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Some topological properties of fuzzy strong b-metric spaces

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Abstract. In this study, we investigate topological properties of fuzzy strong b-metric spaces defined in [13]. Firstly, we prove Baire's theorem for these spaces. Then we define the product of two fuzzy strong b-metric spaces defined with same continuous t-norms and show that $X_1 \times X_2$ is a complete fuzzy strong b-metric space if and only if X_1 and X_2 are complete fuzzy strong b-metric spaces. Finally it is proven that a subspace of a separable fuzzy strong b-metric space is separable.

Keywords: Fuzzy strong b-metric space, strong b-metric space, complete, separable.

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1. Introduction and Preliminaries

The notion of strong b-metric space is obtained by modifying the "relaxed triangle inequality" in the definition of b-metric (or metric type) space [2, 3, 6, 8, 10].

Definition 1.1 [11] Let X be a non-empty set, $K \geqslant 1$ and $D : X \times X \longrightarrow [0, \infty)$ be a function such that for all $x, y, z \in X$,
1) $D(x, y) = 0$ if and only if $x = y$,
2) $D(x, y) = D(y, x)$,
3) $D(x, z) \leqslant D(x, y) + KD(y, z)$.
Then D is called a strong b-metric on X and (X, D, K) is called a strong b-metric space.

In these spaces, the strong b-metric D is continuous and an open ball is open set [11] where these are not true in general for b-metric spaces [1].

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After introducing the theory of fuzzy sets by Zadeh [15], fuzzy analogy of metric spaces were applied by different authors from different points of view [4, 5, 7, 9, 12].

In [13], Öner introduced and studied the notion of fuzzy strong b-metric spaces which is the fuzzy analogy of strong b-metric spaces and a generalization of fuzzy metric space introduced by George and Veeramani [7].

Definition 1.2 [14] A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions:

1) $*$ is associative and commutative,
2) $*$ is continuous,
3) $a * 1 = a$ for all $a \in [0,1]$,
4) $a * b \leqslant c * d$ whenever $a \leqslant c$ and $b \leqslant d$, $\quad a, b, c, d \in [0,1]$.

Definition 1.3 [13] Let X be a non-empty set, $K \geq 1$, $*$ is a continuous t-norm and M be a fuzzy set on $X \times X \times (0, \infty)$ such that for all $x, y, z \in X$ and $t, s > 0$,

1) $M(x, y, t) > 0$,
2) $M(x, y, t) = 1$ if and only if $x = y$,
3) $M(x, y, t) = M(y, x, t)$,
4) $M(x, y, t) * M(y, z, s) \leqslant M(x, z, t + Ks)$,
5) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous.

Then M is called a fuzzy strong b-metric on X and $(X, M, *, K)$ is called a fuzzy strong b-metric space.

For $t > 0$, open balls and closed balls with center x and radius $r \in (0, 1)$ were defined in [13] as follows:

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\},$$
$$B[x, r, t] = \{y \in X : M(x, y, t) \geqslant 1 - r\}$$

and it was proven that every fuzzy strong b-metric spaces $(X, M, *, K)$ induces a Hausdorff and first countable topology τ_M on X which open balls are open and closed balls are closed and the family of sets $\{B(x, r, t) : x \in X, 0 < r < 1, t > 0\}$ form a base.

Proposition 1.4 [13] $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is nondecreasing for all $x, y \in X$.

Definition 1.5 [13] Let $(X, M, *, K)$ be a fuzzy strong b-metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then

i) $\{x_n\}$ is said to converge to x if for any $t > 0$ and any $r \in (0, 1)$ there exists a natural number n_0 such that $M(x_n, x, t) > 1 - r$ for all $n \geqslant n_0$. We denote this by $\lim_{n \longrightarrow \infty} x_n = x$ or $x_n \longrightarrow x$ as $n \to \infty$.

ii) $\{x_n\}$ is said to be a Cauchy sequence if for any $r \in (0, 1)$ and any $t > 0$ there exists a natural number n_0 such that $M(x_n, x_m, t) > 1 - r$ for all $n, m \geqslant n_0$.

iii) $(X, M, *, K)$ is said to be a complete fuzzy strong b-metric space if every Cauchy sequence is convergent.

Theorem 1.6 [13] Let $(X, M, *, K)$ be a fuzzy strong b-metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. $\{x_n\}$ converges to x if and only if $M(x_n, x, t) \longrightarrow 1$ as $n \longrightarrow \infty$, for each $t > 0$.

In this study, we investigate the further topological properties of fuzzy strong b-metric spaces. Firstly, we prove Baire's theorem for these spaces. Then we define the product of two fuzzy strong b-metric spaces defined with same continuous t-norms and show that $X_1 \times X_2$ is a complete fuzzy strong b-metric space if and only if X_1 and X_2 are complete fuzzy strong b-metric spaces. Finally, it is proven that a subspace of a separable fuzzy strong b-metric space is separable.

2. Main results

Theorem 2.1 (Baire's theorem). Let $(X, M, *, K)$ be a complete fuzzy strong b-metric space. Then the intersection of a countable number of dense open sets is dense.

Proof. Let X be the given complete fuzzy strong b-metric space, B_0 be a nonempty open set and $D_1, D_2, D_3, \dots$ be dense open sets in X. Since D_1 is dense in X, $B_0 \cap D_1 \neq \emptyset$. Let $x_1 \in B_0 \cap D_1$. Since $B_0 \cap D_1$ is open, there exist $0 < r_1 < 1$ and $t_1 > 0$ such that $B(x_1, r_1, t_1) \subset B_0 \cap D_1$. Choose $r_1' < r_1$ and $t_1' = \min\{t_1, 1\}$ such that $B[x_1, r_1', t_1'] \subset B_0 \cap D_1$. Let $B_1 = B(x_1, r_1', t_1')$. Since D_2 is dense in X, $B_1 \cap D_2 \neq \emptyset$. Let $x_2 \in B_1 \cap D_2$. Since $B_1 \cap D_2$ is open, there exist $0 < r_2 < 1/2$ and $t_2 > 0$ such that $B(x_2, r_2, t_2) \subset B_1 \cap D_2$. Choose $r_2' < r_2$ and $t_2' = \min\{t_2, 1/2\}$ such that $B[x_2, r_2', t_2'] \subset B_1 \cap D_2$. Let $B_2 = B(x_2, r_2', t_2')$. Similarly, proceeding by induction, we can find a $x_n \in B_{n-1} \cap D_n$. Since $B_{n-1} \cap D_n$ is open, there exist $0 < r_n < 1/n$ and $t_n > 0$ such that $B(x_n, r_n, t_n) \subset B_{n-1} \cap D_n$. Choose $r_n' < r_n$ and $t_n' = \min\{t_n, 1/n\}$ such that $B[x_n, r_n', t_n'] \subset B_{n-1} \cap D_n$. Let $B_n = B(x_n, r_n', t_n')$. Now we claim that $\{x_n\}$ is a Cauchy sequence. For a given $t > 0$ and $0 < \varepsilon < 1$, choose n_0 such that $1/n_0 < t, 1/n_0 < \varepsilon$. Then for $n, m \geqslant n_0$

$$M(x_n, x_m, t) \geqslant M\left(x_n, x_m, \frac{1}{n_0}\right) \geqslant 1 - \frac{1}{n_0} \geqslant 1 - \varepsilon.$$

Therefore, $\{x_n\}$ is Cauchy sequence. Since X is complete, there exists $x \in X$ such that $x_n \to x$. But $x_k \in B[x_n, r_n', t_n']$ for all $k \geqslant n$. Since $B[x_n, r_n', t_n']$ is closed, $x \in B[x_n, r_n', t_n'] \subset B_{n-1} \cap D_n$ for all n. Thus, $B_0 \cap (\bigcap_{n=1}^{\infty} D_n) \neq \emptyset$. Hence, $\bigcap_{n=1}^{\infty} D_n$ is dense in X. ■

Proposition 2.2 Let $(X_1, M_1, *, K_1)$ and $(X_2, M_2, *, K_2)$ be fuzzy strong b-metric spaces. For $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$, consider

$$M((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) * M_2(x_2, y_2, t).$$

Then $(X_1 \times X_2, M, *, K)$ is a fuzzy strong b-metric space where $K = \max\{K_1, K_2\}$.

Proof. 1) Since $M_1(x_1, y_1, t) > 0$ and $M_2(x_2, y_2, t) > 0$ this implies that

$$M_1(x_1, y_1, t) * M_2(x_2, y_2, t) > 0.$$

Therefore, $M((x_1, x_2), (y_1, y_2), t) > 0$.
2) Suppose that $(x_1, x_2) = (y_1, y_2)$. This implies that $x_1 = y_1$ and $x_2 = y_2$. Hence, for all $t > 0$, we have $M_1(x_1, y_1, t) = 1$ and $M_2(x_2, y_2, t) = 1$. It follows that

$$M((x_1, x_2), (y_1, y_2), t) = 1.$$

Conversely, suppose that $M((x_1, x_2), (y_1, y_2), t) = 1$. This implies that

$$M_1(x_1, y_1, t) * M_2(x_2, y_2, t) = 1.$$

Since $0 < M_1(x_1, y_1, t) \leqslant 1$ and $0 < M_2(x_2, y_2, t) \leqslant 1$, it follows that $M_1(x_1, y_1, t) = 1$ and $M_2(x_2, y_2, t) = 1$. Thus, $x_1 = y_1$ and $x_2 = y_2$. Therefore $(x_1, x_2) = (y_1, y_2)$.

3) To prove that $M((x_1,x_2),(y_1,y_2),t) = M((y_1,y_2),(x_1,x_2),t)$. We observe that

$$M_1(x_1,y_1,t) = M_1(y_1,x_1,t),$$
$$M_2(x_2,y_2,t) = M_2(y_2,x_2,t).$$

It follows that for all $(x_1,x_2),(y_1,y_2) \in X_1 \times X_2$ and $t > 0$,

$$M((x_1,x_2),(y_1,y_2),t) = M((y_1,y_2),(x_1,x_2),t).$$

4) Since $(X_1,M_1,*,K_1)$ and $(X_2,M_2,*,K_2)$ are fuzzy strong b-metric spaces, we have

$$M_1(x_1,z_1,t+K_1s) \geqslant M_1(x_1,y_1,t) * M_1(y_1,z_1,s),$$
$$M_2(x_2,z_2,t+K_2s) \geqslant M_2(x_2,y_2,t) * M_2(y_2,z_2,s)$$

for all $(x_1,x_2),(y_1,y_2),(z_1,z_2) \in X_1 \times X_2$ and $t,s > 0$. Since $K = max\{K_1,K_2\}$, we get

$$\begin{aligned} M((x_1,x_2),(z_1,z_2),t+Ks) &= M_1(x_1,z_1,t+Ks) * M_2(x_2,z_2,t+Ks) \\ &\geqslant M_1(x_1,z_1,t+K_1s) * M_2(x_2,z_2,t+K_2s) \\ &\geqslant M_1(x_1,y_1,t) * M_1(y_1,z_1,s) * M_2(x_2,y_2,t) * M_2(y_2,z_2,s) \\ &\geqslant M_1(x_1,y_1,t) * M_2(x_2,y_2,t) * M_1(y_1,z_1,s) * M_2(y_2,z_2,s) \\ &\geqslant M((x_1,x_2),(y_1,y_2),t) * M((y_1,y_2),(z_1,y_2),s). \end{aligned}$$

5) Note that $M_1(x_1,y_1,t)$ and $M_2(x_2,y_2,t)$ are continuous with respect to t and $*$ is continuous. It follows that

$$M((x_1,x_2),(y_1,y_2),t) = M_1(x_1,y_1,t) * M_2(x_2,y_2,t)$$

is also continuous. ■

Proposition 2.3 Let $(X_1,M_1,*,K_1)$ and $(X_2,M_2,*,K_2)$ be fuzzy strong b-metric spaces. Then $(X_1 \times X_2,M,*,K)$ is complete if and only if $(X_1,M_1,*,K_1)$ and $(X_2,M_2,*,K_2)$ are complete.

Proof. Suppose that $(X_1,M_1,*,K_1)$ and $(X_2,M_2,*,K_2)$ are complete fuzzy strong b-metric spaces. Let $\{a_n\}$ be a Cauchy sequence in $X_1 \times X_2$. Note that $a_n = (x_1^n, x_2^n)$ and $a_m = (x_1^m, x_2^m)$. Also, $M(a_n,a_m,t)$ converges to 1. Hence, $M((x_1^n,x_2^n),(x_1^m,x_2^m),t)$ converges to 1 for each $t > 0$. It follows that $M_1(x_1^n,x_1^m,t) * M_2(x_2^n,x_2^m,t)$ converges to 1 for each $t > 0$. Thus, $M_1(x_1^n,x_1^m,t)$ converges to 1 and also, $M_2(x_2^n,x_2^m,t)$ converges to 1. Therefore, $\{x_1^n\}$ is a Cauchy sequence in $(X_1,M_1,*,K_1)$ and $\{x_2^n\}$ is a Cauchy sequence in $(X_2,M_2,*,K_2)$. Since $(X_1,M_1,*,K_1)$ and $(X_2,M_2,*,K_2)$ are complete fuzzy strong b-metric spaces, there exists $x_1 \in X_1$ and $x_2 \in X_2$ such that $M_1(x_1^n,x_1,t)$ converges to 1 and $M_2(x_2^n,x_2,t)$ converges to 1 for each $t > 0$. Let $a = (x_1,x_2)$. Then $a \in X_1 \times X_2$. It follows that $M(a_n,a,t)$ converges to 1 for each $t > 0$. This shows that $(X_1 \times X_2,M,*,K)$ is complete.

Conversely, suppose that $(X_1 \times X_2,M,*,K)$ is complete. We shall show that $(X_1,M_1,*,K_1)$ and $(X_2,M_2,*,K_2)$ are complete. Let $\{x_1^n\}$ and $\{x_2^n\}$ be Cauchy sequences in $(X_1,M_1,*,K_1)$ and $(X_2,M_2,*,K_2)$, respectively. Thus, $M_1(x_1^n,x_1^m,t)$ converges to 1 and $M_2(x_2^n,x_2^m,t)$ converges to 1 for each $t > 0$. It follows that

$$M((x_1^n,x_2^n),(x_1^m,x_2^m),t) = M_1(x_1^n,x_1^m,t) * M_2(x_2^n,x_2^m,t)$$

converges to 1. Then (x_1^n, x_2^n) is a Cauchy sequence in $X_1 \times X_2$. Since $(X_1 \times X_2, M, *, K)$ is complete, there exists $(x_1, x_2) \in X_1 \times X_2$ such that $M((x_1^n, x_2^n), (x_1, x_2), t)$ converges to 1. Clearly, $M_1(x_1^n, x_1, t)$ converges to 1 and $M_2(x_2^n, x_2, t)$ converges to 1. Hence, $(X_1, M_1, *)$ and $(X_2, M_2, *)$ are complete. This completes the proof. ■

Proposition 2.4 A subspace of a separable fuzzy strong b-metric space $(X, M, *, K)$ is separable.

Proof. Let X be the given separable fuzzy strong b-metric space and Y be a subspace of X. Let $A = \{x_n : n \in \mathbb{N}\}$ be a countable dense subset of X. For arbitrary but fixed $n, k \in \mathbb{N}$, if there are points $x \in X$ such that $M(x_n, x, 1/k) > 1 - 1/k$, choose one of them and denote it by x_{nk}. Let $B = \{x_{nk} : n, k \in \mathbb{N}\}$. Then B is countable. Now, we claim that $Y \subset \overline{B}$. Let $y \in Y$. Given r with $0 < r < 1$ and $t > 0$ we can find a $k \in \mathbb{N}$ such that $(1 - 1/k) * (1 - 1/k) > 1 - r$ and $1/k < t/2K$. Since A is dense in X, there exists an $m \in \mathbb{N}$ such that $M(x_m, y, 1/k) > 1 - 1/k$. But, by definition of B, there exists x_{mk} such that $M(x_{mk}, x_m, 1/k) > 1 - 1/k$. Now, we have

$$\begin{aligned} M(x_{mk}, y, t) &\geqslant M\left(x_{mk}, x_m, \frac{t}{2}\right) * M\left(x_m, y, \frac{t}{2K}\right) \\ &\geqslant M\left(x_{mk}, x_m, \frac{1}{k}\right) * M\left(x_m, y, \frac{1}{k}\right) \\ &\geqslant \left(1 - \frac{1}{k}\right) * \left(1 - \frac{1}{k}\right) \\ &> 1 - r. \end{aligned}$$

Thus, $y \in \overline{B}$. Hence, Y is separable. ■

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