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2-Banach stability results for the radical cubic functional equation related to quadratic mapping

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Abstract. The aim of this paper is to introduce and solve the generalized radical cubic functional equation related to quadratic functional equation

$$f\left(\sqrt[3]{ax^3+by^3}\right)+f\left(\sqrt[3]{ax^3-by^3}\right)=2a^2f(x)+2b^2f(y),\quad x,y\in\mathbb{R},$$

for a mapping f from $\mathbb{R}$ into a vector space. We also investigate some stability and hyperstability results for the considered equation in 2-Banach spaces by using an analogue theorem of Brzdęk in [17].

Keywords: Stability, hyperstability, 2-Banach spaces, radical functional equation.

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1. Introduction

Throughout this paper, we will denote the set of natural numbers by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$, $\mathbb{R}_+=[0,\infty)$ the set of non negative real numbers and $\mathbb{R}_0=\mathbb{R}\backslash\{0\}$. By $\mathbb{N}_m$ for $m\in\mathbb{N}$, we will denote the set of all natural numbers greater than or equal to m.

The notion of linear 2-normed spaces was introduced by Gähler [21, 22] in the middle of 1960. We need to recall some basic facts concerning 2-normed spaces and some preliminary results.

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Definition 1.1 Let X be a real linear space with $dimX > 1$ and $\|.,.\| : X \times X \longrightarrow \mathbb{R}_+$ be a function satisfying the following properties:

(1) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
(2) $\|x, y\| = \|y, x\|$,
(3) $\|\lambda x, y\| = |\lambda| \|x, y\|$,
(4) $\|x + y, z\| \leqslant \|x, z\| + \|y, z\|$,

for all $x, y, z \in X$ and $\lambda \in \mathbb{R}$. Then the function $\|.,.\|$ is called a 2-norm on X and the pair $(X, \|.,.\|)$ is called a linear 2-normed space. Sometimes the condition (4) called the triangle inequality.

***Example* 1.2** For $x = (x_1, x_2)$, $y = (y_1, y_2) \in X = \mathbb{R}^2$, the Euclidean 2-norm $\|x, y\|_{\mathbb{R}^2}$ is defined by

$$\|x, y\|_{\mathbb{R}^2} = |x_1 y_2 - x_2 y_1| .$$

Definition 1.3 A sequence $\{x_k\}$ in a 2-normed space X is called a convergent sequence if there is an $x \in X$ such that

$$\lim_{k\to\infty} \|x_k - x, y\| = 0,$$

for all $y \in X$. If $\{x_k\}$ converges to x, write $x_k \longrightarrow x$ with $k \longrightarrow \infty$ and call x the limit of $\{x_k\}$. In this case, we also write $\lim_{k\to\infty} x_k = x$.

Definition 1.4 A sequence $\{x_k\}$ in a 2-normed space X is said to be a Cauchy sequence with respect to the 2-norm if

$$\lim_{k,l\to\infty} \|x_k - x_l, y\| = 0,$$

for all $y \in X$. If every Cauchy sequence in X converges to some $x \in X$, then X is said to be complete with respect to the 2-norm. Any complete 2-normed space is said to be a 2-Banach space.

Now, we state the following results as lemma (see [27] for the details).

Lemma 1.5 Let X be a 2-normed space. Then,

(1) $\big|\|x, z\| - \|y, z\|\big| \leqslant \|x - y, z\|$ for all $x, y, z \in X$,
(2) if $\|x, z\| = 0$ for all $z \in X$, then $x = 0$,
(3) for a convergent sequence x_n in X,

$$\lim_{n\longrightarrow\infty} \|x_n, z\| = \left\|\lim_{n\longrightarrow\infty} x_n, z\right\|$$

for all $z \in X$.

The first stability problem of functional equation was raised by Ulam [31] in 1940. This included the following question concerning the stability of group homomorphisms. Let $(G_1, *_1)$ be a group and let $(G_2, *_2)$ be a metric group with a metric $d(.,.)$. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d\big(h(x *_1 y), h(x) *_2 h(y)\big) < \delta$$

for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with

$$d\big(h(x), H(x)\big) < \varepsilon$$

for all $x \in G_1$?
If the answer is affirmative, we say that the equation of homomorphism

$$h(x *_1 y) = h(x) *_2 H(y)$$

is stable. Since then, this question has attracted the attention of many researchers. In 1941, Hyers [23] gave a first partial answer to Ulam's question and introduced the stability result as follows:

Theorem 1.6 [23] Let E_1 and E_2 be two Banach spaces and $f : E_1 \to E_2$ be a function such that

$$\|f(x+y) - f(x) - f(y)\| \leqslant \delta$$

for some $\delta > 0$ and for all $x, y \in E_1$. Then the limit

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in E_1$, and $A : E_1 \to E_2$ is the unique additive function such that

$$\|f(x) - A(x)\| \leqslant \delta$$

for all $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the function A is linear.

Later, Aoki [10] and Bourgin [11] considered the problem of stability with unbounded Cauchy differences. Rassias [29] attempted to weaken the condition for the bound of the norm of Cauchy difference

$$\|f(x+y) - f(x) - f(y)\|$$

and proved a generalization of Theorem 1.6 using a direct method (cf. Theorem 1.7):

Theorem 1.7 [29] Let E_1 and E_2 be two Banach spaces. If $f : E_1 \to E_2$ satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leqslant \theta\big(\|x\|^p + \|y\|^p\big)$$

for some $\theta \geqslant 0$, for some $p \in \mathbb{R}$ with $0 \leqslant p < 1$, and for all $x, y \in E_1$, then there exists a unique additive function $A : E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \leqslant \frac{2\theta}{2 - 2^p}\|x\|^p$$

for each $x \in E_1$. If, in addition, $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the function A is linear.

After then, Rassias [28, 30] motivated Theorem 1.7 as follows:

Theorem 1.8 Let E_1 be a normed space, E_2 be a Banach space, and $f : E_1 \to E_2$ be a function. If f satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leqslant \theta\big(\|x\|^p + \|y\|^p\big) \tag{1}$$

for some $\theta \geqslant 0$, for some $p \in \mathbb{R}$ with $p \neq 1$, and for all $x, y \in E_1 - \{0_{E_1}\}$, then there exists a unique additive function $A : E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \leqslant \frac{2\theta}{|2 - 2^p|}\|x\|^p \tag{2}$$

for each $x \in E_1 - \{0_{E_1}\}$.

Note that Theorem 1.8 reduces to Theorem 1.6 when $p = 0$. For $p = 1$, the analogous result is not valid. Also, Brzdęk [12] showed that estimation (2) is optimal for $p \geqslant 0$ in the general case.

Recently, Brzdęk [13] showed that Theorem 1.8 can be significantly improved; namely, in the case $p < 0$, each $f : E_1 \to E_2$ satisfying (1) must actually be additive, this result is called the hyperstability of Cauchy functional equation. However, the term of hyperstability was introduced for the first time probably in [26], and it was developed with fixed point theorem of Brzdęk in [17].

In 2013, Brzdęk [15] improved, extended and complemented several earlier classical stability results concerning the additive Cauchy equation (in particular, Theorem 1.8). Over the last few years, many mathematicians have investigated various generalizations, extensions and applications of the Hyers-Ulam stability of a number of functional equations (see, for instance, [1, 4, 16, 18] and references therein); in particular, the stability problem of the radical functional equations in various spaces was proved in [7–9, 19, 20, 24, 25].

An analogue of [17, Theorem1] in 2-Banach spaces was stated and proved in [3].

Theorem 1.9 [3] Let X be a nonempty set, $\big(Y, \|\cdot,\cdot\|\big)$ be a 2-Banach space, $g : X \to Y$ be a surjective mapping and let $f_1, ..., f_r : X \to X$ and $L_1, ..., L_r : X \to \mathbb{R}_+$ be given mappings. Suppose that $\mathcal{T} : Y^X \to Y^X$ and $\Lambda : \mathbb{R}_+^{X\times X} \to \mathbb{R}_+^{X\times X}$ are two operators satisfying the conditions

$$\Big\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x), g(z)\Big\| \leqslant \sum_{i=1}^{r} L_i(x)\Big\|\xi\big(f_i(x)\big) - \mu\big(f_i(x)\big), g(z)\Big\| \tag{3}$$

for all $\xi, \mu \in Y^X andforallx, z \in X$, and

$$\Lambda\delta(x,z) := \sum_{i=1}^{r} L_i(x)\delta\big(f_i(x), z\big), \quad \delta \in \mathbb{R}_+^{X\times X},\ x, z \in X. \tag{4}$$

If there exist functions $\varepsilon : X \times X \to \mathbb{R}_+$ and $\varphi : X \to Y$ such that

$$\Big\|\mathcal{T}\varphi(x) - \varphi(x), g(z)\Big\| \leqslant \varepsilon(x,z) \tag{5}$$

and

$$\varepsilon^*(x,z) := \sum_{n=0}^{\infty} \big(\Lambda^n \varepsilon\big)(x,z) < \infty \tag{6}$$

for all $x, z \in X$, then

$$\lim_{n\to\infty} \big((\mathcal{T}^n \varphi)\big)(x) \tag{7}$$

exists for each $x \in X$. Moreover, the function $\psi : X \to Y$ defined by

$$\psi(x) := \lim_{n\to\infty} \big((\mathcal{T}^n \varphi)\big)(x) \tag{8}$$

is a fixed point of $\mathcal{T}$ with

$$\big\|\varphi(x) - \psi(x), g(z)\big\| \leqslant \varepsilon^*(x,z) \tag{9}$$

for all $x, z \in X$.

The following functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \ \ x, y \in \mathbb{R}, \tag{10}$$

where $f : \mathbb{R} \to X$, is called a quadratic functional equation. In particular, every solution of equation (10) is said to be a quadratic function. It is well known that a function $f : E_1 \to E_2$ between two real linear spaces E_1 and E_2 is quadratic if and only if there exists a unique symmetric biadditive function $B : E_1 \times E_1 \to E_2$ such that $f(x) = B(x,x)$ for all $x \in E_1$. The biadditive function B is given by

$$B(x,y) = \frac{1}{4}[f(x+y) + f(x-y)], \quad x, y \in E_1.$$

In this paper, we introduce and achieve the solutions of the following general radical cubic functional equation related to quadratic functional equation:

$$f\left(\sqrt[3]{ax^3 + by^3}\right) + f\left(\sqrt[3]{ax^3 - by^3}\right) = 2a^2 f(x) + 2b^2 f(x) \tag{11}$$

with $a, b \in \mathbb{Q}$ such that $a \neq 0$ and $b \neq 0$. Furthermore we investigate the generalized Hyers-Ulam-Rassias problem stability, in the spirit of Gavrouta, in 2-Banach spaces by using Theorem [3].

2. Solution of equation (11)

In this section, we give the general solution of functional equation (11).

Lemma 2.1 Let X be a linear space and $a, b \in \mathbb{Q} \setminus \{0\}$. A function $f : \mathbb{R} \to X$ satisfies the functional equation

$$f(ax+by) + f(ax-by) = 2a^2 f(x) + 2b^2 f(y), \ x, y \in \mathbb{R} \tag{12}$$

if and only if

$$f(x) = Q(x) + f(0), \quad x \in \mathbb{R}$$

with $Q : \mathbb{R} :\to X$ is a quadratic function and $f(0)$ satisfies $f(0) = (a^2 + b^2)f(0)$.

Proof. Indeed, it's easy to check that if $f(x) = Q(x) + f(0)$ with $f(0) = (a^2 + b^2)f(0)$ then f satisfies the equation (12).
On the other hand, let $f : \mathbb{R} \to X$ a solution of the equation (12) and $Q : \mathbb{R} \to X$ a function such that $Q(x) = f(x) - f(0)$ for $x \in \mathbb{R}$ with $f(0) = (a^2 + b^2)f(0)$. If we replace y by 0 and x by ax, then we get $Q(ax) = a^2 Q(x)$. Furthermore, replacing x by 0 in the equation (12) we get $f(by) + f(-by) = 2f(0) + 2b^2 Q(y)$ and replacing x by 0 and y by $-y$ we get $f(-by) + f(by) = 2f(0) + 2b^2 Q(-y)$, then g is even. Finely, from (12), we get $Q(ax + by) + Q(ax - by) = Q(ax) + Q(by)$ for all $x, y \in \mathbb{R}$. This complete the proof. ■

The proof of the following theorem has been patterned on the reasoning in [14].

Theorem 2.2 Let X be a linear space. A function $f : \mathbb{R} \to X$ satisfies the functional equation (11) if and only if $f(x) = Q(x^3)$ for all $x \in \mathbb{R}$ such that Q is solution of the functional equation (12)

Proof. It's not hard to see that if $f(x) = Q(x^3)$ then f is solution of the equation (11).
On the other hand, if f is solution of (11), then

$$\begin{aligned} Q(ax + by) + Q(ax - by) =& f\big(\sqrt[3]{a\sqrt[3]{x^3} + b\sqrt[3]{y^3}}\big) + f\big(\sqrt[3]{a\sqrt[3]{x^3} - b\sqrt[3]{y^3}}\big) \\ =& 2a^2 f\big(\sqrt[3]{x^3}\big) + 2b^2 f\big(\sqrt[3]{y^3}\big) \\ =& 2a^2 Q(x) + 2b^2 Q(y) \end{aligned}$$

for all $x, y \in \mathbb{R}$. ■

3. Stability results of the radical cubic functional equation (11)

In the following two theorems and by using the Theorem 1.9, we investigate the generalized Hyers-Ulam stability of the functional equation (11) in 2-Banach spaces.
Hereafter, we assume that $\big(X, \|\cdot, \cdot\|\big)$ is a 2-Banach space and $a, b \in \mathbb{Q}\backslash\{0\}$.

Theorem 3.1 Let $h_1, h_2 : \mathbb{R}^2 \to \mathbb{R}_+$ be two functions such that

$$\begin{aligned} \mathcal{U} = \{n \in \mathbb{N}\colon \alpha_n = & 2a^2\lambda_1(\frac{n+1}{a})\lambda_2(\frac{n+1}{a}) + 2b^2\lambda_1(\frac{-n}{b})\lambda_2(\frac{-n}{b}) \\ & + \lambda_1(2n+1)\lambda_2(2n+1) < 1\} \neq \phi \end{aligned}$$

for $a, b \in \mathbb{Q}\backslash\{0\}$ be an infinite set, where

$$\lambda_i(\rho n) = \inf\left\{t \in \mathbb{R}_+ \colon h_i(\rho n x^3, z) \leqslant t\, h_i(x^3, z), \;\; x, z \in \mathbb{R}\right\}, \;\; \rho \in \mathbb{R}$$

for all $n \in \mathbb{N}$, where $i = 1, 2$. Assume that $f : \mathbb{R} \to X$ satisfies the inequality

$$\left\|f\left(\sqrt[3]{ax^3 + by^3}\right) + f\left(\sqrt[3]{ax^3 - by^3}\right) - 2a^2 f(x) - 2b^2 f(y), g(z)\right\| \leqslant h_1(x^3, z) h_2(y^3, z) \quad (13)$$

for all $x, y, z \in \mathbb{R}_0$ and $f(0) = (a^2 + b^2)f(0)$ where $g : \mathbb{R} \to X$ is a surjective mapping with $g(0) = 0$. Then there exists a unique function $\mathcal{T}_m : \mathbb{R}_0 \to X$ satisfies the equation (11) such that

$$\left\| f(x) - \mathcal{T}_m(x), g(z) \right\| \leqslant \beta h_1(x^3, z) h_2(x^3, z), \quad x, z \in \mathbb{R}_0, \tag{14}$$

where

$$\beta = \inf_{n \in \mathcal{U}} \left\{ \frac{\lambda_1(\frac{n+1}{a})\lambda_2(\frac{-n}{b})}{1 - \alpha_n} \right\}.$$

Proof. Replace x by $\sqrt[3]{\frac{m+1}{a}}x$ and y by $\sqrt[3]{\frac{-m}{b}}x$ in inequality (14), where $x, y \in \mathbb{R}_0$ and $m \in \mathbb{N}$. Then we get

$$\begin{aligned} &\left\| 2a^2 f\left(\sqrt[3]{\frac{m+1}{a}}x\right) + 2b^2 f\left(\sqrt[3]{\frac{-m}{b}}x\right) - f\left(\sqrt[3]{2m+1}x\right) - f(x), g(z) \right\| \\ &\leqslant h_1\left(\frac{m+1}{a}x^3, z\right) h_2\left(\frac{-m}{b}x^3, z\right) \\ &\leqslant \lambda_1\left(\frac{m+1}{a}\right) \lambda_2\left(\frac{-m}{b}\right) h_1\left(x^3, z\right) h_2\left(x^3, z\right) \end{aligned} \tag{15}$$

for all $x, z \in \mathbb{R}_0$. For each $m \in \mathbb{N}$, we define operators $\mathcal{T} : X^{\mathbb{R}_0} \to X^{\mathbb{R}_0}$ by

$$\mathcal{T}\xi(x) = 2a^2 \xi\left(\sqrt[3]{\frac{m+1}{a}}x\right) + 2b^2 \xi\left(\sqrt[3]{\frac{-m}{b}}x\right) - \xi\left(\sqrt[3]{2m+1}x\right), \ \ \xi \in X^{\mathbb{R}_0}, \ x \in \mathbb{R}_0$$

and $\varepsilon : \mathbb{R}_0 \times \mathbb{R}_0 \to \mathbb{R}_+$ by

$$\varepsilon(x, z) = h_1\left(\frac{m+1}{a}x^3, z\right) h_2\left(\frac{-m}{b}x^3, z\right), \quad m \in \mathbb{N}, x, z \in \mathbb{R}_0.$$

Observe that

$$\varepsilon(x, z) \leqslant \lambda_1\left(\frac{m+1}{a}\right) \lambda_2\left(\frac{-m}{b}\right) h_1(x^3, z) h_2(x^3, z) \tag{16}$$

for all $x, z \in \mathbb{R}_0$ and all $m \in \mathbb{N}$. Then the inequality (15) become as

$$\left\| \mathcal{T}f(x) - f(x), g(z) \right\| \leqslant \varepsilon(x, z), \quad x, z \in \mathbb{R}_0.$$

Furthermore, for every $x, z \in \mathbb{R}_0$, $\xi, \mu \in X^{\mathbb{R}_0}$, we obtain

$$\begin{aligned}
&\Big\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x), g(z)\Big\| \\
&= \Bigg\|2a^2\xi\left(\sqrt[3]{\frac{m+1}{a}}x\right) + 2b^2\xi\left(\sqrt[3]{\frac{-m}{b}}x\right) - \xi\left(\sqrt[3]{2m+1}x\right) \\
&\quad - 2a^2\mu\left(\sqrt[3]{\frac{m+1}{a}}x\right) - 2b^2\mu\left(\sqrt[3]{\frac{-m}{b}}x\right) + \mu\left(\sqrt[3]{2m+1}x\right), g(z)\Bigg\| \\
&\leqslant 2a^2\Bigg\|(\xi-\mu)\left(\sqrt[3]{\frac{m+1}{a}}x\right), g(z)\Bigg\| + 2b^2\Bigg\|(\xi-\mu)\left(\sqrt[3]{\frac{-m}{b}}x\right), g(z)\Bigg\| \\
&\quad + \Big\|(\xi-\mu)\left(\sqrt[3]{2m+1}x\right), g(z)\Big\|.
\end{aligned}$$

This brings us to define the operator $\Lambda : \mathbb{R}_+^{\mathbb{R}_0\times\mathbb{R}_0} \to \mathbb{R}_+^{\mathbb{R}_0\times\mathbb{R}_0}$ by

$$\Lambda\delta(x,z) = 2a^2\delta\left(\sqrt[3]{\frac{m+1}{a}}x, z\right) + 2b^2\delta\left(\sqrt[3]{\frac{-m}{b}}x, z\right) + \delta\left(\sqrt[3]{2m+1}x, z\right) \tag{17}$$

for all $x, z \in \mathbb{R}_0$, where $\delta \in \mathbb{R}_+^{\mathbb{R}_0\times\mathbb{R}_0}$. Then, For each $m \in \mathbb{N}$, the above operator has the form described in (4) with $f_1(x) = \sqrt[3]{\frac{m+1}{a}}x$, $f_2(x) = \sqrt[3]{\frac{-m}{b}}x$, $f_3(x) = \sqrt[3]{2m+1}x$ and $L_1(x) = 2a^2$, $L_2(x) = 2b^2$, $L_3 = 1$ for all $x \in \mathbb{R}_0$. By induction, we will show that for each $x, z \in \mathbb{R}_0$, $n \in \mathbb{N}$ and $m \in \mathcal{U}$, we have

$$\Lambda^n\varepsilon(x,z) \leqslant \lambda_1\left(\frac{m+1}{a}\right)\lambda_2\left(\frac{-m}{b}\right)\alpha_m^n h_1(x^3,z)h_2(x^3,z) \tag{18}$$

for $n = 0$, inequality (18) is exactly (16). Next we will assume that (18) holds for $n = k$, where $k \in \mathbb{N}$. Then we have

$$\begin{aligned}
(\Lambda^{k+1}\varepsilon)(x,z) &= \Lambda\Big((\Lambda^k\varepsilon)(x,z)\Big) \\
&= 2a^2(\Lambda^k\varepsilon)(\sqrt[3]{\frac{m+1}{a}}x, z) + 2b^2(\Lambda^k\varepsilon)(\sqrt[3]{\frac{-m}{b}}x, z) + (\Lambda^k\varepsilon)(\sqrt[3]{2m+1}x, z) \\
&\leqslant \lambda_1\left(\frac{m+1}{a}\right)\lambda_2\left(\frac{-m}{b}\right)\alpha_m^k\Big(2a^2h_1\left(\frac{m+1}{a}x^3, z\right)h_2\left(\frac{m+1}{a}x^3, z\right) \\
&\quad + 2b^2h_1\left(\frac{-m}{b}x^3, z\right)h_2\left(\frac{-m}{b}x^3, z\right) + h_1\left((2m+1)x^3, z\right)h_2\left((2m+1)x^3, z\right)\Big) \\
&\leqslant \lambda_1\left(\frac{m+1}{a}\right)\lambda_2\left(\frac{-m}{b}\right)\alpha_m^k\Big(2a^2\lambda_1\left(\frac{m+1}{a}\right)\lambda_2\left(\frac{m+1}{a}\right) \\
&\quad + 2b^2\lambda_1\left(\frac{-m}{b}\right)\lambda_2\left(\frac{-m}{b}\right) + \lambda_1\left((2m+1)\right)\lambda_2\left((2m+1)\right)\Big)h_1\left(x^3, z\right)h_2\left(x^3, z\right) \\
&= \lambda_1\left(\frac{m+1}{a}\right)\lambda_2\left(\frac{-m}{b}\right)\alpha_m^{k+1}h_1\left(x^3, z\right)h_2\left(x^3, z\right)
\end{aligned}$$

for all $x, z \in \mathbb{R}_0$, $m \in \mathcal{U}$. This shows that (18) holds for $n = k + 1$. We conclude that the inequality (18) holds for all $n \in \mathbb{N}$. Since $\alpha_m < 1$ for each $m \in \mathcal{U}$, we get

$$\begin{aligned}\varepsilon^*(x,z) = \sum_{n=0}^{\infty} \big(\Lambda^n \varepsilon\big)(x,z) &\leqslant \lambda_1\left(\frac{m+1}{a}\right)\lambda_2\left(\frac{-m}{b}\right)h_1(x^3,z)h_2(x^3,z)\sum_{n=0}^{\infty}\alpha_m^n \\ &= \frac{\lambda_1\left(\frac{m+1}{a}\right)\lambda_2\left(\frac{-m}{b}\right)h_1(x^3,z)h_2(x^3,z)}{1-\alpha_m} < \infty\end{aligned}$$

for all $x, z \in \mathbb{R}_0$, $m \in \mathcal{U}$. Therefore, according to Theorem 1.9 with $\varphi = f$ and $X = \mathbb{R}_0$ and by using the surjectivity of g, we get that the limit $\mathcal{T'}_m(x) = \lim_{n\to\infty}\big(\mathcal{T}^n f\big)(x)$ exists for each $x \in \mathbb{R}_0$ and $m \in \mathcal{U}$, and

$$\big\|f(x) - \mathcal{T'}_m(x), g(z)\big\| \leqslant \frac{\lambda_1\left(\frac{m+1}{a}\right)\lambda_2\left(\frac{-m}{b}\right)h_1(x^3,z)h_2(x^3,z)}{1-\alpha_m} \quad x, z \in \mathbb{R}_0,\ m \in \mathcal{U}. \tag{19}$$

We define $\mathcal{T}_m : \mathbb{R} \to X$ by $\mathcal{T}_m(x) = \mathcal{T'}_m(x)$ for all $x \in \mathbb{R}_0$ and $\mathcal{T}_m(0) = (a^2+b^2)\mathcal{T}_m(0)$. To prove that $\mathcal{T}_m$ satisfies the functional equation (11), we should prove the following inequality

$$\begin{aligned}&\left\|\mathcal{T}^n f\left(\sqrt[3]{ax^3+by^3}\right) + \mathcal{T}^n f\left(\sqrt[3]{ax^3-by^3}\right) - 2a^2\mathcal{T}^n f(x) - 2b^2\mathcal{T}^n f(y), g(z)\right\| \\ &\leqslant \alpha_m^n h_1(x^3,z)h_2(y^3,z)\end{aligned} \tag{20}$$

for every $x, y, z \in \mathbb{R}_0$, $n \in \mathbb{N}$, and $m \in \mathcal{U}$. We proceed by induction, so, since the case $n = 0$ is just (13), take $k \in \mathbb{N}$ and assume that (22) holds for $n = k$ and every $x, y, z \in \mathbb{R}_0$, $m \in \mathcal{U}$. Then, for each $x, y, z \in \mathbb{R}_0$ and $m \in \mathcal{U}$, we get

$$\begin{aligned}&\left\|\mathcal{T}^{k+1} f\left(\sqrt[3]{ax^3+by^3}\right) + \mathcal{T}^{k+1} f\left(\sqrt[3]{ax^3-by^3}\right) - 2a^2\mathcal{T}^{k+1} f(x) - 2b^2\mathcal{T}^{k+1} f(y), g(z)\right\| \\
&= \Bigg\|2a^2\mathcal{T}^k f\left(\sqrt[3]{\frac{m+1}{a}}\sqrt[3]{ax^3+by^3}\right) + 2b^2\mathcal{T}^k f\left(\sqrt[3]{\frac{-m}{b}}\sqrt[3]{ax^3+by^3}\right) - \mathcal{T}^k f\left(\sqrt[3]{2m+1}\sqrt[3]{ax^3+by^3}\right) \\
&\quad + 2a^2\mathcal{T}^k f\left(\sqrt[3]{\frac{m+1}{a}}\sqrt[3]{ax^3-by^3}\right) + 2b^2\mathcal{T}^k f\left(\sqrt[3]{\frac{-m}{b}}\sqrt[3]{ax^3-by^3}\right) - \mathcal{T}^k f\left(\sqrt[3]{2m+1}\sqrt[3]{ax^3-by^3}\right) \\
&\quad - 4a^4\mathcal{T}^k f\left(\sqrt[3]{\frac{m+1}{a}}x\right) - 4a^2b^2\mathcal{T}^k f\left(\sqrt[3]{\frac{-m}{b}}x\right) + 2a^2\mathcal{T}^k f\left(\sqrt[3]{2m+1}x\right) \\
&\quad - 4a^2b^2\mathcal{T}^k f\left(\sqrt[3]{\frac{m+1}{a}}y\right) - 4b^4\mathcal{T}^k f\left(\sqrt[3]{\frac{-m}{b}}y\right) + 2b^2\mathcal{T}^k f\left(\sqrt[3]{2m+1}y\right), g(z)\Bigg\| \\
&\leqslant 2a^2\left\|\mathcal{T}^k f\left(\sqrt[3]{\frac{m+1}{a}}\sqrt[3]{ax^3+by^3}\right) + \mathcal{T}^k f\left(\sqrt[3]{\frac{m+1}{a}}\sqrt[3]{ax^3-by^3}\right) - 2a^2\mathcal{T}^k f\left(\sqrt[3]{\frac{m+1}{a}}x\right) - 2b^2\mathcal{T}^k f\left(\sqrt[3]{\frac{m+1}{a}}y\right), g(z)\right\| \\
&\quad + 2b^2\left\|\mathcal{T}^k f\left(\sqrt[3]{\frac{-m}{b}}\sqrt[3]{ax^3+by^3}\right) + \mathcal{T}^k f\left(\sqrt[3]{\frac{-m}{b}}\sqrt[3]{ax^3-by^3}\right) - 2a^2\mathcal{T}^k f\left(\sqrt[3]{\frac{-m}{b}}x\right) - 2b^2\mathcal{T}^k f\left(\sqrt[3]{\frac{-m}{b}}y\right), g(z)\right\| \\
&\quad + \left\|\mathcal{T}^k f\left(\sqrt[3]{2m+1}\sqrt[3]{ax^3+by^3}\right) + \mathcal{T}^k f\left(\sqrt[3]{2m+1}\sqrt[3]{ax^3-by^3}\right) - 2a^2\mathcal{T}^k f\left(\sqrt[3]{2m+1}x\right) - 2b^2\mathcal{T}^k f\left(\sqrt[3]{2m+1}y\right), g(z)\right\| \\
&\leqslant 2a^2\alpha_m^k h_1\Big(\frac{m+1}{a}x^3, z\Big)h_2\Big(\frac{m+1}{a}y^3, z\Big) + 2b^2\alpha_m^k h_1\Big(\frac{-m}{b}x^3, z\Big)h_2\Big(\frac{-m}{b}y^3, z\Big) + \alpha_m^k h_1((2m+1)x^3, z)h_2((2m+1)y^3, z) \\
&\leqslant \alpha_m^k\Big(2a^2\lambda_1\left(\frac{m+1}{a}\right)\lambda_2\left(\frac{m+1}{a}\right) + 2b^2\lambda_1\left(\frac{-m}{b}\right)\lambda_2\left(\frac{-m}{b}\right) + \lambda_1(2m+1)\lambda_2(2m+1)\Big)h_1(x^3,z)h_2(y^3,z) \\
&= \alpha_m^{k+1}h_1(x^3,z)h_2(y^3,z).\end{aligned}$$

Thus, by induction, we have shown that (20) holds for every $x, y, z \in \mathbb{R}_0$, $n \in \mathbb{N}$, and

$m \in \mathcal{U}$. Letting $n \to \infty$ in (19), we obtain the equality

$$\mathcal{T}'_m\left(\sqrt[3]{ax^3+by^3}\right)+\mathcal{T}'_m\left(\sqrt[3]{ax^3-by^3}\right)=2a^2\mathcal{T}'_m(x)+2b^2\mathcal{T}'_m(y) \tag{21}$$

for $x, y \in \mathbb{R}_0$ and $m \in \mathcal{U}$. This implies that $\mathcal{T}_m : \mathbb{R} \to X$ is a solution of the equation (11). Now, we will show that $\mathcal{T}_m$ is the unique solution of (11). Indeed, let $\mathcal{F}_m : \mathbb{R} \to X$ an other solution of (11) satisfying the inequality

$$\|\mathcal{F}_m(x)-f(x), g(z)\| \leqslant \theta h_1(x^3,z)h_2(x^3,z), \quad x, z \in \mathbb{R}_0 \tag{22}$$

and $\mathcal{F}_m(0) = (a^2+b^2)\mathcal{F}_m(0)$ with some $\theta > 0$. Then, for each $m \in \mathcal{U}$ fixed and $x, z \in \mathbb{R}_0$, we get

$$\begin{aligned}\left\|\mathcal{F}_m(x)-\mathcal{T}_m(x), g(z)\right\| =& \left\|\mathcal{F}_m(x)-f(x), g(z)\right\|+\left\|\mathcal{T}_m(x)-f(x), g(z)\right\| \\ &\leqslant \theta h_1(x^3,z)h_2(x^3,z)+\frac{\lambda_1\left(\frac{m+1}{a}\right)\lambda_2\left(\frac{-m}{b}\right)}{1-\alpha_m}h_1(x^3,z)h_2(x^3,z) \\ &\leqslant \theta_0 h_1(x^3,z)h_2(x^3,z)\sum_{n=0}^{\infty}\alpha_m^n \end{aligned}\tag{23}$$

with $\theta_0 = \frac{\left((1-\alpha_m)\theta+\lambda_1\left(\frac{m+1}{a}\right)\lambda_2\left(\frac{-m}{b}\right)\right)}{1-\alpha_m}$. We exclude the case that $h_1(x^3,z) \equiv 0$ or $h_2(x^3,z) \equiv 0$, which is trivial. Next, for each $j \in \mathbb{N}_0$, we show that

$$\left\|\mathcal{T}_m(x)-\mathcal{F}_m(x), g(z)\right\| \leqslant \theta h_1(x^3,z)h_2(x^3,z)\sum_{n=j}^{\infty}\alpha_m^n, \quad x, z \in \mathbb{R}_0. \tag{24}$$

The case $j = 0$ is exactly (23). We fix $k \in \mathbb{N}$ and assume that (24) holds for $j = k$. Then, for each $x, z \in \mathbb{R}_0$, we get

$$\begin{aligned}&\|\mathcal{T}_m(x)-\mathcal{F}_m(x), g(z)\| \\ &= \|2a^2\mathcal{T}_m\left(\sqrt[3]{\frac{m+1}{a}}x\right)+2b^2\mathcal{T}_m\left(\sqrt[3]{\frac{-m}{b}}x\right)-\mathcal{T}_m\left(\sqrt[3]{2m+1}x\right) \\ &-2a^2\mathcal{F}_m\left(\sqrt[3]{\frac{m+1}{a}}x\right)-2b^2\mathcal{F}_m\left(\sqrt[3]{\frac{-m}{b}}x\right)+\mathcal{F}_m\left(\sqrt[3]{2m+1}x\right), g(z)\| \\ &\leqslant 2a^2\|\mathcal{T}_m\left(\sqrt[3]{\frac{m+1}{a}}x\right)-\mathcal{F}_m\left(\sqrt[3]{\frac{m+1}{a}}x\right), g(z)\|+2b^2\|\mathcal{T}_m\left(\sqrt[3]{\frac{-m}{b}}x\right)-\mathcal{F}_m\left(\sqrt[3]{\frac{-m}{b}}x\right), g(z)\| \\ &+\|\mathcal{T}_m\left(\sqrt[3]{2m+1}x\right)-\mathcal{F}_m\left(\sqrt[3]{2m+1}x\right), g(z)\| \\ &\leqslant \Big(2a^2\theta_0 h_1\left(\frac{m+1}{a}x^3,z\right)h_2\left(\frac{m+1}{a}x^3,z\right)+2b^2\theta_0 h_1\left(\frac{-m}{b}x^3,z\right)h_2\left(\frac{-m}{b}x^3,z\right) \\ &+\theta_0 h_1((2m+1)x^3,z)h_2((2m+1)x^3,z)\Big)\sum_{n=k}^{\infty}\alpha_m^n \\ &\leqslant \theta_0\Big(2a^2\lambda_1(\frac{m+1}{a})\lambda_2(\frac{m+1}{a})+2b^2\lambda_1(\frac{-m}{b})\lambda_2(\frac{-m}{b})+\lambda_1(2m+1)\lambda_2(2m+1)\Big)h_1(x^3,z)h_2(x^3,z)\sum_{n=k}^{\infty}\alpha_m^n \\ &\leqslant \theta_0 h_1(x^3,z)h_2(x^3,z)\sum_{n=k}^{\infty}\alpha_m^{n+1} \\ &= \theta_0 h_1(x^3,z)h_2(x^3,z)\sum_{n=k+1}^{\infty}\alpha_m^n.\end{aligned}$$

This shows that (24) holds for $j = k+1$. Now, letting $j \to \infty$ in (24), we get $\mathcal{T}_m = \mathcal{F}_m$. This implies the uniqueness of $\mathcal{T}_m$. ■

With an analogous proof of the above theorem, we can prove the following theorem.

Theorem 3.2 Let $h : \mathbb{R}_0^2 \to \mathbb{R}_+$ be a function such that

$$\mathcal{U} = \left\{ n \in \mathbb{N} \colon \alpha_n = 2a^2\lambda(\frac{n+1}{a}) + 2b^2\lambda(\frac{-n}{b}) + \lambda(2n+1) < 1 \right\} \neq \phi$$

be an infinite set, where

$$\lambda(\rho n) = \inf \left\{ t \in \mathbb{R}_+ \colon h(\delta n x^3, z) \leqslant t\, h(x^3, z),\ \ x, z \in \mathbb{R}_0 \right\},\ \ \rho \in \mathbb{R}$$

for all $n \in \mathbb{N}$. Assume that $f : \mathbb{R} \to X$ satisfies the inequality

$$\left\| f\left(\sqrt[3]{ax^3 + by^3}\right) + f\left(\sqrt[3]{ax^3 - by^3}\right) - 2a^2 f(x) - 2b^2 f(y), g(z) \right\| \leqslant h(x^3, z) + h(y^3, z) \quad (25)$$

for all $x, y, z \in \mathbb{R}_0$ and $f(0) = (a^2 + b^2)f(0)$ where $g : \mathbb{R} \to X$ be a surjective mapping with $g(0) = 0$. Then there exists a unique function $\mathcal{T}_m : \mathbb{R} \to X$ satisfies the equation (11) such that

$$\left\| f(x) - \mathcal{T}_m(x), g(z) \right\| \leqslant \lambda_0 h(x^3, z),\ \ x, z \in \mathbb{R}_0, \quad (26)$$

where

$$\lambda_0 = \inf_{n \in \mathcal{U}} \left\{ \frac{\lambda(\frac{n+1}{a}) + \lambda(\frac{-n}{b})}{1 - \alpha_n} \right\}.$$

Proof. Replacing in (25) x by $\sqrt[3]{\frac{m+1}{a}}x$ and y by $\sqrt[3]{\frac{-m}{b}}x$, where $x \in \mathbb{R}_0, m \in \mathbb{N}$. Then we get

$$\begin{aligned} &\left\| 2a^2 f\left(\sqrt[3]{\tfrac{m+1}{a}}x\right) + 2b^2 f\left(\sqrt[3]{\tfrac{-m}{b}}x\right) - f\left(\sqrt[3]{2m+1}x\right) - f(x), g(z) \right\| \\ &\qquad \leqslant \left(\lambda\left(\tfrac{m+1}{a}\right) + \lambda\left(\tfrac{-m}{b}\right)\right) h(x^3, z),\ \ x, z \in \mathbb{R}_0. \end{aligned} \quad (27)$$

For each $m \in \mathbb{N}$, we define operators:

$$\mathcal{T}\xi(x) = 2a^2\xi\left(\sqrt[3]{\frac{m+1}{a}}x\right) + 2b^2\xi\left(\sqrt[3]{\frac{-m}{b}}x\right) - \xi\left(\sqrt[3]{2m+1}x\right),\ \ \xi \in X^{\mathbb{R}_0},\ x \in \mathbb{R}_0,$$

$$\Lambda\delta(x) = 2a^2\delta\left(\sqrt[3]{\frac{m+1}{a}}x, z\right) + 2b^2\delta\left(\sqrt[3]{\frac{-m}{b}}x, z\right) + \delta\left(\sqrt[3]{2m+1}x, z\right), \delta \in {\mathbb{R}_+}^{\mathbb{R}_0 \times \mathbb{R}_0}, x \in \mathbb{R}_0 \quad (28)$$

$$\varepsilon(x, z) = \left(\lambda\left(\frac{m+1}{a}\right) + \lambda\left(\frac{-m}{b}\right)\right) h(x^3, z),\ \ x, z \in \mathbb{R}_0.$$

As in theorem 3.1, we observe that inequality (27) take the form

$$\|f(x) - \mathcal{T}_m f(x), g(z)\| \leqslant \varepsilon(x,z), \quad x, z \in \mathbb{R}_0.$$

■

Next, we give some following corollaries obtained from our main results.

Corollary 3.3 Let $h_1, h_2 : \mathbb{R}_0^2 \to \mathbb{R}_+$ be as in Theorem 3.1 such that

$$\lim_{n\to\infty} \inf \sup_{x,z\in\mathbb{R}_0} \frac{\eta_n(x,z)}{h_1(x^3,z)h_2(x^3,z)} = 0, \tag{29}$$

where

$$\begin{aligned}\eta_n(x,z) = 2a^2 h_1\left(\frac{n+1}{a}x^3, z\right) h_2\left(\frac{n+1}{a}x^3, z\right) + 2b^2 h_1\left(\frac{-n}{b}x^3, z\right) h_2\left(\frac{-n}{b}x^3, z\right) \\ + h_1\left((2n+1)x^3, z\right) h_2\left((2n+1)x^3, z\right); \ \ x, z \in \mathbb{R}_0, \ n \in \mathbb{N}.\end{aligned}$$

Assume that $f : \mathbb{R} \to X$ satisfies (11)and $f(0) = (a^2 + b^2)f(0)$, and $g : \mathbb{R} \to X$ is a surjective mapping with $g(0) = 0$. Then there exist a unique radical cubic function $\mathcal{T}_m : \mathbb{R} \to X$ with $\mathcal{T}_m(0) = (a^2 + b^2)\mathcal{T}_m(0)$ and a unique constant $\kappa \in \mathbb{R}_+$ with

$$\left\|f(x) - \mathcal{T}_m(x), g(z)\right\| \leqslant \kappa\, h_1(x^3,z)h_2(x^3,z), \quad x, z \in \mathbb{R}_0. \tag{30}$$

Proof. By the definition of $\lambda_i(n)$ $(i = 1, 2)$ in the th 3.1, we can see

$$\begin{aligned}2a^2\lambda_1\left(\frac{n+1}{a}\right)\lambda_2\left(\frac{n+1}{a}\right) &= \sup_{x,z\in\mathbb{R}_0} \frac{2a^2 h_1\left(\frac{n+1}{a}x^3, z\right) h_2\left(\frac{n+1}{a}x^3, z\right)}{h_1(x^3,z)h_2(x^3,z)} \\ &\leqslant \sup_{x,z\in\mathbb{R}_0} \frac{\eta_n(x,z)}{h_1(x^3,z)h_2(x^3,z)}\end{aligned} \tag{31}$$

and

$$\begin{aligned}2b^2\lambda_1\left(\frac{-n}{b}\right)\lambda_2\left(\frac{-n}{b}\right) &= \sup_{x,z\in\mathbb{R}_0} \frac{2b^2 h_1\left(\frac{-n}{b}x^3, z\right) h_2\left(\frac{-n}{b}x^3, z\right)}{h_1(x^3,z)h_2(x^3,z)} \\ &\leqslant \sup_{x,z\in\mathbb{R}_0} \frac{\eta_n(x,z)}{h_1(x^3,z)h_2(x^3,z)}\end{aligned} \tag{32}$$

and

$$\begin{aligned}\lambda_1(2n+1)\lambda_2(2n+1) &= \sup_{x,z\in\mathbb{R}_0} \frac{h_1((2n+1)x^3,z)h_2((2n+1)x^3,z)}{h_1(x^3,z)h_2(x^3,z)} \\ &\leqslant \sup_{x,z\in\mathbb{R}_0} \frac{\eta_n(x,z)}{h_1(x^3,z)h_2(x^3,z)}.\end{aligned} \tag{33}$$

Combining inequalities (31), (32) and (33), we get

$$
\begin{aligned}
&2a^2\lambda_1\left(\frac{n+1}{a}\right)\lambda_2\left(\frac{n+1}{a}\right)+2b^2\lambda_1\left(\frac{-n}{b}\right)\lambda_2\left(\frac{-n}{b}\right)+\lambda_1(2n+1)\lambda_2(2n+1)\\
&\leqslant 3\sup_{x,z\in\mathbb{R}_0}\frac{\eta_n(x,z)}{h_1(x^3,z)h_2(x^3,z)}. \qquad (34)
\end{aligned}
$$

Putting

$$\gamma_n=\sup_{x,z\in\mathbb{R}_0}\frac{\eta_n(x,z)}{h_1(x^3,z)h_2(x^3,z)}.$$

From (29), there is a subsequence $\{\gamma_{n_k}\}$ of a sequence $\{\gamma_n\}$ such that $\lim_{k\to\infty}\gamma_{n_k}=0$, that is,

$$\lim_{k\to\infty}\sup_{x,z\in\mathbb{R}_0}\frac{\eta_{n_k}(x,z)}{h_1(x^3,z)h_2(x^3,z)}=0. \qquad (35)$$

From (35) and (34), we obtain

$$\lim_{k\to\infty}2a^2\lambda_1\left(\frac{n_k+1}{a}\right)\lambda_2\left(\frac{n_k+1}{a}\right)+2b^2\lambda_1\left(\frac{-n_k}{b}\right)\lambda_2\left(\frac{-n_k}{b}\right)+\lambda_1(2n_k+1)\lambda_2(2n_k+1)=0.$$

This implies

$$\lim_{k\to\infty}\lambda_1\left(\frac{n_k+1}{a}\right)\lambda_2\left(\frac{n_k+1}{a}\right)=0$$

and hence,

$$\lim_{k\to\infty}\frac{\lambda_1\left(\frac{n_k+1}{a}\right)\lambda_2\left(\frac{-n_k}{b}\right)}{1-2a^2\lambda_1\left(\frac{n_k+1}{a}\right)\lambda_2\left(\frac{n_k+1}{a}\right)-2b^2\lambda_1\left(\frac{-n_k}{b}\right)\lambda_2\left(\frac{-n_k}{b}\right)-\lambda_1(2n_k+1)\lambda_2(2n_k+1)}=0,$$

which means that β defined in Theorem 3.1 is equal to κ. This complete the proof. ■

By a similar proof we can prove the following corollary where $\kappa=1$.

Corollary 3.4 Let $h1:\mathbb{R}_0^2\to\mathbb{R}_+$ be as in Theorem 3.2 such that

$$\lim_{n\to\infty}\inf\sup_{x,z\in\mathbb{R}_0}\frac{2a^2h\left(\frac{n+1}{a}x^3,z\right)+2b^2h\left(\frac{-n}{b}x^3,z\right)+h((2n+1)x^3,z)}{h(x^3,z)}=0$$

for all $x,z\in\mathbb{R}_0$ and $n\in\mathbb{N}$. Assume that $f:\mathbb{R}\to X$ satisfies (11), $f(0)=(a^2+b^2)f(0)$ and $g:\mathbb{R}\to X$ a surjective mapping with $g(0)=0$. Then there exist a unique radical cubic function $\mathcal{T}_m:\mathbb{R}\to X$ with $\mathcal{T}_m(0)=(a^2+b^2)\mathcal{T}_m(0)$ and a unique constant $\kappa\in\mathbb{R}_+$ with

$$\left\|f(x)-\mathcal{T}_m(x),g(z)\right\|\leqslant\kappa\,h_1(x^3,z)h_2(x^3,z),\quad x,z\in\mathbb{R}_0.$$

4. Applications

According to above theorems, by defining $h_1, h_2, h : \mathbb{R}^2 \to (0, \infty)$ as follows: $h_1(x^3, z) = c_1|q_1(x^3)|^p|z|^{r_1}$, $h_2(x^3, z) = c_2|q_2(x^3)|^q|z|^{r_2}$ and $h(x^3, z) = c|q(x^3)|^p|z|^r$ for all $x, z \in \mathbb{R}_0$, where $c_1, c_2, c \geqslant 0$, $r_1, r_2, r > 0$ and q_1, q_2 and q are quadratic mappings, we derive some particular cases.

Corollary 4.1 Let X be 2-Banach space. Assume that a function $f : \mathbb{R}_0 \to X$ verify the inequality

$$\left\|f\left(\sqrt[3]{ax^3+by^3}\right)+f\left(\sqrt[3]{ax^3-by^3}\right)-2a^2f(x)-2b^2f(y), g(z)\right\| \leqslant c|q_1(x^3)|^p|q_2(y^3)|^q|z|^r \tag{36}$$

for all $x, y, z \in \mathbb{R}_0$ and $a, b \in \mathbb{Q}\setminus\{0\}$ with $c = c_1 \times c_2 \geqslant 0$, $p+q<0$ and $r = r_1 + r_2 > 0$, where $g : \mathbb{R} \to X$ is a surjective mapping. Then f is a solution of the equation (11) on $\mathbb{R}_0$.

Proof. For each $m \in \mathbb{N}$ and $a, b \in \mathbb{Q}\setminus\{0\}$ we define $\lambda_1(m)$ as in Theorem 3.1

$$\begin{aligned}
\lambda_1\left(\frac{m+1}{a}\right) &= \inf\left\{t \in \mathbb{R}_+ : h_1\left(\frac{m+1}{a}x^3, z\right) \leqslant th_1(x^3, z)\right\}\\
&= \inf\left\{t \in \mathbb{R}_+ : c_1\left|q_1\left(\frac{m+1}{a}x^3\right)\right|^p|z|^{r_1} \leqslant tc_1|q_1(x^3)|^p|z|^{r_1}\right\}\\
&= \inf\left\{t \in \mathbb{R}_+ : c_1\left|\frac{m+1}{a}\right|^{2p}|q_1(x^3)|^p|z|^{r_1} \leqslant tc_1|q_1(x^3)|^p|z|^{r_1}\right\}\\
&= \left(\frac{m+1}{a}\right)^{2p}.
\end{aligned}$$

for $x, z \in \mathbb{R}\setminus\{0\}$. Also, for $m \in \mathbb{N}$, we have $\lambda_2\left(\frac{-m}{b}\right) = \left(\frac{m}{b}\right)^{2q}$. It's clear that there exists $m_0 \in \mathbb{N}$ such that for each $m \geqslant m_0$ we get

$$\begin{aligned}
\alpha_m &= 2a^2\lambda_1\left(\frac{m+1}{a}\right)\lambda_2\left(\frac{m+1}{a}\right) + 2b^2\lambda_1\left(\frac{-m}{b}\right)\lambda_2\left(\frac{-m}{b}\right)\\
&\quad + \lambda_1(2m+1)\lambda_2(2m+1)\\
&= 2a^2\left(\frac{m+1}{a}\right)^{2(p+q)} + 2b^2\left(\frac{m}{b}\right)^{2(p+q)} + (2m+1)^{2(p+q)} < 1.
\end{aligned}$$

According to theorem 3.1, there exists a unique radical cubic function $\mathcal{T}_m : \mathbb{R}\setminus\{0\} \to X$ such that :

$$\|\mathcal{T}_m(x) - f(x), g(z)\| \leqslant c\beta|q_1(x^3)|^p|q_2(x^3)|^q|z|^r,$$

where

$$\beta = \inf_{m\in\mathcal{U}}\left\{\frac{\lambda_1\left(\frac{m+1}{a}\right)\lambda_2\left(\frac{-m}{b}\right)}{1-\alpha_m}\right\} = \inf_{m\in\mathcal{U}}\left\{\frac{\left(\frac{m+1}{a}\right)^{2p}\left(\frac{-m}{b}\right)^{2q}}{1-\alpha_m}\right\}.$$

On the other hand, since $p+q<0$, one of p and q must be negative. Assume that $p<0$. Then

$$\lim_{m\to\infty}\lambda_1\left(\frac{m+1}{a}\right)\lambda_2\left(\frac{-m}{b}\right)=\lim_{m\to\infty}\left(\frac{m+1}{a}\right)^{2(p+q)}=0 \tag{37}$$

We get the desired results. ■

Corollary 4.2 Let X be 2-Banach space. Assume that a function $f:\mathbb{R}\backslash\{0\}\to X$ verify the in inequality

$$\|f\left(\sqrt[3]{ax^3+by^3}\right)+f\left(\sqrt[3]{ax^3-by^3}\right)-2a^2f(x)-2b^2f(y),g(z)\|\leqslant c|q(x^3)|^p+|q(y^3)|^p|z|^r, \tag{38}$$

for all $x,y,z\in\mathbb{R}\setminus\{0\}$ and $a,b\in\mathbb{Q}\setminus\{0\}$ with $c\geqslant 0$, $p<0$ and $r\in\mathbb{R}$, where $g:\mathbb{R}\to X$ is a surjective mapping. Then f is a solution of the equation (11) on $\mathbb{R}\setminus\{0\}$.

Proof. The proof is similar to the proof of Corollary 4.2. ■

In the following corollaries, we get the hyperstability results for the inhomogeneous general radical cubic functional equation.

Corollary 4.3 Let X be a 2-banach space, $G:\mathbb{R}^2\to X$ be a function such that $G(0,0)=0$ and $c,p,q,r\in\mathbb{R}$ with $c\geqslant 0$, $p+q<0$ and $r>0$. Assume that $G:\mathbb{R}^2\to X$ and $f:\mathbb{R}\to X$ satisfy the inequality:

$$\|f\left(\sqrt[3]{ax^3+by^3}\right)+f\left(\sqrt[3]{ax^3-by^3}\right)-2a^2f(x)-2b^2f(y)-G(x,y),g(z)\|\leqslant c|q_1(x^3)|^p|q_2(y^3)|^q|z|^r \tag{39}$$

for all $x,y,z\in\mathbb{R}\backslash\{0\}$ and $a,b\in\mathbb{Q}\backslash\{0\}$ with $f(0)=(a^2+b^2)f(0)$, where $g:X\to X$ is a surjective mapping. If the functional equation

$$f\left(\sqrt[3]{ax^3+by^3}\right)+f\left(\sqrt[3]{ax^3-by^3}\right)=2a^2f(x)+2b^2f(y)+G(x,y) \tag{40}$$

for all $x,y\in\mathbb{R}\backslash\{0\}$ and $a,b\in\mathbb{Q}\backslash\{0\}$ has a solution $f_0:\mathbb{R}\to X$, then f is a solution of (40).

Proof. Let $\phi:\mathbb{R}\to X$ be a function defined by $\phi(x)=f(x)-f_0(x)$ for all $x\in\mathbb{R}$. then,

$$\begin{aligned}
&\|\phi\left(\sqrt[3]{ax^3+by^3}\right)+\phi\left(\sqrt[3]{ax^3-by^3}\right)-2a^2\phi(x)-2b^2\phi(y),g(z)\|\\
&=\|f\left(\sqrt[3]{ax^3+by^3}\right)+f\left(\sqrt[3]{ax^3-by^3}\right)-2a^2f(x)-2b^2f(y)-G(x,y)\\
&\quad-f_0\left(\sqrt[3]{ax^3+by^3}\right)-f_0\left(\sqrt[3]{ax^3-by^3}\right)+2a^2f_0(x)+2b^2f_0(y)+G(x,y),g(z)\|\\
&\leqslant\|f\left(\sqrt[3]{ax^3+by^3}\right)+f\left(\sqrt[3]{ax^3-by^3}\right)-2a^2f(x)-2b^2f(y)-G(x,y),g(z)\|\\
&\quad+\|f_0\left(\sqrt[3]{ax^3+by^3}\right)+f_0\left(\sqrt[3]{ax^3-by^3}\right)-2a^2f_0(x)-2b^2f_0(y)-G(x,y),g(z)\|\\
&=\|f\left(\sqrt[3]{ax^3+by^3}\right)+f\left(\sqrt[3]{ax^3-by^3}\right)-2a^2f(x)-2b^2f(y)-G(x,y),g(z)\|\\
&\leqslant c|q_1(x^3)|^p|q_2(y^3)|^q|z|^r,\quad x,y\in\mathbb{R}\backslash\{0\}.
\end{aligned}$$

It follows from corollary 4.1 that ϕ is a solution of equation (11). Moreover,

$$\begin{aligned}
&f\left(\sqrt[3]{ax^3+by^3}\right)+f\left(\sqrt[3]{ax^3-by^3}\right)-2a^2f(x)-2b^2f(y)-G(x,y)\\
&=\phi\left(\sqrt[3]{ax^3+by^3}\right)+\phi\left(\sqrt[3]{ax^3-by^3}\right)-2a^2\phi(x)-2b^2\phi(y)\\
&+f_0\left(\sqrt[3]{ax^3+by^3}\right)+f_0\left(\sqrt[3]{ax^3-by^3}\right)-2a^2f_0(x)-2b^2f_0(y)-G(x,y)\\
&=0,
\end{aligned}$$

which means f is a solution of (40). ■

Corollary 4.4 Let X be a 2-banach space, $G:\mathbb{R}^2\to X$ be a function such that $G(0,0)=0$ and $c,p,r\in\mathbb{R}$ with $c\geqslant 0$, $p<0$ and $r>0$. Assume that $G:\mathbb{R}^2\to X$ and $f:\mathbb{R}\to X$ satisfy the inequality:

$$\|f\left(\sqrt[3]{ax^3+by^3}\right)+f\left(\sqrt[3]{ax^3-by^3}\right)-2a^2f(x)-2b^2f(y)-G(x,y),g(z)\|\leqslant c\left(|q(x^3)|^p+|q(y^3)|^p\right)|z|^r \tag{41}$$

for all $x,y,z\in\mathbb{R}\backslash\{0\}$ and $a,b\in\mathbb{Q}\backslash\{0\}$ with $f(0)=(a^2+b^2)f(0)$, where $g:X\to X$ is a surjective mapping. If the functional equation

$$f\left(\sqrt[3]{ax^3+by^3}\right)+f\left(\sqrt[3]{ax^3-by^3}\right)=2a^2f(x)+2b^2f(y)+G(x,y),\quad x,y\in\mathbb{R}\backslash\{0\} \tag{42}$$

has a solution $f_0:\mathbb{R}\to X$, then f is a solution of (42).

Proof. With an analogous proof of Corollary 4.3, we find the desired result. ■

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