - z)} \varphi(t)d\nu(t)) = 0$ or $\int_0^{\rho(Tz - z)} \varphi(t)d\nu(t) = 0$. Thus, $\int_0^{\rho(Tz - z)} \varphi(t)d\nu(t) = 0$; that is, $\rho(Tz - z) = 0$. Therefore, $z = Tz$. Moreover, $T(X_\rho) \subseteq h(X_\rho)$. Hence, there exists a point $z_1 \in X_\rho$ such that

$$z = Tz = hz_1. \tag{11}$$

From (3), we obtain

$$\int_0^{\rho(T^2x_n - Tz_1)} \varphi(t)d\nu(t) \leq F \left(\int_0^{\rho(hTx_n - hz_1)} \varphi(t)d\nu(t), \psi \left(\int_0^{\rho(hTx_n - hz_1)} \varphi(t)d\nu(t) \right) \right) + \phi \left(\int_0^{\rho(hTx_n - T^2x_n)} \varphi(t)d\nu(t) \right) \left(\int_0^{\rho(hz_1 - T^2x_n)} \varphi(t)d\nu(t) \right),$$

as $n \rightarrow \infty$ and using (11), we obtain

$$\int_0^{\rho(z - Tz_1)} \varphi(t)d\nu(t) \leq F(0, \psi(0)) \leq 0. \tag{12}$$

It follows from (12) that a contradiction. Therefore, by properties of φ , we get $\int_0^{\rho(z - Tz_1)} \varphi(t)d\nu(t) = 0$, from which it follows that $\rho(z - Tz_1) = 0$ or $z = Tz_1 = hz_1$.

Also, $hz = hTz_1 = Thz_1 = Tz = z$ (see [18]). Therefore, z is a common fixed point of T and h . In addition, if one consider h to be continuous instead of T , then by similar argument as above, one can prove that $Tz = hz = z$. For uniqueness, suppose that ($w \neq z$) be another common fixed point of T and h then from (3), we get

$$\begin{aligned} \int_0^{\rho(c(z-w))} \varphi(t) d\nu(t) &= \int_0^{\rho(c(Tz-Tw))} \varphi(t) d\nu(t) \\ &\leq F \left(\int_0^{\rho(hz-hw)} \varphi(t) d\nu(t), \psi \left(\int_0^{\rho(hz-hw)} \varphi(t) d\nu(t) \right) \right) \\ &\quad + \phi \left(\int_0^{\rho(hz-Tz)} \varphi(t) d\nu(t) \right) \left(\int_0^{\rho(hw-Tz)} \varphi(t) d\nu(t) \right) \\ &\leq F \left(\int_0^{\rho(z-w)} \varphi(t) d\nu(t), \psi \left(\int_0^{\rho(z-w)} \varphi(t) d\nu(t) \right) \right). \end{aligned}$$

So, $\psi \left(\int_0^{\rho(z-w)} \varphi(t) d\nu(t) \right) = 0$ or $\int_0^{\rho(z-w)} \varphi(t) d\nu(t) = 0$. Thus, $\int_0^{\rho(z-w)} \varphi(t) d\nu(t) > 0$; that is, $\rho(z-w) = 0$. Leading to a contradiction again. Therefore, by the condition on φ , we get $\int_0^{\rho(z-w)} \varphi(t) d\nu(t) = 0$, from which it follows that $\rho(z-w) = 0$ or $z = w$. Hence, T and h have a unique common fixed point. ■

The following theorem is another version of Theorem 2.1, by adding the restrictions that $T, h : B \rightarrow B$, where B is a ρ -closed and ρ -bounded subset of X_ρ .

Theorem 2.2 Let X_ρ be a ρ -complete modular space, where ρ satisfies the Δ_2 -condition and B is a ρ -closed and ρ -bounded subset of X_ρ , $\psi \in \Phi_u$ and $F \in \mathcal{C}$. Suppose that $T, h : B \rightarrow B$ are ρ -compatible mappings such that $T(X_\rho) \subseteq h(X_\rho)$ and

$$\begin{aligned} \int_0^{\rho(Tx-Ty)} \varphi(t) d\nu(t) &\leq F \left(\int_0^{\rho(hx-hy)} \varphi(t) d\nu(t), \psi \left(\int_0^{\rho(hx-hy)} \varphi(t) d\nu(t) \right) \right) \\ &\quad + \phi \left(\int_0^{\rho(hx-Tx)} \varphi(t) d\nu(t) \right) \left(\int_0^{\rho(hy-Tx)} \varphi(t) d\nu(t) \right) \end{aligned}$$

for all $x, y \in X_\rho$, and $\nu, \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are monotone increasing functions such that $\phi(0) = 0$. Let $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a Lebesgue–Stieltjes integrable mapping which is summable and nonnegative such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t) d\nu(t) > 0$. If one of h or T is continuous, then there exists a unique common fixed point of h and T .

Proof. Let $x \in B$ and $m, n \in \mathbb{N}$. Then, we have

$$\begin{aligned} \int_0^{\rho(Tx_{n+m}-Tx_m)} \varphi(t)d\nu(t) &\leq F \left(\int_0^{\rho(hx_{n+m}-hx_m)} \varphi(t)d\nu(t), \psi \left(\int_0^{\rho(hx_{n+m}-hx_m)} \varphi(t)d\nu(t) \right) \right) \\ &\quad + \phi \left(\int_0^{\rho(hx_{n+m}-Tx_{n+m})} \varphi(t)d\nu(t) \right) \left(\int_0^{\rho(hx_m-Tx_{n+m})} \varphi(t)d\nu(t) \right) \\ &\leq F \left(\int_0^{\rho(Tx_{n+m-1}-Tx_{m-1})} \varphi(t)d\nu(t), \psi \left(\int_0^{\rho(Tx_{n+m-1}-Tx_{m-1})} \varphi(t)d\nu(t) \right) \right) \\ &\quad + \phi \left(\int_0^{\rho(c(Tx_{n+m-1}-Tx_{n+m}))} \varphi(t)d\nu(t) \right) \left(\int_0^{\rho(Tx_{m-1}-Tx_{n+m})} \varphi(t)d\nu(t) \right). \end{aligned}$$

Taking the limit as $n, m \rightarrow \infty$ and using (5), we have

$$\begin{aligned} &\lim_{n,m \rightarrow \infty} \int_0^{\rho(Tx_{n+m}-Tx_m)} \varphi(t)d\nu(t) \\ &\leq \lim_{n,m \rightarrow \infty} F \left(\int_0^{\rho(Tx_{n+m-1}-Tx_{m-1})} \varphi(t)d\nu(t), \psi \left(\int_0^{\rho(Tx_{n+m-1}-Tx_{m-1})} \varphi(t)d\nu(t) \right) \right) \\ &\leq \lim_{n,m \rightarrow \infty} \int_0^{\rho(Tx_{n+m-1}-Tx_{m-1})} \varphi(t)d\nu(t). \end{aligned}$$

Since B is ρ -bounded, then $\lim_{n,m \rightarrow \infty} \int_0^{\rho(Tx_{n+m}-Tx_m)} \varphi(t)d\nu(t) \rightarrow r \geq 0$. So with $n, m \rightarrow \infty$, we have $r \leq F(r, \psi(r))$. Hence, $\psi(r) = 0$ or $r = 0$, which implies that $\lim_{n,m \rightarrow \infty} \rho(Tx_{n+m} - Tx_m) = 0$. Therefore, by Δ_2 -condition, $(Tx_n)_{n \in \mathbb{N}}$ is ρ -Cauchy. Since X_ρ is ρ -complete, then there exists $z \in X_\rho$ such that $\rho(Tx_n - z) \rightarrow 0$ as $n \rightarrow \infty$. If T is continuous, then $T^2x_n \rightarrow Tz$ and $Thx_n \rightarrow Tz$. Since $\rho(hTx_n - Thx_n) \rightarrow 0$, then $hTx_n \rightarrow Tz$ by ρ -compatibility. We now prove that z is a fixed point of T . If not, we have from (3) that

$$\begin{aligned} \int_0^{\rho(T^2x_n-Tx_n)} \varphi(t)d\nu(t) &\leq F \left(\int_0^{\rho(hTx_n-hx_n)} \varphi(t)d\nu(t), \psi \left(\int_0^{\rho(hTx_n-hx_n)} \varphi(t)d\nu(t) \right) \right) \\ &\quad + \phi \left(\int_0^{\rho(hTx_n-T^2x_n)} \varphi(t)d\nu(t) \right) \left(\int_0^{\rho(hx_n-T^2x_n)} \varphi(t)d\nu(t) \right). \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, we get

$$\int_0^{\rho(c(Tz-z))} \varphi(t) d\nu(t) \leq F \left(\int_0^{\rho(Tz-z)} \varphi(t) d\nu(t), \psi \left(\int_0^{\rho(Tz-z)} \varphi(t) d\nu(t) \right) \right).$$

So, $\psi \left(\int_0^{\rho(Tz-z)} \varphi(t) d\nu(t) \right) = 0$ or $\int_0^{\rho(Tz-z)} \varphi(t) d\nu(t) = 0$. Thus, $\int_0^{\rho(Tz-z)} \varphi(t) d\nu(t) = 0$, leading to a contradiction again. Therefore, $z = Tz$. Moreover, $T(X_\rho) \subseteq h(X_\rho)$ and there exists a point $z_1 \in X_\rho$ such that $z = Tz = hz_1$. From (3), we obtain

$$\begin{aligned} \int_0^{\rho(T^2x_n - Tz_1)} \varphi(t) d\nu(t) &\leq F \left(\int_0^{\rho(hTx_n - hz_1)} \varphi(t) d\nu(t), \psi \left(\int_0^{\rho(hTx_n - hz_1)} \varphi(t) d\nu(t) \right) \right) \\ &\quad + \phi \left(\int_0^{\rho(hTx_n - T^2x_n)} \varphi(t) d\nu(t) \right) \left(\int_0^{\rho(hz_1 - T^2x_n)} \varphi(t) d\nu(t) \right), \end{aligned}$$

as $n \rightarrow \infty$ and using (11), we get $\int_0^{\rho(z - Tz_1)} \varphi(t) d\nu(t) \leq F(0, \psi(0)) = 0$, so that (12), we

have a contradiction. Therefore, by the condition on φ , we get $\int_0^{\rho(z - Tz_1)} \varphi(t) d\nu(t) = 0$, from which it follows that $\rho(z - Tz_1) = 0$ or $z = Tz_1 = hz_1$. Also, $hz = hTz_1 = Thz_1 = Tz = z$. Therefore, z is a common fixed point of T and h . In addition, if one considers h to be continuous instead of T , then one can prove $Tz = hz = z$ by similar argument as above. For uniqueness, suppose that $(z \neq w)$ are two arbitrary common fixed point of T and h , then from (3), we obtain

$$\begin{aligned} \int_0^{\rho(z-w)} \varphi(t) d\nu(t) &= \int_0^{\rho(Tz - Tw)} \varphi(t) d\nu(t) \\ &\leq F \left(\int_0^{\rho(hz - hw)} \varphi(t) d\nu(t), \psi \left(\int_0^{\rho(hz - hw)} \varphi(t) d\nu(t) \right) \right) \\ &\quad + \phi \left(\int_0^{\rho(hz - Tz)} \varphi(t) d\nu(t) \right) \left(\int_0^{\rho(hw - Tz)} \varphi(t) d\nu(t) \right) \\ &\leq F \left(\int_0^{\rho(z-w)} \varphi(t) d\nu(t), \psi \left(\int_0^{\rho(z-w)} \varphi(t) d\nu(t) \right) \right). \end{aligned}$$

So, $\psi \left(\int_0^{\rho(z-w)} \varphi(t) d\nu(t) \right) = 0$ or $\int_0^{\rho(z-w)} \varphi(t) d\nu(t) = 0$. Thus, $\int_0^{\rho(z-w)} \varphi(t) d\nu(t) = 0$; that is

$\rho(z - w) = 0$. Leading to a contradiction again. Therefore, by the conditions on φ , we get $\int_0^{\rho(z-w)} \varphi(t) d\nu(t) = 0$, from which it follows that $\rho(z - w) = 0$ or $z = w$. Hence, T and h have a unique common fixed point. ■

Now, we generalize (Theorem 3.3, [24]) in C -class function to obtain a common fixed point for ρ -compatible mappings in modular spaces involving altering distances of integral type.

Theorem 2.3 Let X_ρ be a ρ -complete modular space, where ρ satisfies the Δ_2 -condition. Suppose $T, h : X_\rho \rightarrow X_\rho$ are two ρ -compatible mappings such that $T(X_\rho) \subseteq h(X_\rho)$ and

$$\psi\left(\int_0^{\rho(Tx-Ty)} u(t)dt\right) \leq F(\psi(\theta(x, y)), \phi(\theta(x, y))) \tag{13}$$

for each $x, y \in X_\rho$ with nonnegative real numbers ζ, β, γ such that $2\zeta + \beta + 2\gamma < 1$, where $\phi \in \Phi_u, F \in \mathcal{C}$ and ψ is altering distance, and

$$\begin{aligned} \theta(x, y) = & \frac{\zeta}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(l(hx-Tx)+l(hy-Ty))} u(t)dt + \frac{\beta}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(l(hx-hy))} u(t)dt \\ & + \frac{\gamma}{\beta + 2\zeta + 2\gamma} \int_0^{\max\{\rho(l(hx-Ty)), \rho(l(hy-Tx))\}} u(t)dt, \end{aligned} \tag{14}$$

where $u(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a Lebesgue-integrable mapping which is summable, subadditive on each subset of \mathbb{R}^+ , and for each $\epsilon > 0, \int_0^\epsilon u(t)dt > 0$. If one of h or T is continuous, then there exists a unique fixed point of h and T .

Proof. Let x be an arbitrary point of X_ρ and generate inductively the sequence $(Tx_n)_{n \in \mathbb{N}}$ as $Tx_n = hx_{n+1}$ for each $n \in \mathbb{N}$. By (14), we have

$$\begin{aligned} \theta(x_{n+1}, x_n) = & \frac{\zeta}{\beta + 2\zeta + 2\gamma} \int_0^{\rho((hx_{n+1}-Tx_{n+1})+(hx_n-Tx_n))} u(t)dt + \frac{\beta}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(hx_{n+1}-hx_n)} u(t)dt \\ & + \frac{\gamma}{\beta + 2\zeta + 2\gamma} \int_0^{\max\{\rho(hx_{n+1}-Tx_n), \rho(hx_n-Tx_{n+1})\}} u(t)dt \\ = & \frac{\zeta}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(Tx_n-Tx_{n+1}+Tx_{n-1}-Tx_n)} u(t)dt \\ & + \beta \int_0^{\rho(Tx_n-Tx_{n-1})} u(t)dt + \gamma \int_0^{\max\{\rho(Tx_n-Tx_n), \rho(Tx_{n-1}-Tx_{n+1})\}} u(t)dt. \end{aligned}$$

By subadditive of u , we obtain

$$\begin{aligned} \theta(x_{n+1}, x_n) &= \frac{\zeta}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(Tx_n - Tx_{n+1})} u(t)dt + \frac{\zeta}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(Tx_{n-1} - Tx_n)} u(t)dt \\ &+ \frac{\beta}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(Tx_n - Tx_{n-1})} u(t)dt + \frac{\gamma}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(Tx_{n-1} - Tx_{n+1})} u(t)dt. \end{aligned}$$

Moreover,

$$\begin{aligned} \rho(Tx_{n-1} - Tx_{n+1}) &\leq \rho\left(\frac{2}{2}(Tx_{n-1} - Tx_n)\right) + \rho\left(\frac{2}{2}(Tx_n - Tx_{n+1})\right) \\ &\leq \rho(2(Tx_{n-1} - Tx_n)) + \rho(2(Tx_n - Tx_{n+1})), \end{aligned}$$

which implies

$$\begin{aligned} \theta(x_{n+1}, x_n) &\leq \frac{\zeta}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(Tx_n - Tx_{n+1})} u(t)dt + \frac{\zeta}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(Tx_{n-1} - Tx_n)} u(t)dt \\ &+ \frac{\beta}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(Tx_n - Tx_{n-1})} u(t)dt + \frac{\gamma}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(Tx_{n-1} - Tx_n)} u(t)dt \\ &+ \frac{\gamma}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(Tx_n - Tx_{n+1})} u(t)dt. \end{aligned} \tag{15}$$

From (13) and (15), we have

$$\begin{aligned} \psi\left(\int_0^{\rho(Tx_{n+1} - Tx_n)} u(t)dt\right) &\leq F(\psi(\theta(x_{n+1}, x_n)), \phi(\theta(x_{n+1}, x_n))) \\ &\leq \psi(\theta(x_{n+1}, x_n)) = \psi\left(\begin{pmatrix} \frac{\zeta}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(Tx_n - Tx_{n+1})} u(t)dt \\ + \frac{\zeta}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(Tx_{n-1} - Tx_n)} u(t)dt \\ + \frac{\beta}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(Tx_n - Tx_{n-1})} u(t)dt \\ + \frac{\gamma}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(Tx_{n-1} - Tx_n)} u(t)dt \\ + \frac{\gamma}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(Tx_n - Tx_{n+1})} u(t)dt. \end{pmatrix}\right) \end{aligned} \tag{16}$$

By the fact ψ is non-decreasing, we get

$$\begin{aligned} \int_0^{\rho(Tx_{n+1}-Tx_n)} u(t)dt &\leq (\theta(x_{n+1}, x_n)) \\ &\leq \left(\frac{\zeta + \gamma}{\beta + 2\zeta + 2\gamma}\right) \int_0^{\rho(Tx_{n+1}-Tx_n)} u(t)dt + \left(\frac{\zeta + \beta + \gamma}{\beta + 2\zeta + 2\gamma}\right) \int_0^{\rho(Tx_n-Tx_{n-1})} u(t)dt, \end{aligned}$$

which implies that

$$\int_0^{\rho(Tx_{n+1}-Tx_n)} u(t)dt \leq \int_0^{\rho(Tx_n-Tx_{n-1})} u(t)dt.$$

Thus,

$$\int_0^{\rho(Tx_{n+1}-Tx_n)} u(t)dt \rightarrow r \geq 0.$$

Taking the limit in (16) as $n \rightarrow \infty$ yields $\psi(r) \leq F(\psi(r), \phi(r))$. So $\psi(r) = 0$ or $\phi(r) = 0$. Thus,

$$\lim_{n \rightarrow \infty} \int_0^{\rho(Tx_{n+1}-Tx_n)} u(t)dt = 0.$$

Therefore,

$$\lim_{n \rightarrow \infty} \rho(Tx_{n+1} - Tx_n) \rightarrow 0. \tag{17}$$

Now, we show that $(Tx_n)_{n \in \mathbb{N}}$ is ρ -Cauchy. If not, then there exists an $\epsilon > 0$ and two sequences of integers $\{n(s)\}, \{m(s)\}$ with $n(s) > m(s) \geq s$ such that

$$\rho(Tx_{n(s)} - Tx_{m(s)}) \geq \epsilon \tag{18}$$

for $s = 1, 2, \dots$. We can assume that

$$\rho(Tx_{n(s)-1} - Tx_{m(s)}) < \epsilon. \tag{19}$$

Again, from (14), we have

$$\begin{aligned}
\theta(x_{m(s)}, x_{n(s)}) &= \frac{\zeta}{\beta + 2\zeta + 2\gamma} \int_0^{\rho((hx_{m(s)} - Tx_{m(s)}) + (hx_{n(s)} - Tx_{n(s)}))} u(t) dt \\
&+ \frac{\beta}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(hx_{m(s)} - hx_{n(s)})} u(t) dt \\
&+ \frac{\gamma}{\beta + 2\zeta + 2\gamma} \int_0^{\max\{\rho(hx_{m(s)} - Tx_{n(s)}), \rho(hx_{n(s)} - Tx_{m(s)})\}} u(t) dt \\
&= \frac{\zeta}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(Tx_{m(s)-1} - Tx_{m(s)} + Tx_{n(s)-1} - Tx_{n(s)})} u(t) dt \\
&+ \frac{\beta}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(Tx_{m(s)-1} - Tx_{n(s)-1})} u(t) dt \\
&+ \frac{\gamma}{\beta + 2\zeta + 2\gamma} \int_0^{\max\{\rho(Tx_{m(s)-1} - Tx_{n(s)}), \rho(Tx_{n(s)-1} - Tx_{m(s)})\}} u(t) dt. \quad (20)
\end{aligned}$$

Moreover,

$$\begin{aligned}
\rho(Tx_{m(s)-1} - Tx_{n(s)-1}) &\leq \rho\left(\frac{2}{2}(Tx_{m(s)-1} - Tx_{m(s)})\right) + \rho\left(\frac{2}{2}(Tx_{m(s)} - Tx_{n(s)-1})\right) \\
&\leq \rho(2(Tx_{m(s)-1} - Tx_{m(s)})) + \rho(2(Tx_{m(s)} - Tx_{n(s)-1})).
\end{aligned}$$

Using the Δ_2 -condition and (17), we obtain

$$\lim_{s \rightarrow \infty} \rho(2(Tx_{m(s)-1} - Tx_{m(s)})) = 0. \quad (21)$$

Therefore,

$$\lim_{s \rightarrow \infty} \int_0^{\rho(Tx_{m(s)-1} - Tx_{n(s)-1})} u(t) dt \leq \int_0^{\epsilon} u(t) dt. \quad (22)$$

Also,

$$\begin{aligned}
\rho(l(Tx_{m(s)-1} - Tx_{n(s)})) &\leq \rho\left(\frac{2}{2}(Tx_{m(s)-1} - Tx_{m(s)})\right) + \rho\left(\frac{2}{2}(Tx_{m(s)} - Tx_{n(s)})\right) \\
&\leq \rho(2(Tx_{m(s)-1} - Tx_{m(s)})) + \rho(2c(Tx_{m(s)} - Tx_{n(s)})).
\end{aligned}$$

Using the Δ_2 -condition and (18), (19) and (21), we have

$$\lim_{s \rightarrow \infty} \int_0^{\max\{\rho(Tx_{m(s)-1}-Tx_{n(s)}), \rho(Tx_{n(s)-1}-Tx_{m(s)})\}} u(t) dt \leq \int_0^\epsilon u(t) dt. \tag{23}$$

Taking the limit as $s \rightarrow \infty$ in (20), using (17), (21) and (23), we have

$$\lim_{s \rightarrow \infty} \theta(x_{m(s)}, x_{n(s)}) \leq \left(\frac{\beta + \gamma}{\beta + 2\zeta + 2\gamma}\right) \int_0^\epsilon u(t) dt. \tag{24}$$

On the other hand, by (13), we have

$$\psi\left(\int_0^{\rho(Tx_{m(s)}-Tx_{n(s)})} u(t) dt\right) \leq \psi(\theta(x_{m(s)}, x_{n(s)})) - \phi(\theta(x_{m(s)}, x_{n(s)})).$$

Taking $s \rightarrow \infty$ and using the continuity of ψ and ϕ , we have from (18) and (24) that

$$\begin{aligned} \psi\left(\int_0^\epsilon u(t) dt\right) &\leq \psi\left(\int_0^{\rho(Tx_{m(s)}-Tx_{n(s)})} u(t) dt\right) \\ &\leq F\left(\psi\left(\left(\frac{\beta + \gamma}{\beta + 2\zeta + 2\gamma}\right) \int_0^\epsilon u(t) dt\right), \phi\left(\left(\frac{\beta + \gamma}{\beta + 2\zeta + 2\gamma}\right) \int_0^\epsilon u(t) dt\right)\right) \\ &\leq \psi\left(\int_0^\epsilon u(t) dt\right). \end{aligned}$$

So $\psi\left(\left(\frac{\beta + \gamma}{\beta + 2\zeta + 2\gamma}\right) \int_0^\epsilon u(t) dt\right) = 0$ or $\phi\left(\left(\frac{\beta + \gamma}{\beta + 2\zeta + 2\gamma}\right) \int_0^\epsilon u(t) dt\right) = 0$. Thus, $\frac{\beta + \gamma}{\beta + 2\zeta + 2\gamma} \int_0^\epsilon u(t) dt = 0$.

Therefore, we get $\int_0^\epsilon u(t) dt = 0$, which is a contradiction. Thus, $(Tx_n)_{n \in \mathbb{N}}$ is ρ -Cauchy.

Since X_ρ is ρ -complete, then there exists $z \in X_\rho$ such that $\rho(Tx_n - z) \rightarrow 0$ as $n \rightarrow \infty$. If T is continuous, then $T^2x_n \rightarrow Tz$ and $Thx_n \rightarrow Tz$. Since $\rho(hTx_n - Thx_n) \rightarrow 0$, then $hTx_n \rightarrow Tz$ by ρ -compatibility. We now prove that z is a fixed point of T . From (14), we have

$$\begin{aligned} \theta(Tx_n, x_n) &= \frac{\zeta}{\beta + 2\zeta + 2\gamma} \int_0^{\rho((hTx_n - T^2x_n) + (hx_n - Tx_n))} u(t) dt + \frac{\beta}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(hTx_n - hx_n)} u(t) dt \\ &\quad + \frac{\gamma}{\beta + 2\zeta + 2\gamma} \int_0^{\max\{\rho(hTx_n - Tx_n), \rho(hx_n - T^2x_n)\}} u(t) dt. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, yields

$$\lim_{n \rightarrow \infty} \theta(Tx_n, x_n) = \left(\frac{\beta + \gamma}{\beta + 2\zeta + 2\gamma} \right) \int_0^{\rho(Tz-z)} u(t) dt. \quad (25)$$

Again, from (13), $\psi\left(\int_0^{\rho(T^2x_n - Tx_n)} u(t) dt\right) \leq F(\psi(\theta(Tx_n, x_n)), \phi(\theta(Tx_n, x_n)))$, as $n \rightarrow \infty$ and using (25), we get

$$\begin{aligned} \psi\left(\int_0^{\rho(Tz-z)} u(t) dt\right) &\leq F\left(\psi\left(\frac{\beta + \gamma}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(Tz-z)} u(t) dt\right), \phi\left(\frac{\beta + \gamma}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(Tz-z)} u(t) dt\right)\right) \\ &\leq \psi\left(\frac{\beta + \gamma}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(Tz-z)} u(t) dt\right). \end{aligned}$$

Thus, $\psi\left(\frac{\beta + \gamma}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(Tz-z)} u(t) dt\right) = 0$ or $\phi\left(\frac{\beta + \gamma}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(Tz-z)} u(t) dt\right) = 0$. Hence, $\frac{\beta + \gamma}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(Tz-z)} u(t) dt = 0$. Therefore, we get $\rho(Tz - z) = 0$ or $z = Tz$. Moreover, $T(X_\rho) \subseteq h(X_\rho)$. Hence, there exists a point $z_1 \in X_\rho$ such that

$$z = Tz = hz_1. \quad (26)$$

From (14), we have

$$\begin{aligned} \theta(Tx_n, z_1) &= \frac{\zeta}{\beta + 2\zeta + 2\gamma} \int_0^{\rho((hTx_n - T^2x_n) + (hz_1 - Tx_1))} u(t) dt + \frac{\beta}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(hTx_n - hz_1)} u(t) dt \\ &\quad + \frac{\gamma}{\beta + 2\zeta + 2\gamma} \int_0^{\max\{\rho(hTx_n - Tx_1), \rho(hz_1 - T^2x_n)\}} u(t) dt, \end{aligned}$$

as $n \rightarrow \infty$, yields

$$\begin{aligned} \lim_{n \rightarrow \infty} \theta(Tx_n, z_1) &= \zeta \int_0^{\rho(Tz - Tz + hz_1 - Tx_1)} u(t) dt \\ &\quad + \beta \int_0^{\rho(Tz - hz_1)} u(t) dt + \gamma \int_0^{\max\{\rho(Tz - Tx_1), \rho(hz_1 - Tz)\}} u(t) dt. \end{aligned}$$

Using (26), we get

$$\lim_{n \rightarrow \infty} \theta(Tx_n, z_1) = (\zeta + \gamma) \int_0^{\rho(z-Tz_1)} u(t)dt. \tag{27}$$

Again, from (13), we have

$$\psi\left(\int_0^{\rho(T^2x_n-Tz_1)} u(t)dt\right) \leq F(\psi(\theta(Tx_n, z_1)), \phi(\theta(Tx_n, z_1))).$$

Taking the limit as $n \rightarrow \infty$ and using (26) and (27), we get

$$\begin{aligned} & \psi\left(\int_0^{\rho(z-Tz_1)} u(t)dt\right) \\ & \leq F\left(\psi\left(\left(\frac{\zeta + \gamma}{\beta + 2\zeta + 2\gamma}\right) \int_0^{\rho(z-Tz_1)} u(t)dt\right), \phi\left(\left(\frac{\zeta + \gamma}{\beta + 2\zeta + 2\gamma}\right) \int_0^{\rho(z-Tz_1)} u(t)dt\right)\right) \\ & \leq \psi\left(\left(\frac{\zeta + \gamma}{\beta + 2\zeta + 2\gamma}\right) \int_0^{\rho(z-Tz_1)} u(t)dt\right). \end{aligned}$$

Thus, $\psi\left(\left(\frac{\beta + \gamma}{\beta + 2\zeta + 2\gamma}\right) \int_0^{\rho(z-Tz_1)} u(t)dt\right) = 0$ or $\phi\left(\left(\frac{\beta + \gamma}{\beta + 2\zeta + 2\gamma}\right) \int_0^{\rho(z-Tz_1)} u(t)dt\right) = 0$. Therefore,

$\frac{\beta + \gamma}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(z-Tz_1)} u(t)dt = 0$, which we have $\rho(z - Tz_1) = 0$. Hence, $z = Tz_1 = hz_1$ and also, $hz = hTz_1 = Thz_1 = Tz = z$. In addition, if one consider h to be a continuous in stead of T , then one can prove $hz = Tz = z$ by similar argument. Finally, suppose that z and w are two arbitrary common fixed point of T and h , with $w \neq z$. Then, from (14), we get

$$\begin{aligned} \theta(z, w) &= \frac{\zeta}{\beta + 2\zeta + 2\gamma} \int_0^{\rho((hz-Tz)+(hw-Tw))} u(t)dt \\ &+ \frac{\beta}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(hz-hw)} u(t)dt + \frac{\gamma}{\beta + 2\zeta + 2\gamma} \int_0^{\max\{\rho(hz-Tw), \rho(hw-Tz)\}} u(t)dt \\ &= \frac{\beta + \gamma}{\beta + 2\zeta + 2\gamma} \int_0^{\rho(z-w)} u(t)dt. \end{aligned}$$

From (13), we have

$$\psi\left(\int_0^{\rho(Tz-Tw)} u(t)dt\right) \leq F(\psi(\theta(z,w)), \phi(\theta(z,w))).$$

Hence,

$$\begin{aligned} \psi\left(\int_0^{\rho(z-w)} u(t)dt\right) &= \psi\left(\int_0^{\rho(Tz-Tw)} u(t)dt\right) \\ &\leq F\left(\psi\left(\frac{\beta+\gamma}{\beta+2\zeta+2\gamma} \int_0^{\rho(z-w)} u(t)dt\right), \phi\left(\frac{\beta+\gamma}{\beta+2\zeta+2\gamma} \int_0^{\rho(z-w)} u(t)dt\right)\right) \\ &\leq \psi\left(\int_0^{\rho(c(z-w))} u(t)dt\right). \end{aligned}$$

Therefore, $\psi\left(\frac{\beta+\gamma}{\beta+2\zeta+2\gamma} \int_0^{\rho(z-w)} u(t)dt\right) = 0$ or $\phi\left(\frac{\beta+\gamma}{\beta+2\zeta+2\gamma} \int_0^{\rho(z-w)} u(t)dt\right) = 0$. Thus, $\frac{\beta+\gamma}{\beta+2\zeta+2\gamma} \int_0^{\rho(z-w)} u(t)dt = 0$. Hence, we get $\rho(z-w) = 0$ or $z = w$. This completes the proof. \blacksquare

If we take $F(\psi, \phi) = \psi(t) = \frac{t}{2}$ and $\Phi(t) = \frac{t}{4}$ in Theorem 2.3, then we have the following corollary:

Corollary 2.4 [21] Let X_ρ be a ρ -complete modular space, where ρ satisfies the Δ_2 -condition. Suppose $c, l \in \mathbb{R}^+$, $c > l$ and $T, h : X_\rho \rightarrow X_\rho$ are two ρ -compatible mappings such that $T(X_\rho) \subseteq h(X_\rho)$ and

$$\begin{aligned} \int_0^{\rho(c(Tx-Ty))} u(t)dt &\leq \zeta \int_0^{\rho(l(hx-Tx)+l(hy-Ty))} u(t)dt + \beta \int_0^{\rho(l(hx-hy))} u(t)dt \\ &\quad + \gamma \int_0^{\max\{\rho(l(hx-Ty)), \rho(l(hy-Tx))\}} u(t)dt \end{aligned}$$

for all $x, y \in X_\rho$ with nonnegative real numbers ζ, β, γ such that $2\zeta + \beta + 2\gamma < 1$, where ψ and Φ are altering distances, and $u(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a Lebesgue-integrable mapping which is summable, subadditive on all subset of \mathbb{R}^+ and nonnegative such that $\int_0^\epsilon u(t)dt > 0$ for each $\epsilon > 0$. If one of h or T is continuous, then there exists a unique common fixed point of h and T .

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