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Applications of fuzzy e-open sets

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Abstract. The aim of this paper is to introduce and study the notions of fuzzy upper e-limit set, fuzzy lower e-limit set and fuzzy e-continuously convergent functions. Properties and basic relationships among fuzzy upper e-limit set, fuzzy lower e-limit set and fuzzy e-continuity are investigated via fuzzy e-open sets.

Keywords: Fuzzy e-open sets, fuzzy lower e-limit set, fuzzy upper e-limit set, fuzzy e-continuously convergent functions.

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1. Introduction

Ever since the introduction of fuzzy sets by Zadeh [25], the fuzzy concepts has involved almost all branches of mathematics. One of the most important conclusions of advanced scientific research into the very basic question related to the quintessence of natural science and philosophy is that our universe is fundamentally and irreducibly fuzzy. This notion of fuzziness is central to the work of written and El-Naschie to mention only two well-known names working on the frontiers of fundamentally and irreducibly fuzzy. This notion of fuzziness is central to the work of written and El-Naschie to mention only two well-known names working on the frontiers of fundamental research in quantum gravity and high energy particle physics. Based on the concept of fuzzy sets, Chang [3] introduced and developed the concept of fuzzy topological spaces. Since then various

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important notions in the classical topology such as continuous functions [3] have been extended to fuzzy topological spaces. Fuzzy continuity is one of the main topics in fuzzy topology. Mukherjee and Debnath [17] has defined fuzzy δ-open set and fuzzy δ-closed set. In 2006, Ekici [6] introduced fuzzy upper and lower s-limit sets. Seenivasan and Kamala [20] defined the concepts of fuzzy e-open set and fuzzy e-continuous mappings in fuzzy topological spaces. The initiations of e-open sets, e^*-open sets, a-open sets, e-continuity and e-compactness and related studies in topological spaces are due to Ekici ([7–11]).

In this paper, we introduce and study the notions of fuzzy upper e-limit set, fuzzy lower e-limit set and fuzzy e-continuously convergent functions. Properties and basic relationships among fuzzy upper e-limit set, fuzzy lower e-limit set and fuzzy e-continuity are investigated via fuzzy e-open sets.

2. Preliminaries

Most of the concepts, notations and definitions which we have used in this paper are standard by now. But, for the sake of completeness we recall some definitions and results used in the sequel. A fuzzy set in X is called a fuzzy point [3] if it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is α ($0 < \alpha \leqslant 1$), we denote this fuzzy point by x_α, where the point x is called its support. A fuzzy point x_α for $\alpha \in I_0$ is an element of I^X such that

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

The set of all fuzzy points in X is denoted by $P_\alpha(X)$. A fuzzy point x_α is said to be contained in a fuzzy set μ or to belong to μ, denoted by $x_\alpha \in \mu$ if $\alpha \leqslant \mu(x)$. A fuzzy point x_α is said to be quasi-coincident [18] with a fuzzy set μ in X, denoted by $x_\alpha q\mu$, if $\alpha + \mu(x) > 1$. A fuzzy set μ in a fuzzy topological space X is said to be quasi-coincident [18] (q-coincident, in short) with a fuzzy set ρ in X, denoted by $\mu q\rho$, if there exists some $x \in X$ such that $\mu(x) + \rho(x) > 1$. If μ is not q-coincident with ρ, we write $\mu\overline{q}\rho$. A fuzzy set μ in a fuzzy topological space X is called a fuzzy open neighborhood [18] (or a nbd, for short) of a fuzzy point x_α in X if there exists a fuzzy open set v of X such that $x_\alpha \in v \leqslant \mu$. The family N_{x_α} of all nbds of x_α is called the system of nbds of x_α. A fuzzy set μ in a fuzzy topological space X is called a fuzzy open Q-neighborhood [18] of a fuzzy point x_α in X if there exists $\rho \in \tau$ such that $x_\alpha q\rho$ and $\rho \leqslant \mu$. The family of all fuzzy open Q-neighborhoods of the fuzzy point x_α in X is $N^Q_{x_\alpha}$.

Let λ be a fuzzy subset of a space X. The fuzzy closure of λ and fuzzy interior of λ are denoted by $Cl(\lambda)$ and $Int(\lambda)$, respectively. A fuzzy subset λ of space X is called fuzzy regular open [1] (resp. fuzzy regular closed) if $\lambda = Int(Cl(\lambda))$ (resp. $\lambda = Cl(Int(\lambda))$). The fuzzy δ-interior [20] of fuzzy subset λ of X is the union of all fuzzy regular open sets contained in λ. A fuzzy subset λ is called fuzzy δ-open [13] if $\lambda = \delta Int(\lambda)$. The complement of fuzzy δ-open set is called fuzzy δ-closed (i.e. $\lambda = \delta Cl(\lambda)$). The fuzzy δ-closure of λ and the fuzzy δ-interior of λ are denoted by $\delta Cl(\lambda)$ and $\delta Int(\lambda)$. A fuzzy subset λ of a space X is called fuzzy δ-preopen [2] if $\lambda \leqslant int(\delta Cl(\lambda))$. The complement of a fuzzy δ-preopen set is called fuzzy δ-pre-closed.

A map $f : X \to Y$ is called fuzzy continuous [19] if for each fuzzy point x_α in X and each fuzzy open nbd V of $f(x_\alpha)$, there exists fuzzy open nbd U of x_α such that $f(U) \leqslant V$. A map $f : X \to Y$ is called fuzzy continuous [19] if the inverse image of every fuzzy open subset of Y is fuzzy open subset of X. $FC(X,Y)$ denote the family of all fuzzy continuous functions of an fts X into another fts Y. Let (X,τ) be an fts. A fuzzy point $x_\alpha \in Cl(\mu)$ [18] if each Q-neighborhood η of x_α is quasi-coincident with μ, we have $\eta q\mu$.

Let I be a directed set. Let χ be the collection of all fuzzy points of an ordered set X. The function $S : I \to \chi$ is called a fuzzy net [18] in X. For every $i \in I$, $S(I)$ is often denoted by s_i and hence, a net S is often denoted by $\{s_i : i \in I\}$.

Let $S = \{s_i : i \in I\}$ be a fuzzy net in X. Then S is said to be quasi-coincident with μ if for each $i \in I$, s_i is quasi-coincident with μ. A fuzzy net $\{g_j : j \in J\}$ in X, is called a fuzzy subnet [18] of a fuzzy net $\{s_i : i \in I\}$ in X if there is a function $N : J \to I$ such that (i) $g_i = S_{N_{(j)}}$ and (ii) for the element $i_0 \in I$, there is $j_0 \in J$ such that if $j \geqslant j_0$, $j \in J$, then $N(j) \geqslant i_0$. A fuzzy net $\{S(n) : n \in D\}$ in an fts X is said to be fuzzy converges [16] to x_α if for each fuzzy open nbd v of x_t there is some $n_0 \in D$ such that $n \geqslant n_0$ implies $S(n) \in v$. A fuzzy net $\{f_m : m \in M\}$ in $FC(X,Y)$ is said to be fuzzy continuously converges [12] to $f \in FC(X,Y)$ if for every x_α in X and for every fuzzy open nbd V of $f(x_\alpha)$ in Y there exists an element $m_0 \in M$ and a fuzzy open nbd U of X_α in X such that $f_m(U) \leqslant V$, for every $m \in M$, $m \geqslant m_0$. A fuzzy set μ in a fuzzy topological space X is called a fuzzy e-Q-nbd [22] of a fuzzy point x_α in X if there exists a fuzzy e-open set V in X such that $x_\alpha qV \leqslant \mu$. If in addition, μ is fuzzy e-open then μ will be referred to as a fuzzy e-open Q-nbd of x_α. A fuzzy set μ in a fuzzy topological space (X,τ) is called fuzzy e-neighborhood [22] of a fuzzy point x_α if there exists $\rho \in eO(X)$ such that $x_\alpha \in \rho \leqslant \mu$.

A fuzzy point x_α in a fuzzy topological space X is called a fuzzy e-cluster point [23] of a fuzzy set μ in X if every fuzzy e-q-nbd of x_α is q-coincident with μ. The union of all fuzzy e-cluster points of μ is called the fuzzy e-closure of μ and is denoted by $eCl(\mu)$. A fuzzy set λ in a fuzzy topological space X is called fuzzy e-open [20] if $A \leqslant Int(\delta Cl(A)) \vee Cl(\delta Int(A))$. The complement of fuzzy e-open set is called fuzzy e-closed. (i.e. $Int(\delta Cl(A)) \wedge Cl(\delta Int(A)) \leqslant A$). Let λ be a fuzzy set of a fuzzy topological space X. $eInt(\lambda) = \bigvee\{\mu \in I^X : \mu \leqslant \lambda, \mu \text{ is a } feo \text{ set}\}$ is called the fuzzy e-interior [20] of λ. $eCl(\lambda) = \bigwedge\{\mu \in I^X : \mu \geqslant \lambda, \mu \text{ is a } fec \text{ set }\}$ is called the fuzzy e-closure [20] of λ. Let $f : (X,\tau_1) \to (Y,\tau_2)$ be a mapping from a fts (X,τ_1) to another (Y,τ_2). Then f is called fuzzy e-continuous [20] iff $f^{-1}(\lambda)$ is a feo set in X for any fuzzy open set λ in Y.

Theorem 2.1 [23] For a fuzzy topological space X, the following conditions are equivalent:

(*i*) X is fuzzy e-regular.
(*ii*) for each fuzzy point x_α and each fuzzy e-open set U in X, q-coincident with x_α, there exists a fuzzy open set V in X such that $x_\alpha qV \leqslant eClV \leqslant U$.

3. Fuzzy e-continuously converge

Now, we introduce the following definition.

Definition 3.1 A function $f : (X,\tau) \to (Y,\sigma)$ is said to be fuzzy e-continuous if for every fuzzy point x_α in X and for every fuzzy e-q-neighborhood μ of $f(x_\alpha)$, there exists a fuzzy e-q-neighborhood ρ of x_α such that $f(\rho) \leqslant \mu$.

The family of all fuzzy e-continuous functions from (X,τ) into (Y,σ) is denoted by $eC(X,Y)$.

Definition 3.2 Let (X,τ) be a fuzzy topological space and let $\{p_i : i \in I\}$ be a net of fuzzy points in X. We say that the fuzzy net $\{p_i : i \in I\}$ fuzzy e-converges to a fuzzy point p of X if for every fuzzy e-q-nbd μ of p in X there exists $i_0 \in I$ such that $p_i q\mu$ for every $i \in I$ and $i \geqslant i_0$.

Theorem 3.3 Let μ be a fuzzy set of a fuzzy topological space (X, τ). Then, a fuzzy point $x_\alpha \in eCl(\mu)$ if and only if for every $\rho \in eO(X)$ for which $x_\alpha q\rho$ we have $\rho q\mu$.

Proof. The fuzzy point $x_\alpha \in eCl(\mu)$ if and only if $x_\alpha \in \rho$ for every fuzzy e-closed set ρ of X for which $\mu \leqslant \rho$. Equivalently, $x_\alpha \in eCl(\mu)$ if and only if $\alpha \leqslant 1 - \rho(x)$ for every fuzzy e-open set ρ for which $\mu \leqslant 1 - \rho$. Thus, $x_\alpha \in eCl(\mu)$ if and only if $\rho(x) \leqslant 1 - \alpha$, for every fuzzy e-open set ρ for which $\rho \leqslant 1 - \mu$. So, $x_\alpha \in eCl(\mu)$ if and only if for every fuzzy e-open set ρ of X such that $\rho(x) > 1 - \alpha$ we have ρ not less than $1 - \mu$. Therefore, $x_\alpha \in eCl(\mu)$ if and only if for every fuzzy e-open set ρ of X such that $\rho(x) + \alpha > 1$ we have $\rho q\mu$. Thus, $x_\alpha \in eCl(\mu)$ if and only if for every fuzzy e-open set ρ of X for which $x_\alpha q\rho$ we have $\rho q\mu$. ■

Theorem 3.4 Let $f : (X, \tau) \to (Y, \sigma)$ be a fuzzy e-continuous function, x_α be a fuzzy point in X and μ, ρ be fuzzy e-q-neighborhoods of x_α and $f(x_\alpha)$, respectively such that $f(\mu) \nleq \rho$. Then there exists a fuzzy point x_θ in X such that $x_\theta q\mu$ and $f(x_\theta)\overline{q}\rho$.

Proof. Since $f(\mu)$ not less than or equal to ρ, we have μ not less than or equal to $f^{-1}(\rho)$. Thus, there exists $x \in Y$ such that $\mu(x) > f^{-1}(\rho(x))$ or $\mu(x) - f^{-1}(\rho(x)) > 0$ and therefore $\mu(x) + 1 - f^{-1}(\rho(x)) > 1$ or $\mu(x) + (f^{-1}(\rho))^c(x)) > 1$. Let $(f^{-1}(\rho))^c(x)) = r$. Clearly, for the fuzzy point x_α we have $x_\alpha q\mu$ and $x_\alpha \in (f^{-1}(\rho))^c$. Hence, for the fuzzy point $x_\alpha = x_\theta$, we have $x_\theta q\mu$ and $f(x_\theta)\overline{q}\rho$. ■

Definition 3.5 A net $\{f_i | i \in I\}$ in $eC(X, Y)$ fuzzy e-continuously converges to $f \in eC(X, Y)$ if and only if for every net $\{p_j | j \in J\}$ in X which fuzzy e-converges to a fuzzy point p in X we have that the fuzzy net $\{f_i(p_j) | (i, j) \in I \times J\}$ fuzzy e-converges to the fuzzy point $f(p)$ in Y.

Theorem 3.6 A function $f : (X, \tau) \to (Y, \sigma)$ is fuzzy e-continuous if and only if for every fuzzy point x_α of X and for every net $\{p_i | i \in I\}$ of X which fuzzy e-converges to x_α, the net $\{f(p_i) | i \in I\}$ of Y fuzzy e-converges to $f(x_\alpha)$.

Proof. Straightforward. ■

Theorem 3.7 A net $\{f_i | i \in I\}$ in $eC(X, Y)$ fuzzy e-continuously converges to $f \in eC(X, Y)$ if and only if for every fuzzy point x_α in X and for every fuzzy e-q-neighborhood ρ of $f(x_\alpha)$ in Y there exists an element $i_0 \in I$ and a fuzzy e-q-neighborhood μ of x_α in X such that $f_i(\mu) \leqslant \rho$ for every $i \in I$ with $i \geqslant i_0$.

Proof. Let x_α be a fuzzy point in X and ρ be a e-q-neighborhood of $f(x_\alpha)$ in Y such that for every $i \in I$ and for every fuzzy e-q-neighborhood μ of x_α in X we can choose a fuzzy point x_μ in X by Theorem 3.4 such that $x_\mu q\mu$ and $f_i(x_\mu)\overline{q}\rho$. Clearly, the fuzzy net $\{x_\mu | \mu \in N(x_\alpha)\}$ fuzzy e-converges to x_α, but the fuzzy net $\{f_i(x_\mu), (\mu, i) \in N(x_\alpha) \times I\}$ does not fuzzy e-converges to $f(x_\alpha)$ in Y.

Conversely, let $\{p_j | j \in J\}$ be a fuzzy net in $eC(X, Y)$ which fuzzy e-converges to the fuzzy point x_α in X and let ρ be an arbitrary fuzzy e-q-neighborhood of $f(x_\alpha)$ in Y. By assumption there exists a fuzzy e-q-neighborhood μ of x_α in X and an element $i_0 \in I$ such that $f_i(\mu) \leqslant \rho$ for every $i \in I$ with $i \geqslant i_0$. Since the fuzzy net $\{p_j | j \in J\}$ fuzzy e-converges to x_α in X, there exists $j_0 \in J$ such that $p_j q\mu$, for every $j \in J$ with $j \geqslant j_0$,. Let $(i_0, j_0) \in I \times J$. Then for every $(i, j) \in I \times J$, $(i, j) \geqslant (i_0, j_0)$, we have $f_i(p_j) q f_i(\mu)$ and $f_i(\mu) \leqslant \rho$, i.e., $f_i(p_j) q f_i(\mu) \leqslant \rho$. Thus, the fuzzy net $\{f_i(p_j) | (i, j) \in I \times J\}$ fuzzy

e-converges to $f(x_\alpha)$ and the fuzzy net $\{f_i|i \in I\}$ fuzzy e-continuously converges to f. ■

Definition 3.8 [21] A fuzzy set μ of a fuzzy topological space X is called fuzzy e-generalized closed set or $f\tilde{e}$-closed (in short, fegc) if $eCl(\mu) \leqslant \rho$ whenever $\mu \leqslant \rho$ and ρ is feo in X.

Definition 3.9 A fuzzy topological space X is called fuzzy e-T_1 if every fuzzy point is fec.

Theorem 3.10 A fuzzy topological space X is fuzzy e-T_1 if and only if for each $x \in X$ and each $\alpha \in [0, 1]$ there exists a feo set μ such that $\mu(x) = 1-\alpha$ and $\mu(y) = 1$ for $y \neq x$.

Proof. Assume X is fuzzy e-T_1. Let $\alpha = 0.$ and set $\mu = X$. Then μ is feo set such that $\mu(x) = 1 - 0$ and $\mu(y) = 1$ for $y \neq x$. Now, let $\alpha \in (0, 1]$, $x \in X$ and $\mu = (x_\alpha)^c$. Hence x_α is fec and the set μ is feo such that $\mu(x) = 1 - \alpha$ and $\mu(y) = 1$ for $y \neq x$.

Conversely, let x_α be an arbitrary fuzzy point of X. We prove that the fuzzy point x_α is fec. By assumption, there exists a feo set μ such that $\mu(x) = 1 - \alpha$ and $\mu(y) = 1$ for $y \neq x$. Now, $\mu(x) + \alpha = 1$ implies $\mu(x)\bar{q}x_\alpha$ or $\mu(x)qx^c_\alpha$. Clearly, $\mu^c = x_\alpha$. Thus, the fuzzy point x_α is fec and X is fuzzy e-T_1. ■

Definition 3.11 A fuzzy topological space X is called fuzzy quasi e-T_1 if for any fuzzy points x_α and y_β for which $supp(x_\alpha) = x \neq supp(y_\beta) = y$, there exists a feo set μ such that $x_\alpha \in \mu$ and $y_\beta \notin \mu$ and another e-open set ρ such that $x_\alpha \notin \rho$ and $y_\beta \in \rho$.

Definition 3.12 A fuzzy topological space X is called a fuzzy e-T_2 if for any fuzzy points x_α and y_β for which $supp(x_\alpha) \neq supp(y_\beta)$, there exists two fuzzy e-q-neighborhoods ρ and μ of x_α and y_β, respectively, such that $\rho \wedge \mu = 0$.

Definition 3.13 [18] A fuzzy point x_α is called weak (resp. strong) if $\alpha \leqslant \frac{1}{2}$ (resp. $\alpha > \frac{1}{2}$).

Theorem 3.14 If X is a fuzzy quasi e-T_1 fuzzy topological space and x_α a weak fuzzy point in X, then $(x_\alpha)^c$ is a fuzzy e-neighborhood of each fuzzy point y_β with $x \neq y$.

Proof. Let $x \neq y$, x_α and y_β be fuzzy points of X. Since X is fuzzy quasi e-T_1, there exists a fuzzy e-open set μ of X such that $y_\beta \in \mu$ and $x_\alpha \notin \mu$. This implies that $\alpha > \mu(x)$. Since x_α is a weak fuzzy point, $\alpha \leqslant \frac{1}{2}$. Thus $\mu(x) < \alpha \leqslant \frac{1}{2}$ implies $\mu(x) \leqslant \frac{1}{2}$. So, $\mu(x) = 1 - \alpha$. Therefore, $\mu(y) \leqslant 1 = (x_\alpha)^c(y)$ for every $y \in X \backslash \{x\}$. Consequently, $\mu \leqslant (x_\alpha)^c$ and the fuzzy point $(x_\alpha)^c$ is a fuzzy e-neighborhood of y_β. ■

Definition 3.15 A fuzzy topological space X is called a fuzzy e-regular if there exists $\mu,\ \eta\ \in eO(X)$ such that $x_\alpha \in \mu$, $\rho \leqslant \eta$ and $\mu \wedge \eta = 0$ for any fuzzy point x_α and a fuzzy e-closed set ρ not containing x_α.

Theorem 3.16 If X is a fuzzy e-regular space, then there exists a fuzzy e-open set ρ containing x_α such that $eCl(\rho) \leqslant \mu$ for any strong fuzzy point x_α and any fuzzy e-open set μ containing x_α.

Proof. Suppose that x_α be any strong fuzzy point contained in $\mu \in eO(X)$. Then $x_\alpha \in \mu$. Since α is strong fuzzy point, $\alpha > \frac{1}{2}$ and $x_\alpha \in \rho$. Then $\frac{1}{2} < \alpha \leqslant \rho(x)$. Thus, the complement of μ; that is, the set μ^c is a fuzzy e-closed set which does not contain the fuzzy point x_α. Since X is a fuzzy e-regular space, there exists $\rho, \eta \in eO(X)$ such that $x_\alpha \in \rho$ and $\mu^c \leqslant \eta$ with $\rho \wedge \eta = 0$. Hence, we have $\rho \leqslant \eta^c$ and $eCl(\rho) \leqslant eCl(\eta^c) = \eta^c$. Now, $\mu^c \leqslant \eta$ implies $\eta^c \leqslant \mu$. This means that $eCl(\rho) \leqslant \mu$ which completes the proof. ■

Theorem 3.17 If X is a fuzzy e-regular space, then the strong fuzzy points in X are fuzzy eg-closed.

Proof. Let x_α be any strong fuzzy point in X and μ be a fuzzy e-open set such that $x_\alpha \in \mu$. By Theorem 3.16, there exists $\rho \in eO(X)$ such that $x_\alpha \in \rho$ and $eCl(\rho) \leqslant \mu$. We have $eCl(x_\alpha) \leqslant eCl(\rho) \leqslant \mu$. Thus, $eCl(x_\alpha) \leqslant \mu$ whenever $x_\alpha \in \mu$ (μ is fuzzy e-open). Hence, the fuzzy point x_α is fuzzy eg-closed. ■

Definition 3.18 A fuzzy topological space X is called a weakly fuzzy e-regular if for any weak fuzzy point x_α and a fuzzy e-closed set ρ not containing x_α, there exists $\mu,\ \eta\ \in eO(X)$ such that $x_\alpha \in \mu$, $\rho \leqslant \eta$ and $\mu \wedge \eta = 0$.

Definition 3.19 A fuzzy set μ in a fuzzy topological space X is said to be fuzzy e-nearly crisp if $eCl(\mu) \wedge (eCl(\mu))^c = 0$.

Theorem 3.20 Let X be a fuzzy topological space. If for any weak fuzzy point x_α and $\mu \in eO(X)$ containing x_α, there exists a fuzzy e-open and e-nearly crisp fuzzy set ρ containing x_α such that $eCl(\rho) \leqslant \mu$, then X is weakly fuzzy e-regular.

Proof. Assume that η is a fuzzy e-closed set not containing the weak fuzzy point x_α. Then η^c is a fuzzy e-open set containing x_α. By hypothesis, there exists a fuzzy e-open and e-nearly crisp fuzzy set ρ such that $x_\alpha \in \rho$ and $eCl(\rho) \leqslant \eta^c$. We set $\gamma = eInt(eCl(\rho))$ and $\mu = 1 - eCl(\rho)$. Then γ is fuzzy e-open, $x_\alpha \in \gamma$ and $\eta \leqslant \mu$. We are going to prove that $\mu \wedge \gamma = 0$. Now assume that there exists $y \in X$ such that $(\gamma \wedge \mu)(y) = \alpha \neq 0$. Then $y_\alpha \in \gamma \wedge \mu$ and so, $y_\alpha \in \gamma$ and $y_\alpha \in \mu$. Hence, $y_\alpha \in eCl(\rho)$ and $y_\alpha \in (eCl(\rho))^c$. This is a contradiction, since ρ is fuzzy e-nearly crisp. Therefore, $\mu \wedge \gamma = 0$. Hence, X is fuzzy e-regular. ■

Definition 3.21 Let μ be a fuzzy set of a fuzzy topological space X. A fuzzy point x_α is called a e-boundary point of a fuzzy set μ if and only if $x_\alpha \in eCl(\mu) \wedge (1 - eCl(\mu))$. We denote the fuzzy set $eCl(\mu) \wedge (1 - eCl(\mu))$ by $e\text{-}bd(\mu)$.

Theorem 3.22 Let X be a fuzzy topological space. Suppose that x_α and y_β be weak and strong fuzzy points, respectively. If x_α is fuzzy e-generalized closed, then $y_\beta \in eCl(x_\alpha) \Rightarrow x_\alpha \in eCl(y_\beta)$.

Proof. Suppose that $y_\beta \in eCl(x_\alpha)$ and $x_\alpha \notin eCl(y_\beta)$. Then $eCl(y_\beta) < \alpha$. Also $\alpha \leqslant \frac{1}{2}$. Thus, $eCl(y_\beta)(x) \leqslant 1 - \alpha$ and $\alpha \leqslant 1 - eCl(y_\beta)(x)$. So $x_\alpha \in (eCl(y_\beta))^c$. But x_α is fuzzy e-generalized closed and $(eCl(y_\beta))^c$ is fuzzy e-open. Hence, $eCl(x_\alpha) \leqslant ((eCl(y_\beta))^c$. By assumption, we have $y_\beta \in eCl(x_\alpha)$. Thus, $y_\beta \in (eCl(y_\beta))^c$. We prove that this is a contradiction. Indeed, we have

$$\beta \leqslant 1 - eCl(y_\beta)(y) \quad \text{or} \quad eCl(y_\beta)(y) \leqslant 1 - \beta.$$

Also, $y_\beta \in eCl(y_\beta)$. Thus, $\beta \leqslant 1 - \beta$. But y_β is a strongly fuzzy point; that is, $\beta > \frac{1}{2}$. So the above relation $\beta \leqslant 1 - \beta$ is a contradiction. Hence, $x_\alpha \in eCl(y_\beta)$. ■

Theorem 3.23 Let μ be a fuzzy set of a fuzzy topological space X. Then $\mu \vee e\text{-}bd(\mu) \leqslant eCl(\mu)$.

Proof. Let $x_\alpha \in \mu \vee e\text{-}bd(\mu)$. Then $x_\alpha \in \mu$ or $x_\alpha \in e\text{-}bd(\mu)$. If $x_\alpha \in e\text{-}bd(\mu)$, then $x_\alpha \in eCl(\mu)$. Let us suppose that $x_\alpha \in \mu$. We have

$$eCl(\mu) = \bigwedge\{\rho : \mu \leqslant \rho \ and\ \rho\ is\ fec\}.$$

So if $x_\alpha \in \mu$, then $x_\alpha \in \rho$, for any fec set ρ of X for which $\mu \leqslant \rho$ and $x_\alpha \in eCl(\mu)$. ■

Definition 3.24 A fuzzy point x_α in a fuzzy topological space X is said to be:

(i) well fuzzy e-closed if there exists $y_\beta \in eCl(x_\alpha)$ such that $supp(x_\alpha) \neq supp(y_\beta)$;
(ii) just fuzzy e-closed if the fuzzy set $eCl(x_\alpha)$ is again a fuzzy point.

Clearly, in a fuzzy e-T_1 space every fuzzy point is just fec.

Theorem 3.25 If X is a fuzzy topological space and x_α is a fuzzy e-generalized closed but well e-closed fuzzy point, then X is not fuzzy quasi e-T_1.

Proof. Let X be a fuzzy quasi e-T_1 space. By the fact that x_α is fuzzy well e-closed, there exists a fuzzy point y_β with $supp(x_\alpha) \neq supp(y_\beta)$ such that $y_\beta \in eCl(x_\alpha)$. Then there exists $\mu \in eO(X)$ such that $x_\alpha \in \mu$ and $y_\beta \notin \mu$. Therefore, $eCl(x_\alpha) \leqslant \mu$ and $y_\beta \in \mu$. But this is a contradiction and hence X cannot be fuzzy quasi e-T_1 space. ■

Theorem 3.26 Let X be a fuzzy topological space. If x_α and x_β are two fuzzy points such that $\alpha < \beta$ and x_β is fuzzy e-open, then x_α is just fuzzy e-closed if it is fuzzy eg-closed.

Proof. We prove that the fuzzy set $eCl(x_\alpha)$ is again a fuzzy point. We have $\alpha < \beta$, i.e $x_\alpha \in x_\beta$ and the fuzzy set x_β is fuzzy e-open. Since x_α is fuzzy eg-closed, we have $eCl(x_\alpha) \leqslant x_\beta$. Thus, $eCl(x_\alpha)(x) \leqslant \beta$ and $eCl(x_\alpha)(z) \leqslant 0$, for every $z \in X\backslash\{x\}$. So the fuzzy set $eCl(x_\alpha)$ is a fuzzy point. ■

4. Fuzzy upper and lower e-limit sets

Definition 4.1 Let $\{\mu_i : i \in I\}$ be a net of fuzzy sets in a fuzzy topological space X. Then, by $eF\overline{\lim}_I(\mu_i)$, we denote fuzzy upper e-limit of the net $\{\mu_i : i \in I\}$ in X; that is, the fuzzy set which is the union of all fuzzy points x_α in X such that for every $i_0 \in I$ and for every fuzzy e-q-neighborhood μ of x_α in X there exists an element $i \in I$ for which $i \geqslant i_0$ and $\mu_i q \mu$. In other case, we get $eF\ \overline{\lim}_I(\mu_i) = 0$.

Theorem 4.2 Let $\{\mu_i : i \in I\}$ and $\{\rho_i : i \in I\}$ be two nets of fuzzy sets in X. Then the following properties hold:

(i) The fuzzy upper e-limit is fuzzy e-closed,
(ii) $eF\ \overline{\lim}_I(\mu_i) = eF\overline{\lim}_I(eCl(\mu_i))$,
(iii) If $\mu_i = \mu$ for every $i \in I$, then $eF\ \overline{\lim}_I(\mu_i) = eCl(\mu)$,
(iv) The fuzzy upper e-limit is not affected by changing a finite number of the μ_i,
(v) If $\mu_i \leqslant \rho_i$ for every $i \in I$, then $eF\ \overline{\lim}_I(\mu_i) \leqslant eF\overline{\lim}_I(\rho_i)$,
(vi) $eF\ \overline{\lim}_I(\mu_i) \leqslant eCl(\bigvee\{\mu_i : i \in I\})$,
(vii) $eF\ \overline{\lim}_I(\mu_i \vee \rho_i) = eF\overline{\lim}_I(\mu_i) \vee eF\overline{\lim}_I(\rho_i)$,
$(viii)$ $eF\ \overline{\lim}_I(\mu_i \wedge \rho_i) \leqslant eF\overline{\lim}_I(\mu_i) \wedge eF\overline{\lim}_I(\rho_i)$.

Proof. (i) It is sufficient to prove that $eCl(eF\overline{\lim}_I(\mu_i)) \leqslant eF\overline{\lim}_I(\mu_i)$. Let $x_\alpha \in eCl(eF\overline{\lim}_I(\mu_i))$ and μ be an arbitrary fuzzy e-open q-neighborhood of x_α. Then, we have, $\mu q eF\overline{\lim}_I(\mu_i)$. Hence, there exists an element $x^1 \in X$ such that $\mu(x^1) + eF\overline{\lim}_I(\mu_i)(x^1) > 1$. Let $eF\ \overline{\lim}_I(\mu_i)(x^1) = \alpha$. Then, for the fuzzy point x^1_α in X, we have $x^1_\alpha q \mu$ and $x^1_\alpha \in eF\overline{\lim}_I(\mu_i)$. Thus, for every element $i_0 \in I$, there exists $i \in I$ with $i \geqslant i_0$ such that $\mu_i q \mu$. This means that $x_\alpha \in eF\overline{\lim}_I(\mu_i)$.

(ii) Clearly, it is sufficient to prove that for every e-open set μ the condition $\mu q \mu_i$ is equivalent to $\mu q eCl(\mu_i)$. Let $\mu q \mu_i$. Then there exists an element $x \in X$ such that

$\mu(x) + \mu_i(x) > 1$. Since, $\mu_i \leqslant eCl(\mu_i)$, we have $\mu(x) + eCl(\mu_i)(x) > 1$ and therefore $\mu q eCl(\mu_i)$. Conversely, let $\mu q eCl(\mu_i)$. Then there exists an element $x \in X$ such that $\mu(x) + eCl(\mu_i)(x) > 1$. Let $eCl(\mu_i(x)) = r$. Then $x_r \in eCl(\mu_i)$ and the fuzzy e-open set μ is a fuzzy e-q-neighborhood of x_r. Hence, $\mu q \mu_i$.

(iii) If $\mu_i = \mu$ for every $i \in I$, then by (ii) and Theorem 4.1 of [15],

$$eF\ \overline{\lim}_I(\mu_i) = eF\overline{\lim}_I(eCl(\mu_i)) = eF\overline{\lim}_I(eCl(\mu)) = eCl(\mu).$$

(iv) It follows from Definition 4.1.

(v) It is obvious.

(vi) Let $x_r \in eF\overline{\lim}_I(\mu_i)$ and μ be a fuzzy e-q-neighborhood of x_r in X. Then for every $i_0 \in I$ there exists $i \in I$ with $i \geqslant i_0$ such that $\mu_i q \mu$ and therefore $\bigvee\{\mu_i : i \in I\} q \mu$. Thus, $x_r \in eCl(\bigvee\{\mu_i : i \in I\})$.

(vii) Clearly, $\mu_i \leqslant \mu_i \vee \rho_i$ and $\rho_i \leqslant \mu_i \vee \rho_i$ for every $i \in I$. Hence, by (v), $eF\ \overline{\lim}_I(\mu_i) \leqslant eF\overline{\lim}_I(\mu_i \vee \rho_i)$ and $eF\ \overline{\lim}_I(\rho_i) \leqslant eF\overline{\lim}_I(\mu_i \vee \rho_i)$. Thus, $eF\ \overline{\lim}_I(\mu_i) \vee eF\overline{\lim}_I(\rho_i) \leqslant eF\overline{\lim}_I(\mu_i \vee \rho_i)$. Conversely, let $x_r \in eF\overline{\lim}_I(\mu_i \vee \rho_i)$. We prove that $x_r \in eF\overline{\lim}_I(\mu_i) \vee eF\overline{\lim}_I(\rho_i)$. Let us suppose that $x_r \notin eF\overline{\lim}_I(\mu_i) \vee eF\overline{\lim}_I(\rho_i)$. Then $x_r \notin eF\overline{\lim}_I(\mu_i)$ and $x_r \notin eF\overline{\lim}_I(\rho_i)$. Hence, there exists a fuzzy e-q-neighborhood μ_1 of x_r and an element $i_1 \in I$ such that $\mu_i \overline{q} \mu_1$, for every $i \in I$, $i \geqslant i_1$. Also, there exists a fuzzy e-q-neighborhood μ_2 of x_r and an element $i_2 \in I$ such that $\rho_i \overline{q} \mu_2$, for every $i \in I$, $i \geqslant i_2$. Let $\mu = \mu_1 \vee \mu_2$ and $i_0 \in I$ such that $i_0 \geqslant i_1$ and $i_0 \geqslant i_2$. Then the fuzzy set μ is a fuzzy e-q-neighborhood of x_r and $(\mu_i \vee \rho_i)\overline{q}\mu$ for every $i \in I$, $i \geqslant i_0$. Since, $x_r \in eF\overline{\lim}_I(\mu_i \vee \rho_i)$, this is a contradiction. Thus, $x_r \in eF\overline{\lim}_I(\mu_i) \vee eF\overline{\lim}_I(\rho_i)$.

(viii) Straightforward. ■

Theorem 4.3 Let $\{\mu_i : i \in I\}$ be a net of fuzzy sets in X. Then we have $eF\ \overline{\lim}_I(\mu_i) = \bigwedge\{eCl(\bigvee\{\mu_i : i \geqslant i_0\}) : i_0 \in I\}$.

Proof. Let $x_r \in eF\overline{\lim}_I(\mu_i)$ and $i_0 \in I$. We prove that $x_r \in \{eCl(\bigvee\{\mu_i : i \geqslant i_0\}) : i_0 \in I\}$. Let μ be an arbitrary fuzzy e-q-neighborhood of x_r in X. Then there exists $i \in I$ with $i \geqslant i_0$ such that $\mu q \mu_i$. Thus, $\mu q \bigvee\{\mu_i : i \geqslant i_0\}$ and $x_r \in \{eCl(\bigvee\{\mu_i : i \geqslant i_0\}) : i_0 \in I\}$.

Conversely, let $x_r \in \bigwedge\{eCl(\bigvee\{\mu_i : i \geqslant i_0\}) : i_0 \in I\}$. Then we have $x_r \in eCl(\bigvee\{\mu_i : i \geqslant i_0\})$, for every $i_0 \in I$. We prove that $x_r \in eF\overline{\lim}_I(\mu_i)$. Let μ be an arbitrary fuzzy e-q-neighborhood of x_r in X and let $i_0 \in I$. Then, $\mu q \bigvee\{\mu_i : i \geqslant i_0\}$. We prove that there exists $i \in I$, $i \geqslant i_0$ such that $\mu_i q \mu$. Let us suppose that $\mu \overline{q} \mu_i$, for every $i \in I$, $i \geqslant i_0$. Then, for every $i \in I$, $i \geqslant i_0$ and for every $x \in X$ we have $\mu(x) + \mu_i(x) \leqslant 1$ and therefore

$$\mu(x) + (\bigvee\{\mu_i : i \geqslant i_0\}(x)) \leqslant 1,$$

which is a contradiction. Thus $\mu q \mu_i$. Hence, $x_r \in eF\overline{\lim}_I(\mu_i)$. ■

Theorem 4.4 Let $\{\mu_i : i \in I\}$ be a net of fuzzy e-closed sets in X such that $\mu_{i_1} \leqslant \mu_{i_2}$ if and only if $i_2 \leqslant i_1$. Then $eF\ \overline{\lim}_I(\mu_i) = \bigwedge\{\mu_i : i \in I\}$.

Proof. Let $x_r \in \bigwedge\{\mu_i : i \in I\}$. Then $x_r \in \mu_i$ or $r \leqslant \mu_i(x)$ for every $i \in I$. Let $i_0 \in I$ and μ be a fuzzy e-q-neighborhood of x_r, that is, $r + \mu(x) > 1$. Then there exists $i \in I$ with $i \geqslant i_0$ such that $\mu_i(x) + \mu(x) \geqslant r + \mu(x) > 1$. Hence, $\mu_i q \mu$ and therefore $x_r \in eF\overline{\lim}_I(\mu_i)$.

Conversely, let $x_r \in eF\overline{\lim}_I(\mu_i)$ and let $x_r \notin \{\mu_i : i \in I\}$. Then there exists $i_0 \in i$ such that $x_r \notin \mu_{i_0}$, that is, $r > \mu_{i_0}(x)$. Let $\mu = \mu_{i_0}^c$. This implies $x_r \in \mu_{i_0}^c$. Then μ is a fuzzy e-q-neighborhood of x_r in X and for every $i \geqslant i_0$, $\mu \overline{q} \mu_i$, which is a contradiction. Therefore, $x_r \in \bigwedge\{\mu_i : i \in I\}$. ■

Theorem 4.5 A net $\{f_i : i \in I\}$ in $eC(X,Y)$ fuzzy e-continuously converges to $f \in eC(X,Y)$ if and only if $eF\overline{\lim}_I(f_i^{-1}(\beta)) \leqslant f^{-1}(\beta)$ for every fuzzy e-closed subset β of Y.

Proof. Let $\{f_i : i \in I\}$ be a net in $eC(X,Y)$, which fuzzy e-continuously converges to f and β be an arbitrary fuzzy e-closed subset of Y. Let $x_r \in eF\overline{\lim}_I(f_i^{-1}(\mu))$ and μ be an arbitrary fuzzy e-q-neighborhood of $f(x_r)$ in Y. Since the net $\{f_i : i \in I\}$ fuzzy e-continuously converges to f, there exists a fuzzy e-q-neighborhood ρ of x_r in X and an element $i_0 \in I$ such that $f_i(\rho) \leqslant \mu$ for every $i \in I$ with $i \geqslant i_0$ by Theorem 3.7. On the other hand, there exists an element $i \geqslant i_0$ such that $\rho q f_i^{-1}(\beta)$. Hence, $f_i(\rho) q \beta$ and therefore $\mu q \beta$. This means that $f(x_r) \in eCl(\beta) = \beta$. Thus $x_r \in f^{-1}(\beta)$.

Conversely, let $\{f_i : i \in I\}$ be a net in $eC(X,Y)$ and $f \in eC(X,Y)$ such that $eF\overline{\lim}_I(f_i^{-1}(\beta)) \leqslant f^{-1}(\beta)$ for every fuzzy e-closed subset β of Y. We prove that the net $\{f_i : i \in I\}$ fuzzy e-continuously converges to f. Let x_r be a fuzzy point of X and μ be a fuzzy e-q-neighborhood of $f(x_r)$ in Y. Since $x_r \notin f^{-1}(\mu)$ we have $x_r \notin eF\overline{\lim}_I(f_i^{-1}(\beta))$, where $\beta = \mu^c$. This means that, there exists an element $i_0 \in I$ and a fuzzy e-q-neighborhood ρ of x_r in X such that $f_i^{-1}(\beta)\overline{q}\rho$ for every $i \in I$ with $i \geqslant i_0$. Then we have $\rho \leqslant (f_i^{-1}(\beta))^c = f_i^{-1}(\beta^c) = f_i^{-1}(\mu)$ and therefore, $f_i(\rho) \leqslant \mu$ for every $i \in I$ with $i \geqslant i_0$; that is, the net $\{f_i : i \in I\}$ fuzzy e-continuously converges to f. ■

Theorem 4.6 The following properties hold:

(*i*) If $\{f_i | i \in I\}$ is a net in $eC(X,Y)$ such that $f_i = f$ for every $i \in I$, then the $\{f_i | i \in I\}$ fuzzy e-continuously converges to $f \in eC(X,Y)$.

(*ii*) If $\{f_i | i \in I\}$ is a net in $eC(X,Y)$ which fuzzy e-continuously converges to $f \in eC(X,Y)$ and $\{g_i | i \in J\}$ be a subnet of $\{f_i | i \in I\}$, then the net $\{g_i | i \in J\}$ fuzzy e-continuously converges to f.

(*iii*) If $\{f_i | i \in I\}$ is a net in $eC(X,Y)$ which does not fuzzy e-continuously converges to $f \in eC(X,Y)$, then there exists no subset of $\{f_i | i \in I\}$, which fuzzy continuously converges to f.

Proof. (i) and (ii) are obvious. Now, we prove (iii).
(iii) Since the fuzzy net $\{f_i : i \in I\}$ does not fuzzy e-continuously converges to f by Theorem 4.5, there exits a fuzzy e-closed set $\beta \in Y$ such that $eF\overline{\lim}_I(f_i^{-1}(\beta)) \nleq f^{-1}(\beta)$. Hence, there exists $x \in X$ such that

$$f^{-1}(\beta)(x) \leqslant eF\overline{\lim}_I(f_i^{-1}(\beta))(x).$$

Let $f^{-1}(\beta)(x) = r$. Then, for the fuzzy point x_r, we have $x_r \in f^{-1}(\beta)$ and therefore, $x_r \in eF\overline{\lim}_I(f_i^{-1}(\beta))$. Let μ be an arbitrary fuzzy open q-neighborhood of x_r in X. Let $N = I \times N(x_r)$ and ϕ be a map of N into I defined as follows: If $n = (i,\ \mu) \in N$, then by $\phi(n)$ we denote an element i_0 of I such that $i_0 \geqslant i$ and $f_i^{-1}(\beta) q \mu$. Clearly, the net $\{g_n = f_{\phi(n)} : n \in N\}$ is a subnet of $\{f_i : i \in I\}$. Let $\{h_t : t \in T\}$ be an arbitrary subnet of $\{g_n : n \in N\}$. We prove that the net $\{h_t : t \in T\}$ does not fuzzy e-continuously converge to f. Obviously, for this it is sufficient to prove that $x_r \in eF\overline{\lim}_I(h_t^{-1}(\beta))$. Since the net $\{h_t : t \in T\}$ is a subnet of $\{g_n : n \in N\}$, there exists a map $\chi : T \to N$ such that
(i) $h_t = g_{\chi(t)}$, $\forall t \in T$ and
(ii) For every element $n_1 \in N$, there exists $t \in T$ such that if $t \in T$, $t \geqslant t_1$, then $\chi(t) \geqslant n_1$.

Now, let $t_0 \in T$ and μ be an arbitrary fuzzy open q-neighborhood of x_r in X. We prove that there exists $t \in T$ with $t \geqslant t_0$ such that $h_t^{-1}(\beta) q \mu$. Indeed, let $\chi(t_0) = n_0 = (i_0,\ \mu_0)$, $\gamma = \mu \wedge \mu_0$ and $n_1 = (i_0,\ \gamma_0)$. Then there exists an element $t_1 \in T$, $t_1 \geqslant t_0$ such that if $t \in T$, $t \geqslant t_1$, then $\chi(t) \geqslant n_1 \geqslant n_0$. Let $t \in T$, $t \geqslant t_1$ and $\chi(t) = n = (i,\ \rho)$. Then

(iii) $h_t^{-1}(\beta) = g^{-1}{}_{\chi(t)}(\beta) = f^{-1}_{\phi(\chi(t))}(\beta)$.

(iv) $f^{-1}_{\phi(\chi(t))}(\beta)\ q\ \rho$.

Since $\chi(t) = n = (i,\ \rho) \geqslant n_1 = (i_0,\ \gamma_0)$ we have that $\rho \leqslant \gamma_0 \leqslant \mu$. By the above relation and by relations (iii) and (iv), we have that $h_t^{-1}(\beta)q\rho$ and $h_t^{-1}(\beta)q\mu$, where $t \in T$ with $t \geqslant t_0$. Thus, $x_r \in eF\overline{\lim}_I(h_t^{-1}(\beta))$. ■

Definition 4.7 Let $\{\mu_i : i \in I\}$ be a net of fuzzy sets in a fuzzy topological space X. Then, by $eF\ \underline{\lim}_I(\mu_i)$, we denote the fuzzy lower e-limit of the net $\{\mu_i : i \in I\}$ in X; that is, the fuzzy set which is the union of all fuzzy points x_r in X such that for every fuzzy e-q-neighborhood μ of x_r in X there exists an element $i_0 \in I$ such that $\mu_i q\mu$ for every $i \in I$ and $i \geqslant i_0$. In other case, we get $eF\ \underline{\lim}_I(\mu_i) = 0$.

Theorem 4.8 For the fuzzy upper and lower e-limits, we have $eF\ \underline{\lim}_I(\mu_i) \leqslant eF\overline{\lim}_I(\mu_i)$.

The proof follows from Definitions 4.1 and 4.7.

Theorem 4.9 Let $\{\mu_i : i \in I\}$ be a net of fuzzy sets in X such that $\mu_{i_1} \leqslant \mu_{i_2}$ if and only if $i_1 \leqslant i_2$. Then $eCl(\bigvee\{\mu_i : i \in I\}) = eF\underline{\lim}_I(\mu_i)$.

Proof. Let $x_r \in eCl(\bigvee\{\mu_i : i \in I\})$ and μ be a fuzzy e-q-neighborhood of x_r in X. Then $\mu q \bigvee\{\mu_i : i \in I\}$. Hence, there exists an element $i_0 \in I$ such that $\mu q\mu_0$. By assumption, we have $\mu q\mu_i$ for every $i \in I$ with $i \geqslant i_0$. Thus, $x_r \in eF\underline{\lim}_I(\mu_i)$. This implies $eCl(\bigvee\{\mu_i : i \in I\}) \leqslant eF\underline{\lim}_I(\mu_i)$.

Conversely, let $x_r \in eF\underline{\lim}_I(\mu_i)$ and μ be an arbitrary fuzzy e-q-neighborhood of x_r in X. Then there exists an element $i_0 \in I$ such that $\mu q\mu_i$ for every $i \in I$ with $i \geqslant i_0$. Hence, $\mu q \bigvee\{\mu_i : i \in I\}$ and therefore $x_r \in eCl(\bigvee\{\mu_i : i \in I\})$. Thus $eF\underline{\lim}_I(\mu_i) \leqslant eCl(\bigvee\{\mu_i : i \in I\})$. Hence $eF\underline{\lim}_I(\mu_i) = eCl(\bigvee\{\mu_i : i \in I\})$. ■

Theorem 4.10 Let $\{\mu_i : i \in I\}$ and $\{\rho_i : i \in I\}$ be two nets of fuzzy sets in X. Then the following properties hold:

(*i*) The fuzzy lower e-limit is fuzzy e-closed,
(*ii*) $eF\ \underline{\lim}_I(\mu_i) = eF\underline{\lim}_I(eCl(\mu_i))$,
(*iii*) IF $\mu_i = \mu$ for every $i \in I$, then $eF\ \underline{\lim}_I(\mu_i) = eCl(\mu)$,
(*iv*) The fuzzy lower e-limit is not affected by changing a finite number of the μ_i,
(*v*) If $\mu_i \leqslant \rho_i$ for every $i \in I$, then $eF\ \underline{\lim}_I(\mu_i) \leqslant eF\underline{\lim}_I(\rho_i)$,
(*vi*) $eF\ \underline{\lim}_I(\mu_i) \leqslant eCl(\vee\{\mu_i : i \in I\})$,
(*vii*) $eF\ \underline{\lim}_I(\mu_i \vee \rho_i) \geqslant eF\underline{\lim}_I(\mu_i) \vee eF\underline{\lim}_I(\rho_i)$,
(*viii*) $eF\ \underline{\lim}_I(\mu_i \wedge \rho_i) \leqslant eF\underline{\lim}_I(\mu_i) \wedge eF\underline{\lim}_I(\rho_i)$,
(*ix*) $\wedge\{\mu_i : i \in I\} \leqslant eF\underline{\lim}_I(\mu_i)$,
(*x*) $\vee\{\wedge\{\mu_i : i \geqslant i_0\} : i_0 \in I\} \leqslant eF\underline{\lim}_I(\mu_i)$.

Proof. (i) It is sufficient to prove that $eCl(eF\overline{\lim}_I(\mu_i)) \leqslant eF\underline{\lim}_I(\mu_i)$. Let $x_\alpha \in eCl(eF\underline{\lim}_I(\mu_i))$ and let μ be an arbitrary fuzzy e-open q-neighborhood of x_α. Then we have $\mu qeF\underline{\lim}_I(\mu_i)$. Hence, there exists an element $x^1 \in X$ such that $\mu(x^1) + eF\underline{\lim}_I(\mu_i)(x^1) > 1$. Let $eF\ \underline{\lim}_I(\mu_i)(x^1) = \alpha$. Then, for the fuzzy point x^1_α in X, we have $x^1_\alpha q\mu$ and $x^1_\alpha \in eF\underline{\lim}_I(\mu_i)$. Thus, for every element $i_0 \in I$, there exists $i \geqslant i_0$, $i \in I$ such that $\mu_i q\mu$. This means that $x_\alpha \in eF\underline{\lim}_I(\mu_i)$.

(ii) and (iii) are similar to Theorem 4.2. (iv) follows from the Definition 4.7. (v) is obvious.

(vi) Let $x_r \in eF\underline{\lim}_I(\mu_i)$ and let μ be a fuzzy e-q-neighborhood of x_r in X. Then, for every $i_0 \in I$, there exists $i \in I$ with $i \geqslant i_0$ such that $\mu_i q\mu$ and therefore $\bigvee\{\mu_i : i \in I\}q\mu$. Thus, $x_r \in eCl(\bigvee\{\mu_i : i \in I\})$.

(vii) Let $x_r \in eF\underline{\lim}_I(\mu_i) \bigvee eF\underline{\lim}_I(\rho_i)$. Then either $x_r \in eF\underline{\lim}_I(\mu_i)$ or $x_r \in eF\underline{\lim}_I(\rho_i)$. Let $x_r \in eF\underline{\lim}_I(\mu_i)$. Then, for every fuzzy e-q-neighborhood μ of x_r in X, there exists an element $i_0 \in I$ such that $\mu_i q \mu$, for every $i \in I$, $i \geqslant i_0$. Also $\mu_i \leqslant \mu_i \vee \rho_i$. Thus, $(\mu_i \vee \rho_i) q \mu$ for every $i \in I$, $i \geqslant i_0$ and therefore, $x_r \in eF\underline{\lim}_I(\mu_i \vee \rho_i)$.

(viii) Let $x_r \in eF\underline{\lim}_I(\mu_i \bigwedge \rho_i)$. Then, for every fuzzy e-q-neighborhood μ of x_r in X, there exists an element $i_0 \in I$ such that $\mu_i q \mu$ for every $i \in I$ with $i \geqslant i_0$. Also, $\mu_i \wedge \rho_i \leqslant \mu_i$ and $\mu_i \wedge \rho_i \leqslant \rho_i$. By (v), $eF\underline{\lim}_I(\mu_i \bigwedge \rho_i) \leqslant eF\underline{\lim}_I(\mu_i)$ and $eF\underline{\lim}_I(\mu_i \bigwedge \rho_i) \leqslant eF\underline{\lim}_I(\rho_i)$. Thus, $eF\ \underline{\lim}_I(\mu_i \wedge \rho_i) \leqslant eF\underline{\lim}_I(\mu_i) \wedge eF\underline{\lim}_I(\rho_i)$.

(ix) Let $x_r \in \bigwedge\{\mu_i : i \in I\}$. We prove that $x_r \in eF\underline{\lim}_I(\mu_i)$. Let us suppose that $x_r \notin eF\underline{\lim}_I(\mu_i)$. Then there exists a fuzzy e-q-neighborhood μ of x_r such that for every $i \in I$ there exists $i_0 \geqslant i$ for which $\mu_i \overline{q} \mu$. This means that $\mu_{i_0}(x) + \mu(x) \leqslant 1$ for every $x \in X$. Now, since $x_r \in \bigwedge\{\mu_i : i \in I\}$ and μ is a fuzzy e-q-neighborhood of x_r we have $r \leqslant \mu_i(x)$ for every $i \in I$ and $r + \mu(x) > 1$. Thus, $\mu_i(x) + \mu(x) > 1$, for every $i \in I$. By the above, this is a contradiction. Hence, $x_r \in eF\underline{\lim}_I(\mu_i)$.

(x) Let $x_r \in \{\bigwedge\{\mu_i : i \geqslant i_0\} : i_0 \in I\}$. Then there exists $i_0 \in I$ such that $x_r \in \bigwedge\{\mu_i : i \geqslant i_0\}$. Hence, $x_r \in \mu_i$ for every i, $i \geqslant i_0$ and therefore, $r \leqslant \mu_i(x)$ for every $i \in I$ with $i \geqslant i_0$. We prove that $x_r \in eF\underline{\lim}_I(\mu_i)$. Let μ be an arbitrary fuzzy e-q-neighborhood of x_r in Y. Then we have $x_r q \mu$ or equivalently $r + \mu(x) > 1$. Since $r \leqslant \mu_i(x)$, for every $i \in I$ with $i \geqslant i_0$ we have that $\mu_i(x) + \mu(x) > 1$ for every $i \in I$ with $i \geqslant i_0$. Thus, $\mu_i q \mu$ for every $i \in I$ with $i \geqslant i_0$ and therefore, $x_r \in eF\underline{\lim}_I(\mu_i)$. ■

Definition 4.11 A net $\{\mu_i : i \in I\}$ of fuzzy sets in a fuzzy topological space X is said to be fuzzy e-convergent to the fuzzy set μ if $eF\ \underline{\lim}_I(\mu_i) = eF\overline{\lim}_I(\mu_i) = \mu$. We write eF-$\lim_I(\mu_i) = \mu$.

Theorem 4.12 Let $\{\mu_i : i \in I\}$ be a e-convergent net of fuzzy sets in X.

(i) If $\mu_{i_1} \geqslant \mu_{i_2}$ for $i_1 \leqslant i_2$, then $eF\ \lim_I(\mu_i) = \bigwedge\{eCl(\mu_i) : i \in I\}$.
(ii) If $\mu_{i_1} \leqslant \mu_{i_2}$ for $i_1 \leqslant i_2$, then $eF\ \lim_I(\mu_i) = eCl(\bigvee\{\mu_i : i \in I\})$.

Proof. (i) By Theorems 4.2, 4.4 and 4.10, we have

$$\begin{aligned}\bigwedge\{eCl(\mu_i) : i \in I\} &\leqslant eF\underline{\lim}_I(eCl(\mu_i))\\ &= eF\underline{\lim}_I(\mu_i)\\ &\leqslant eF\overline{\lim}_I(\mu_i)\\ &= eF\overline{\lim}_I(eCl(\mu_i))\\ &= \bigwedge\{eCl(\mu_i) : i \in I\}.\end{aligned}$$

Thus, $eF\ \lim_I(\mu_i) = \bigwedge\{eCl(\mu_i) : i \in I\}$.

(ii) By Theorem 4.2 and 4.9, we have

$$\begin{aligned}eCl(\bigvee\{\mu_i : i \in I\}) &= eF\underline{\lim}_I(\mu_i)\\ &\leqslant eF\overline{\lim}_I(\mu_i)\\ &\leqslant eCl(\bigvee\{\mu_i : i \in I\}).\end{aligned}$$

Thus, $eF\ \lim_I(\mu_i) = eCl(\bigvee\{\mu_i : i \in I\})$. ■

Theorem 4.13 Let $\{\mu_i : i \in I\}$ and $\{\rho_i : i \in I\}$ be two e-convergent net of fuzzy sets in X. Then the following properties hold:

(i) If $\mu_i \leqslant \rho_i$ for every $i \in I$, then $eF\ \lim_I(\mu_i) \leqslant eF \lim_I(\rho_i)$,
(ii) $eF\ \lim_I(\mu_i \vee \rho_i) = eF \lim_I(\mu_i) \vee eF \lim_I(\rho_i)$,
(iii) $eCl(eF \lim_I(\mu_i)) = eF \lim_I(\mu_i) = eF \lim_I(eCl(\mu_i))$,
(iv) If $\mu_i = \mu$ for every $i \in I$, then $eF\ \lim_I(\mu_i) = eCl(\mu)$.

Proof. (i) follows by Theorems 4.2 and 4.10.

(ii) By Theorem 4.2 and 4.10, we have

$$\begin{aligned} eF\ \overline{\lim_I}(\mu_i \vee \rho_i) &= eF\overline{\lim_I}(\mu_i) \vee eF\overline{\lim_I}(\rho_i) \\ &\leqslant eF\underline{\lim_I}(\mu_i) \vee eF\underline{\lim_I}(\rho_i) \\ &\leqslant eF\underline{\lim_I}(\mu_i \vee \rho_i) \\ &\leqslant eF\overline{\lim_I}(\mu_i \vee \rho_i) \\ &= eF\lim_I(\mu_i) \vee eF\lim_I(\rho_i). \end{aligned}$$

Thus, $eF\ \lim_I(\mu_i \vee \rho_i) = eF\lim_I(\mu_i) \vee eF\lim_I(\rho_i)$.

(iii) Take $\mu = eF\lim_I(\mu_i) = eCl(\mu)$. Then, by Theorem 4.10 (iii), $eF\underline{\lim_I}(\mu_i) = eCl(\mu)$. This implies $eCl(eF\lim_I(\mu_i)) = eF\lim_I(\mu_i)$. Then, by Theorem 4.10 (ii), $eF\underline{\lim_I}(\mu_i) = eF\underline{\lim_I}(eCl(\mu_i))$. This implies that $eF\lim_I(\mu_i) = eF\lim_I(eCl(\mu_i))$.

(iv) follows by Theorems 4.2 and 4.10. ■

Theorem 4.14

(i) Let μ_1, $\mu \in I^X$ and μ_2, $\rho \in I^Y$. If $(\mu_1 \times \mu_2)q(\mu \times \rho)$, then $\mu_1 q\mu$ and $\mu_2 q\rho$.

(ii) Let μ_1 and μ_2 be fuzzy e-q-neighborhoods of x_r and y_r in X and Y respectively. Then the fuzzy set $\mu_1 \times \mu_2$ is a fuzzy e-q-neighborhood of $(x,\ y)_r$ in $X \times Y$.

Theorem 4.15 Let $\{\mu_i : i \in I\}$ and $\{\rho_i : i \in I\}$ be two nets of fuzzy sets in X. Then the following properties hold:

(i) $eF\ \overline{\lim}_I(\mu_i \times \rho_i) \leqslant eF\ \overline{\lim}_I(\mu_i) \times eF\ \overline{\lim}_I(\rho_i)$.

(ii) $eF\ \underline{\lim}_I(\mu_i \times \rho_i) \leqslant eF\ \underline{\lim}_I(\mu_i) \times eF\ \underline{\lim}_I(\rho_i)$.

(iii) If $\{\mu_i : i \in I\}$ and $\{\rho_i : i \in I\}$ are e-convergent nets, then $eF\ \lim_I(\mu_i \times \rho_i) \leqslant eF\ \lim_I(\mu_i) \times eF\ \lim_I(\rho_i)$.

Proof. (i) Let $(x,y)_r \in eF\ \overline{\lim}_I(\mu_i \times \rho_i)$. We must prove that $(x,y)_r \in eF\ \overline{\lim}_I(\mu_i) \times eF\ \overline{\lim}_I(\rho_i)$ or equivalently $r \leqslant (eF\overline{\lim}_I(\mu_i) \times eF\overline{\lim}_I(\rho_i))(x,y)$. Let $i_0 \in I$, μ_1 be an arbitrary fuzzy e-q-neighborhood of x_r in X and μ_2 be a constant fuzzy e-q-neighborhood of y_r in Y. Then the fuzzy set $\mu_1 \times \mu_2$ is a fuzzy e-q-neighborhood of $(x,y)_r$ in $X \times Y$. Hence, there exists $i \in I$ with $i \geqslant i_0$ such that $(\mu_1 \times \mu_2)q(\mu_i \times \rho_i)$, we have $\mu_1 q\mu_i$ and $\mu_2 q\rho_i$. Thus, $x_r \in eF\overline{\lim}_I(\mu_i)$. Similarly, we can prove that $y_r \in eF\ \overline{\lim}_I(\rho_i)$. Hence, $(x,y)_r \in eF\overline{\lim}_I(\mu_i) \times eF\ \overline{\lim}_I(\rho_i)$.

(ii) Let $(x,y)_r \in eF\ \underline{\lim}_I(\mu_i \times \rho_i)$. We must prove that $(x,y)_r \in eF\ \underline{\lim}_I(\mu_i) \times eF\ \underline{\lim}_I(\rho_i)$ or equivalently $r \leqslant (eF\underline{\lim}_I(\mu_i) \times eF\underline{\lim}_I(\rho_i))(x,y)$. Let $i_0 \in I$, μ_1 be an arbitrary fuzzy e-q-neighborhood of x_r in X and μ_2 be a constant fuzzy e-q-neighborhood of y_r in Y. Then, the fuzzy set $\mu_1 \times \mu_2$ is a fuzzy e-q-neighborhood of $(x,y)_r$ in $X \times Y$. Hence, there exists $i \in I$ with $i \geqslant i_0$ such that $(\mu_1 \times \mu_2)q(\mu_i \times \rho_i)$ and we have $\mu_1 q\mu_i$ and $\mu_2 q\rho_i$. Thus, $x_r \in eF\underline{\lim}_I(\mu_i)$. Similarly, we can prove that $y_r \in eF\ \underline{\lim}_I(\rho_i)$. Hence, $(x,y)_r \in eF\underline{\lim}_I(\mu_i) \times eF\ \underline{\lim}_I(\rho_i)$.

(iii) Since $\{\mu_i : i \in I\}$ and $\{\rho_i : i \in I\}$ are e-convergent nets, $eF\ \overline{\lim}_I(\mu_i) = eF\ \lim_I(\mu_i)$ and $eF\ \overline{\lim}_I(\rho_i) = eF\ \lim_I(\rho_i)$. Also, $eF\ \overline{\lim}_I(\mu_i \times \rho_i) = eF\ \lim_I(\mu_i \times \rho_i)$. Thus (iii) proved. ■

5. Conclusion

In this paper, fuzzy upper and lower e-limit sets are studied via fuzzy e-open sets. The initiations of e-open sets and related studies in topological spaces are due to Ekici [7–11]. This present paper contains the studies of fuzzy e-openness. Also, the present paper is related to [6] for fuzzy limit sets. So, we introduce and study the notions of fuzzy upper e-limit set, lower e-limit set and fuzzy e-continuously convergent functions. Properties and basic relationships among fuzzy upper e-limit set, fuzzy lower e-limit set and fuzzy e-continuity are investigated using fuzzy e-open sets.

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