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On the X basis in the Steenrod algebra

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Abstract. Let $\mathcal{A}_p$ be the mod p Steenrod algebra, where p is an odd prime, and let $\mathcal{A}$ be the subalgebra $\mathcal{A}$ of $\mathcal{A}_p$ generated by the Steenrod pth powers. We generalize the X-basis in $\mathcal{A}$ to $\mathcal{A}_p$.

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1. Introduction

Let p be an odd prime and $\mathcal{A}_p$ be the mod p Steenrod algebra. This algebra is an important tool in algebraic topology and has been an active area of research for almost 70 years, both in topology and pure algebra. From a topological view, $\mathcal{A}_p$ is the algebra of stable cohomology operations for ordinary cohomology H^* over the field $\mathbb{F}_p$. On the other hand, being a Hopf algebra, it has also a rich algebraic structure with its unique Hopf algebra conjugation map, χ.

Wall [8, Theorem 1] constructed a basis (Wall basis) in the mod 2 Steenrod algebra $\mathcal{A}_2$, which was also discussed as the Arnon basis in [1, Theorem 5 (B)]. Papastavridis [5, Theorem 4.3] generalized the Wall basis to $\mathcal{A}_p$. Also, Arnon [1] introduced the Arnon A and C bases. Karaca [3] developed the C basis for the subalgebra $\mathcal{A}$ of $\mathcal{A}_p$ generated by the Steenrod pth powers (the Bockstein free-part of $\mathcal{A}_p$). Emelyanov and Popelensky [2] generalized Arnon A and B bases to $\mathcal{A}$. In [2], these bases are called the X- and Z-

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bases, respectively. Many other bases for the Steenrod algebra have also been developed. For more details, we refer to [9].

In this paper, we generalize the X-basis in $\mathcal{A}$ to $\mathcal{A}_p$. This result can also be viewed as a generalization of the Arnon A basis in $\mathcal{A}_2$ to $\mathcal{A}_p$.

2. Preliminaries

Definition 2.1 The mod p steenrod algebra $\mathcal{A}_p$ is the free $\mathbb{F}_p$ algebra on the generators β, $\mathcal{P}^i$ for $i > 0$, modulo the Adem a relations and the relation $\beta^2 = 0$. Here, β is the the mod p Bockstein element of degree 1 and $\mathcal{P}^i$ is the Steenrod reduced power operation of degree $2i(p-1)$. The Adem relations in $\mathcal{A}_p$ [6]: If $i < pj$, then

$$\mathcal{P}^i\mathcal{P}^j = \sum_{k=0}^{[\frac{i}{p}]}(-1)^{i+k}\binom{(p-1)(j-k)-1}{i-pk}\mathcal{P}^{i+j-k}\mathcal{P}^k$$

and if $i \leqslant pj$, then

$$\mathcal{P}^i\beta\mathcal{P}^j = \sum_{k=0}^{[\frac{i}{p}]}(-1)^{i+k}\binom{(p-1)(j-k)}{i-pk}\beta\mathcal{P}^{i+j-k}\mathcal{P}^k + \sum_{k=0}^{[\frac{i-1}{p}]}(-1)^{i+k-1}\binom{(p-1)(j-k)-1}{i-pk-1}\mathcal{P}^{i+j-k}\beta\mathcal{P}^k$$

with the binomial coefficients taken modulo p. In the above formulas, $\mathcal{P}^0$ is understood to be the unit. A monomial in $\mathcal{A}_p$ has the form $\beta^{\varepsilon_0}\mathcal{P}^{r_1}\beta^{\varepsilon_1}\cdots\mathcal{P}^{r_k}\beta^{\varepsilon_k}$, where $\varepsilon_i = 0, 1$ and $r_i = 1, 2, \ldots$.

Definition 2.2 [1] Let F be the free (noncommutative) graded algebra over a field k generated by the set of symbols $\{x_i\}_i \in I$ and assume that for any integer N only a finite number of symbols have degree smallers than N. Let $\leqslant$ be any linear ordering on the monomials in F, and U be the two-sided homogeneous ideal in F. A monomial M is minimal with respect to $(U, \leqslant)$ if M is not equivalent, mod U, to a linear combination of monomials smaller than M under $\leqslant$.

Corollary 2.3 [1] Given F, U and $\leqslant$, the set of minimal monic monomials form a vector space basis for F/U.

Definition 2.4 Given finite sequences of positive integers $I = (i_1, \ldots, i_r)$ and $J = (j_1, \ldots, j_s)$, we define the *left lexicographical ordering* as follow: $I <_L J$ if

(1) I is empty and J is not,
(2) I and J are nonempty sets and $i_1 < j_1$,
(3) I and J are nonempty sets, $i_1 = j_1$, and $(i_2, \ldots, i_r) <_L (j_2, \ldots, j_s)$.

For the definition of right lexicographical ordering and properties of both left and right lexicographical orderings, see [2]. Definition 2.4 enables us to define left lexicographical ordering of monomials $\beta^{\varepsilon_0}\mathcal{P}^{r_1}\beta^{\varepsilon_1}\cdots\mathcal{P}^{r_k}\beta^{\varepsilon_k}$. Let $D = \sum D_i$ and $H = \sum H_j$ be linear combinations of some monomials and define $D <_L H$ if $D_i <_L H_j$ for all i, j.

It is worth nothing that the order relation is not well defined in $\mathcal{A}_p$. In the last preceding paragraph, monomials are considered as elements of the free algebra generated by $\{\mathcal{P}^i\}_{i=0}^{\infty}$, and the β. On the other hand, we use the phrase "a basis of monomials" to mean a basis in $\mathcal{A}_p$. Following [2, 7], we give the following definitions.

Definition 2.5 Let $X_k^n = \mathcal{P}^{p^n}\mathcal{P}^{p^{n-1}}\cdots\mathcal{P}^{p^k}$ for $n \geqslant k \geqslant 0$. An X-monomial has the form $(X_{k_r}^{n_r})^{j_r}\cdots(X_{k_1}^{n_1})^{j_1}$ such that $(n_r, k_r) <_L \cdots <_L (n_2, k_2) <_L (n_1, k_1)$ and $1 \leqslant j_r \leqslant p-1$ for each r.

Definition 2.6 Let $Z_k^n = \mathcal{P}^{p^k}\mathcal{P}^{p^{k+1}}\cdots\mathcal{P}^{p^n}$ for $n \geqslant k \geqslant 0$. A Z-monomial has the form

$$(Z_{k_1}^{n_1})^{j_1}\cdots(Z_{k_r}^{n_r})^{j_r}$$

such that $(n_r, k_r) <_L \cdots <_L (n_2, k_2) <_L (n_1, k_1)$ and $1 \leqslant j_r \leqslant p-1$ for each r.

Now, let us consider the Bockstein part of $\mathcal{A}_p$. Following [4], at an odd prime p, $Q_0 = \beta$, and $Q_i \in \mathcal{A}_p$ may be defined inductively by $Q_{i+1} = [\mathcal{P}^{p^i}, Q_i] = \mathcal{P}^{p^i}Q_i - Q_i\mathcal{P}^{p^i}$.

Remark 1 *In [5], the element Z_k^n is denoted by the symbol O_k^n, a Z-monomial is called an "allowable monomial", and a generalized Z-monomial is called a "generalized allowable monomial".*

3. The generalized X- and Z-monomials in the mod p Steenrod algebra

Now, inspired by [5], we give the following definitions.

Definition 3.1

(i) A generalized X-monomial has the form $Q_0^{\varepsilon_0}Q_1^{\varepsilon_i}\cdots Q_m^{\varepsilon_m}(X_{k_r}^{n_r})^{j_r}\cdots(X_{k_1}^{n_1})^{j_1}$,
(ii) A generalized Z-monomial has the form $Q_0^{\varepsilon_0}Q_1^{\varepsilon_i}\cdots Q_m^{\varepsilon_m}(Z_{k_1}^{n_1})^{j_1}\cdots(Z_{k_r}^{n_r})^{j_r}$.

Both of these monomials satisfy the conditions:

(1) $(n_r, k_r) <_L \cdots <_L (n_2, k_2) <_L (n_1, k_1)$,
(2) $1 \leqslant j_r \leqslant p-1$ for each r,
(3) $\varepsilon_i = 0$ or 1 for $0 \leqslant i \leqslant m$.

Papastavridis [5] described relations for the algebra $\mathcal{A}_p$. Moreover, following [8], he defined the concept of height to prove his results. Now, we describe Papastavridis's basis in this present paper's language. See Papas for more details.

Theorem 3.2 [5] The set of all generalized Z-monomials forms an additive basis of $\mathcal{A}_p$.

Lemma 3.3 The number of the generalized X-monomials of a given degree is the same as the dimension of $\mathcal{A}_p$ of the same degree.

Proof. By Theorems 2.3 and 2.4 in [2], we have the same number of Z-monomials and X-monomials at each degree. Following this in cases (i) and (ii) of Definition 3.1, we have the same number of monomials in each degree. Hence, by Theorem 3.2, we deduce that the number of X-monomials having a given degree is equal to the dimension of $\mathcal{A}_p$ of the same degree. ■

Theorem 3.4 The set of all generalized X-monomials forms an additive basis of $\mathcal{A}_p$. The generalized X-monomials are minimal with respect to $<_L$.

Proof. We follow Arnon's argument [1] to deduce Theorem 3.4 from Theorem 3.2. We start proof by recalling the following property of the conjugation:

$$\chi(\mathcal{P}^{p^n}) \equiv -\mathcal{P}^{p^n} + K, \tag{1}$$

where $\equiv$ denotes the equality modulo the Adem relations, and K is a polynomial in lower $\mathcal{P}^j$'s, which means K is a sum of products of $\mathcal{P}^j$'s with $j < p^n$ for each j. Recalling the argument in [2], we know that if one substitutes K for $\mathcal{P}^{p^n}$ in any monomial reduces it lexicographically with respect to both right and left orders. On the other hand, since Q_i's are primitive elements of $\mathcal{A}_p$, these satisfy $\chi(Q_i) = -Q_i$ [4]. In particular, we have $\chi(\beta) = -\beta$. Following this, since χ is aniti-automorphism, (1) gives

$$\chi(\beta\mathcal{P}^{p^n}) \equiv \mathcal{P}^{p^n}\beta - \underbrace{K\beta}_{K'} \text{ and } \chi(\mathcal{P}^{p^n}\beta) \equiv \beta\mathcal{P}^{p^n} - \underbrace{\beta K}_{K''}.$$

Now, applying the same argument above, we see that if one substitutes K' for $\mathcal{P}^{p^n}\beta$ and K'' for $\beta\mathcal{P}^{p^n}$ in any monomial, then the result will be lower in both the right and left lexicographical orderings. Following this, given a monomial M in $\mathcal{P}^{p^n}$ and β, which is not a generalized X monomial, one applies χ, reduces the result to a linear combination of the generalized Z- monomials and applies χ again to obtain an expression of that monomial in terms of lower ones with respect to the $<_L$ ordering. ■

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