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Interval valued fuzzy weak bi-ideals of Γ-near-rings

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Abstract. In this paper, we introduce the concept of interval valued fuzzy weak bi-ideals of Γ-near-rings, which is a generalized concept of fuzzy weak bi-ideals of Γ-near-rings. We also characterize some properties and provide examples of interval valued fuzzy weak bi-ideals of Γ-near-rings.

Keywords: Γ-near-rings, fuzzy weak bi-ideals, interval valued fuzzy weak bi-ideals, homomorphism and anti-homomorphism.

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1. Introduction

Zadeh introduced the concept of fuzzy sets in 1965 [21], and also, generalized it to interval valued fuzzy subsets [22]. Near-ring was introduced by Pilz [14]. Gamma -near-ring was introduced by Satyanarayana [15] in 1984. The concept of bi-ideals of near-ring was applied to Γ-near-rings by Tamizh chelvam et al. [16]. The idea of fuzzy ideals of near-rings was proposed by Kim et al. [9]. Fuzzy ideals in *Gamma*-near-rings proposed by Jun et al. [8] in 1998. Moreover, Manikantan [10] introduced the notion of fuzzy bi-ideals of near-rings and discussed some properties. Meenakumari et al. [12] studied the fuzzy bi-ideals in gamma -near-rings. Cho et al. [20] introduced the concept of weak bi-ideals of near-rings. Thillaigovindan *at al.* [17] studied the interval valued fuzzy quasi-ideals of semigroups. Chinnadurai et al. [3] studied the fuzzy weak bi-ideals of near rings. Thillaigovindan et al. [18] worked on interval valued fuzzy ideals of near-rings.

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In this paper, we define a new notion of an interval valued fuzzy weak bi-ideals of Γ-near-rings, which is a generalized concept of interval valued fuzzy bi-ideals of Γ-near-rings. We also investigate some of its properties and illustrate with examples.

2. Preliminaries

In this section, we list some basic definitions.

Definition 2.1 [14] A near-ring is an algebraic system $(R, +, \cdot)$ consisting of a non empty set R together with two binary operations called $+$ and $\cdot$ such that $(R, +)$ is a group not necessarily abelian and $(R, \cdot)$ is a semigroup connected by the following distributive law: $(x + z) \cdot y = x \cdot y + z \cdot y$ valid for all $x, y, z \in R$. We use the word 'near-ring 'to mean 'right near-ring '. We denote xy instead of $x \cdot y$.

Definition 2.2 [15] A Γ- near-ring is a triple $(M, +, \Gamma)$ where
(i) $(M, +)$ is a group,
(ii) Γ is a nonempty set of binary operators on M such that for each $\alpha \in \Gamma$,
$(M, +, \alpha)$ is a near-ring,
(iii) $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Definition 2.3 [12] A Γ-near-ring M is said to be zero-symmetric if $x\alpha 0 = 0$ for all $x \in M$ and $\alpha \in \Gamma$.

Throughout this paper M denotes a zero-symmetric right Γ- near-ring with atleast two elements.

Definition 2.4 [15] A subset A of a Γ-near-ring M is called a left(resp. right) ideal of M if
(i) $(A, +)$ is a normal subgroup of $(M, +)$, (i.e) $x-y \in A$ for all $x, y \in A$ and $y+x-y \in A$ for $x \in A, y \in M$
(ii) $u\alpha(x + v) - u\alpha v \in A$ (resp. $x\alpha u \in A$) for all $x \in A, \alpha \in \Gamma$ and $u, v \in M$.

Definition 2.5 [15] Let M be a Γ-near-ring. Given two subsets A and B of M, we define $A\Gamma B = \{a\alpha b | a \in A, b \in B \text{ and } \alpha \in \Gamma\}$ and also define another operation $*$ on the class of subset of M define by $A\Gamma * B = \{a\gamma(a' + b) - a\gamma a' | a, a' \in A, \gamma \in \Gamma, b \in B\}$.

Definition 2.6 [16] A subgroup B of $(M, +)$ is called a bi-ideal of M if and only if $B\Gamma M\Gamma B \subseteq B$.

Definition 2.7 [5] A subgroup H of $(M, +)$ is said to be a weak bi-ideal of M if $H\Gamma H\Gamma H \subseteq H$.

The characteristic function of M is denoted by $\mathbf{M}$.

Definition 2.8 [22] If X be any set. A mapping $\eta : X \to D[0, 1]$ is called an interval valued fuzzy subset (briefly, an i.v fuzzy subset) of X, where $D[0, 1]$ denotes the family of closed subintervals of $[0, 1]$ and $\tilde{\eta}(x) = [\eta^-(x), \eta^+(x)]$ for all $x \in X$, where $\eta^-(x)$ and $\eta^+(x)$ are fuzzy subsets of X such that , $\eta^-(x) \leqslant \eta^+(x)$ for all $x \in X$.

Definition 2.9 [17] By an interval number $\tilde{a}$, we mean an interval $[a^-, a^+]$ such that $0 \leqslant a^- \leqslant a^+ \leqslant 1$ and where a^- and a^+ are the lower and upper limits of $\tilde{a}$ respectively. The set of all closed subintervals of $[0, 1]$ is denoted by $D[0, 1]$. We also identify the interval $[a, a]$ by the number $a \in [0, 1]$. For any interval numbers $\tilde{a}_j = [a_j^-, a_j^+], \tilde{b}_j = [b_j^-, b_j^+] \in D[0, 1], j \in \Omega$ we define

$\max^i\{\tilde{a}_j, \tilde{b}_j\} = [\max\{a_j^-, b_j^-\}, \max\{a_j^+, b_j^+\}]$, $\min^i\{\tilde{a}_j, \tilde{b}_j\} = [\min\{a_j^-, b_j^-\}, \max\{a_j^+, b_j^+\}]$, $\inf^i \tilde{a}_j = [\cap_{j\in\Omega} a_j^-, \cap_{j\in\Omega} a_j^+]$, $\sup^i \tilde{a}_j = [\cup_{j\in\Omega} a_j^-, \cup_{j\in\Omega} a_j^+]$, and let
(i) $\tilde{a} \leqslant \tilde{b} \Leftrightarrow a^- \leqslant b^-$ and $a^+ \leqslant b^+$,
(ii) $\tilde{a} = \tilde{b} \Leftrightarrow a^- = b^-$ and $a^+ = b^+$,
(iii) $\tilde{a} < \tilde{b} \Leftrightarrow \tilde{a} \leqslant \tilde{b}$ and $\tilde{a} \neq \tilde{b}$,
(iv) $k\tilde{a} = [ka^-, ka^+]$, whenever $0 \leqslant k \leqslant 1$.

Definition 2.10 [17] Let $\tilde{\eta}$ be an i.v fuzzy subset of X and $[t_1, t_2] \in D[0,1]$. Then the set $\tilde{U}(\tilde{\eta} : [t_1, t_2]) = \{x \in X | \tilde{\eta}(x) \geqslant [t_1, t_2]\}$ is called the upper level subset of $\tilde{\eta}$.

Definition 2.11 [7, 13, 19] If $\tilde{\eta}$ and $\tilde{\lambda}$ are i.v fuzzy subsets of M.
Then $\tilde{\eta} \cap \tilde{\lambda}$, $\tilde{\eta} \cup \tilde{\lambda}$, $\tilde{\eta} + \tilde{\lambda}$, and $\tilde{\eta} * \tilde{\lambda}$ are fuzzy subsets of M defined by:

$$(\tilde{\eta} \cap \tilde{\lambda})(x) = \min^i\{\tilde{\eta}(x),\ \tilde{\lambda}(x)\}.$$

$$(\tilde{\eta} \cup \tilde{\lambda})(x) = \max^i\{\tilde{\eta}(x),\ \tilde{\lambda}(x)\}.$$

$$(\tilde{\eta} + \tilde{\lambda})(x) = \begin{cases} \sup^i_{x=y+z}\{\min^i\{\tilde{\eta}(y),\ \tilde{\lambda}(z)\}\} & \text{if } x \text{ is expressible as } x = y + z \\ 0 & \text{otherwise.} \end{cases}$$

$$(\tilde{\eta} * \tilde{\lambda})(x) = \begin{cases} \sup^i_{x=y\alpha z}\{\min^i\{\tilde{\eta}(y),\ \tilde{\lambda}(z)\}\} & \text{if } x \text{ is expressible as } x = y\alpha z \\ 0 & \text{otherwise.} \end{cases}$$

for $x, y, z \in M$.

Definition 2.12 [5] An i.v fuzzy subset $\tilde{\eta}$ in a Γ-near-ring M is called an i.v fuzzy left (resp. right) ideal of M if
(i) $\tilde{\eta}$ is an i.v fuzzy normal divisor with respect to the addition,
(ii) $\tilde{\eta}(u\alpha(x+v) - u\alpha v) \geqslant \tilde{\eta}(x)$, (resp. $\tilde{\eta}(x\alpha u) \geqslant \tilde{\eta}(x)$ for all $x, u, v \in M$ and $\alpha \in \Gamma$.
The condition (i) of definition 2.12 means that $\tilde{\eta}$ satisfies:
(i) $\tilde{\eta}(x-y) \geqslant \min^i\{\tilde{\eta}(x), \tilde{\eta}(y)\}$,
(ii) $\tilde{\eta}(y + x - y) \geqslant \tilde{\eta}(x)$, for all $x, y \in M$
Note that $\tilde{\eta}$ is an i.v fuzzy left (resp. right) ideal of Γ-near-ring M, then $\tilde{\eta}(0) \geqslant \tilde{\eta}(x)$ for all $x \in M$, where 0 is the zero element of M.

Definition 2.13 [6] An i.v fuzzy subset $\tilde{\eta}$ of M is called an i.v fuzzy bi-ideal of M if
(i) $\tilde{\eta}(x-y) \geqslant \min^i\{\tilde{\eta}(x), \tilde{\eta}(y)\}$ for all $x, y \in M$
(ii) $\tilde{\eta}(x\alpha y\beta z) \geqslant \min^i\{\tilde{\eta}(x), \tilde{\eta}(z)\}$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

3. Interval valued fuzzy weak bi-ideals of Γ-near-rings

In this section, we introduce the notion of i.v fuzzy weak bi-ideal of M and discuss some of its properties.

Definition 3.1 An i.v fuzzy set $\tilde{\eta}$ of M is called an i.v fuzzy weak bi-ideal of M, if
(i) $\tilde{\eta}(x-y) \geqslant \min^i\{\tilde{\eta}(x), \tilde{\eta}(y)\}$ for all $x, y \in M$
(ii) $\tilde{\eta}(x\alpha y\beta z) \geqslant \min^i\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\}$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

***Example* 3.2** Let $M = \{0, a, b, c\}$ be a non-empty set with binary operation $+$ and $\Gamma = \{\alpha, \beta\}$ be a non-empty set of binary operations as shown in the following tables:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

α	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	b	b
c	a	a	c	c

β	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	a	c	b
c	0	a	b	c

Let $\tilde{\eta} : M \to D[0,1]$ be an i.v fuzzy subset defined by $\tilde{\eta}(0) = [0.8, 0.9], \tilde{\eta}(a) = [0.6, 0.7]$, $\tilde{\eta}(b) = \tilde{\eta}(c) = [0.2, 0.3]$. Then $\tilde{\eta}$ is an i.v fuzzy weak bi-ideal of M.

Theorem 3.3 Let $\tilde{\eta}$ be an i.v fuzzy subgroup of M. Then $\tilde{\eta}$ is an i.v fuzzy weak bi-ideal of M if and only if $\tilde{\eta} * \tilde{\eta} * \tilde{\eta} \subseteq \tilde{\eta}$.

Proof. Assume that $\tilde{\eta}$ is an i.v fuzzy weak bi-ideal of M. Let $x, y, z, y_1, y_2 \in M$ and $\alpha, \beta \in \Gamma$ such that $x = y\alpha z$ and $y = y_1\beta y_2$. Then

$$
\begin{aligned}
(\tilde{\eta} * \tilde{\eta} * \tilde{\eta})(x) &= \sup\nolimits^i_{x=y\alpha z}\{\min\nolimits^i\{(\tilde{\eta} * \tilde{\eta})(y), \tilde{\eta}(z)\}\} \\
&= \sup\nolimits^i_{x=y\alpha z}\{\min\nolimits^i\{\sup\nolimits^i_{y=y_1\beta y_2} \min\nolimits^i\{\tilde{\eta}(y_1), \tilde{\eta}(y_2)\}, \tilde{\eta}(z)\}\} \\
&= \sup\nolimits^i_{x=y\alpha z} \sup\nolimits^i_{y=y_1\beta y_2}\{\min\nolimits^i\{\min\nolimits^i\{\tilde{\eta}(y_1), \tilde{\eta}(y_2)\}, \tilde{\eta}(z)\}\} \\
&= \sup\nolimits^i_{x=y_1\beta y_2\alpha z}\{\min\nolimits^i\{\tilde{\eta}(y_1), \tilde{\eta}(y_2), \tilde{\eta}(z)\}\} \\
&\quad (\text{since } \tilde{\eta} \text{ is an i.v fuzzy weak bi-ideal of } M, \tilde{\eta}(y_1\beta y_2\alpha z) \geqslant \min\nolimits^i\{\tilde{\eta}(y_1), \tilde{\eta}(y_2), \tilde{\eta}(z)\}) \\
&\leqslant \sup\nolimits^i_{x=y_1\beta y_2\alpha z} \tilde{\eta}(y_1\beta y_2\alpha z) \\
&= \tilde{\eta}(x).
\end{aligned}
$$

If x can not be expressed as $x = y\alpha z$, then $(\tilde{\eta} * \tilde{\eta} * \tilde{\eta})(x) = \tilde{0} \leqslant \tilde{\eta}(x)$. In both cases $\tilde{\eta} * \tilde{\eta} * \tilde{\eta} \subseteq \tilde{\eta}$. Conversely, assume that $\tilde{\eta} * \tilde{\eta} * \tilde{\eta} \subseteq \tilde{\eta}$. For $x', x, y, z \in M$ and $\alpha, \beta, \alpha_1, \beta_1 \in \Gamma$. Let x' be such that $x' = x\alpha y\beta z$. Then

$$
\begin{aligned}
\tilde{\eta}(x\alpha y\beta z) = \eta(x') &\geqslant (\tilde{\eta} * \tilde{\eta} * \tilde{\eta})(x') \\
&= \sup\nolimits^i_{x'=p\alpha_1 q}\{\min\nolimits^i\{(\tilde{\eta} * \tilde{\eta})(p), \tilde{\eta}(q)\}\} \\
&= \sup\nolimits^i_{x'=p\alpha_1 q}\{\min\nolimits^i\{\sup\nolimits^i_{p=p_1\beta_1 p_2} \min\nolimits^i\{\tilde{\eta}(p_1), \tilde{\eta}(p_2)\}, \tilde{\eta}(q)\}\} \\
&= \sup\nolimits^i_{x'=p_1\beta_1 p_2\alpha_1 q}\{\min\nolimits^i\{\tilde{\eta}(p_1), \tilde{\eta}(p_2), \tilde{\eta}(q)\}\} \\
&\geqslant \min\nolimits^i\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\}.
\end{aligned}
$$

Hence $\tilde{\eta}(x\alpha y\beta z) \geqslant \min^i\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\}$. ■

Lemma 3.4 Let $\tilde{\eta}$ and $\tilde{\lambda}$ be an i.v fuzzy weak bi-ideals of M. Then the products $\tilde{\eta} * \tilde{\lambda}$ and $\tilde{\lambda} * \tilde{\eta}$ are also an i.v fuzzy weak bi-ideals of M.

Proof. Let $\tilde{\eta}$ and $\tilde{\lambda}$ be an i.v fuzzy weak bi-ideals of M and let $\alpha, \alpha_1, \alpha_2 \in \Gamma$. Now

$$\begin{aligned}(\tilde{\eta} * \tilde{\lambda})(x-y) &= \sup^i_{x-y=a\alpha b} \min^i\{\tilde{\eta}(a), \tilde{\lambda}(b)\} \\ &\geqslant \sup^i_{x-y=a_1\alpha_1 b_1 - a_2\alpha_2 b_2 < (a_1-a_2)(b_1-b_2)} \min^i\{\tilde{\eta}(a_1-a_2), \tilde{\lambda}(b_1-b_2)\} \\ &\geqslant \sup^i \min^i\{\min^i\{\tilde{\eta}(a_1), \tilde{\eta}(a_2)\}, \min^i\{\tilde{\lambda}(b_1), \tilde{\lambda}(b_2)\}\} \\ &= \sup^i \min^i\{\min^i\{\tilde{\eta}(a_1), \tilde{\lambda}(b_1)\}, \min^i\{\tilde{\eta}(a_2), \tilde{\lambda}(b_2)\}\} \\ &\geqslant \min^i\{\sup^i_{x=a_1\alpha_1 b_1} \min^i\{\tilde{\eta}(a_1), \tilde{\lambda}(b_1)\}, \sup^i_{y=a_2\alpha_2 b_2} \min^i\{\tilde{\eta}(a_2), \tilde{\lambda}(b_2)\}\} \\ &= \min^i\{(\tilde{\eta} * \tilde{\lambda})(x), (\tilde{\eta} * \tilde{\lambda})(y)\}.\end{aligned}$$

It follows that $\tilde{\eta} * \tilde{\lambda}$ is an i.v fuzzy subgroup of M. Further,

$$\begin{aligned}(\tilde{\eta} * \tilde{\lambda}) * (\tilde{\eta} * \tilde{\lambda}) * (\tilde{\eta} * \tilde{\lambda}) &= \tilde{\eta} * \tilde{\lambda} * (\tilde{\eta} * \tilde{\lambda} * \tilde{\eta}) * \tilde{\lambda} \\ &\subseteq \tilde{\eta} * \tilde{\lambda} * (\tilde{\lambda} * \tilde{\lambda} * \tilde{\lambda}) * \tilde{\lambda}, \text{ since } \tilde{\lambda} \text{ is an i.v fuzzy weak bi-ideal of } M \\ &\subseteq \tilde{\eta} * (\tilde{\lambda} * \tilde{\lambda} * \tilde{\lambda}), \text{ since } \tilde{\lambda} \text{ is an i.v fuzzy weak bi-ideal of } M \\ &\subseteq \tilde{\eta} * \tilde{\lambda}.\end{aligned}$$

Therefore $\tilde{\eta} * \tilde{\lambda}$ is an i.v fuzzy weak bi-ideal of M. Similarly $\tilde{\lambda} * \tilde{\eta}$ is an i.v fuzzy weak bi-ideal of M. ■

Lemma 3.5 Every i.v fuzzy ideal of M is an i.v fuzzy bi-ideal of M.

Proof. Let $\tilde{\eta}$ be an i.v fuzzy ideal of M. Then

$$\tilde{\eta} * \mathbf{M} * \tilde{\eta} \subseteq \tilde{\eta} * \mathbf{M} * \mathbf{M} \subseteq \tilde{\eta} * \mathbf{M} \subseteq \tilde{\eta}$$

since $\tilde{\eta}$ is an i.v fuzzy ideal of M. This implies that $\tilde{\eta} * \mathbf{M} * \tilde{\eta} \subseteq \tilde{\eta}$. Therefore $\tilde{\eta}$ is an i.v fuzzy bi-ideal of M. ■

Theorem 3.6 Every i.v fuzzy bi-ideal of M is an i.v fuzzy weak bi-ideal of M.

Proof. Assume that $\tilde{\eta}$ is an i.v fuzzy bi-ideal of M. Then $\tilde{\eta} * \mathbf{M} * \tilde{\eta} \subseteq \tilde{\eta}$. We have $\tilde{\eta} * \tilde{\eta} * \tilde{\eta} \subseteq \tilde{\eta} * \mathbf{M} * \tilde{\eta}$. This implies that $\tilde{\eta} * \tilde{\eta} * \tilde{\eta} \subseteq \tilde{\eta} * \mathbf{M} * \tilde{\eta} \subseteq \tilde{\eta}$. Therefore $\tilde{\eta}$ is an i.v fuzzy weak bi-ideal of M. ■

Theorem 3.7 Every i.v fuzzy ideal of M is an i.v fuzzy weak bi-ideal of M.

Proof. By Lemma 3.5, every i.v fuzzy ideal of M is an i.v fuzzy bi-ideal of M. By Theorem 3.6, every i.v fuzzy bi-ideal of M is an i.v fuzzy weak bi-ideal of M. Theorefore $\tilde{\eta}$ is an i.v fuzzy weak bi-ideal of M. ■

However the converse of the Theorems 3.6 and 3.7 is not true in general which is demonstrated by the following example.

***Example* 3.8** Let $M = \{0, a, b, c\}$ be a non-emptyset with binary operation $+$ and $\Gamma = \{\alpha, \beta\}$ be a nonempty set of binary operations as shown in the following tables:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

α	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

β	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	a	c	b
c	0	a	b	c

Let $\tilde{\eta} : M \to D[0,1]$ be an i.v fuzzy subset defined by $\tilde{\eta}(0) = [0.7, 0.8], \tilde{\eta}(a) = [0.3, 0.4] = \tilde{\eta}(b)$ and $\tilde{\eta}(c) = [0.5, 0.6]$. Then $\tilde{\eta}$ is an i.v fuzzy weak bi-ideal of M. But $\tilde{\eta}$ is not an i.v fuzzy ideal and bi-ideal of M, since $\tilde{\eta}(b\alpha c) = \tilde{\eta}(b) = [0.3, 0.4] \ngeqslant [0.5, 0.6] = \tilde{\eta}(c), \tilde{\eta}(b\beta c) = \tilde{\eta}(b) = [0.3, 0.4] \ngeqslant [0.5, 0.6] = \tilde{\eta}(c)$ and $\tilde{\eta}(c\alpha b\beta c) = \tilde{\eta}(b) = [0.3, 0.4] \ngeqslant [0.5, 0.6] = \min^i\{\tilde{\eta}(c), \tilde{\eta}(c)\}$.

Theorem 3.9 Let $\{\tilde{\eta}_i | i \in \Omega\}$ be family of i.v fuzzy weak bi-ideals of a Γ- near-ring M, then $\bigcap\limits_{i\in\Omega}\tilde{\eta}_i$ is also an i.v fuzzy weak bi-ideal of M, where Ω is any index set.

Proof. Let $\{\tilde{\eta}_i | i \in \Omega\}$ be a family of i.v fuzzy weak bi-ideals of M. Let $x, y, z \in M$, $\alpha, \beta \in \Gamma$ and $\tilde{\eta} = \bigcap\limits_{i\in\Omega}\tilde{\eta}_i$. Then, $\tilde{\eta}(x) = \bigcap\limits_{i\in\Omega}\tilde{\eta}_i(x) = \left(\inf^i_{i\in\Omega}\tilde{\eta}_i\right)(x) = \inf^i_{i\in\Omega}\tilde{\eta}_i(x)$. Now,

$$
\begin{aligned}
\tilde{\eta}(x-y) &= \inf^i_{i\in\Omega}\ \tilde{\eta}_i(x-y) \\
&\geqslant \inf^i_{i\in\Omega}\ \min^i\{\tilde{\eta}_i(x), \tilde{\eta}_i(y)\} \\
&= \min^i\left\{\inf^i_{i\in\Omega}\tilde{\eta}_i(x), \inf^i_{i\in\Omega}\tilde{\eta}_i(y)\right\} \\
&= \min^i\left\{\bigcap_{i\in\Omega}\tilde{\eta}_i(x), \bigcap_{i\in\Omega}\tilde{\eta}_i(y)\right\} \\
&= \min^i\{\tilde{\eta}(x), \tilde{\eta}(y)\},
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{\eta}(x\alpha y\beta z) &= \inf^i_{i\in\Omega}\ \tilde{\eta}_i(x\alpha y\beta z) \\
&\geqslant \inf^i_{i\in\Omega}\ \min^i\{\tilde{\eta}_i(x), \tilde{\eta}_i(y), \tilde{\eta}_i(z)\} \\
&= \min^i\left\{\inf^i_{i\in\Omega}\tilde{\eta}_i(x), \inf^i_{i\in\Omega}\tilde{\eta}_i(y), \inf^i_{i\in\Omega}\tilde{\eta}_i(z)\right\} \\
&= \min^i\left\{\bigcap_{i\in\Omega}\tilde{\eta}_i(x), \bigcap_{i\in\Omega}\tilde{\eta}_i(y), \bigcap_{i\in\Omega}\tilde{\eta}_i(z)\right\} \\
&= \min^i\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\}.
\end{aligned}
$$

■

Theorem 3.10 Let $\tilde{\eta}$ be an i.v fuzzy subset of M. Then $\tilde{\eta}$ is an i.v fuzzy weak bi-ideal of M if and only if $\tilde{U}(\tilde{\eta} : [t_1, t_2])$ is a weak bi-ideal of M, for all $[t_1, t_2] \in D[0,1]$.

Proof. Assume that $\tilde{\eta}$ is an i.v fuzzy weak bi-ideal of M. Let $[t_1, t_2] \in D[0,1]$ such that $x, y \in \tilde{U}(\tilde{\eta} : [t_1, t_2])$.Then $\tilde{\eta}(x-y) \geqslant \min^i\{\tilde{\eta}(x), \tilde{\eta}(y)\} \geqslant \min^i\{[t_1, t_2], [t_1, t_2]\} = [t_1, t_2]$. Thus $x - y \in \tilde{U}(\tilde{\eta} : [t_1, t_2])$. Let $x, y, z \in \tilde{U}(\tilde{\eta} : [t_1, t_2])$ and $\alpha, \beta \in \Gamma$. We have $\tilde{\eta}(x\alpha y\beta z) \geqslant \min^i\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\} \geqslant \min^i\{[t_1, t_2], [t_1, t_2], [t_1, t_2]\} = [t_1, t_2]$. Therefore $x\alpha y\beta z \in \tilde{U}(\tilde{\eta} : [t_1, t_2])$. Hence $\tilde{U}(\tilde{\eta} : [t_1, t_2])$ is a weak bi-ideal of M.

Conversely, assume $\tilde{U}(\tilde{\eta} : [t_1, t_2])$ is a weak bi-ideal of M, for all $[t_1, t_2] \in D[0, 1]$. Let $x, y \in M$. Suppose $\tilde{\eta}(x - y) < \min^i\{\tilde{\eta}(x), \tilde{\eta}(y)\}$. Choose $[0, 0] < [t_1, t_2] \leqslant [1, 1]$ such that $\tilde{\eta}(x - y) < [t_1, t_2] < \min^i\{\tilde{\eta}(x), \tilde{\eta}(y)\}$. This implies that $\tilde{\eta}(x) > [t_1, t_2], \tilde{\eta}(y) > [t_1, t_2]$ and $\tilde{\eta}(x - y) < [t_1, t_2]$. Then we have $x, y \in \tilde{U}(\tilde{\eta} : [t_1, t_2])$, but $x - y \notin \tilde{U}(\tilde{\eta} : [t_1, t_2])$ a contradiction. Thus, $\tilde{\eta}(x - y) \geqslant \min^i\{\tilde{\eta}(x), \tilde{\eta}(y)\}$. If there exist $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ such that $\tilde{\eta}(x\alpha y\beta z) < \min^i\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\}$.Choose $[t_1, t_2]$ such that $\tilde{\eta}(x\alpha y\beta z) < [t_1, t_2] < \min^i\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\}$. Then $\tilde{\eta}(x) > [t_1, t_2], \tilde{\eta}(y) > [t_1, t_2], \tilde{\eta}(z) > [t_1, t_2]$ and $\tilde{\eta}(x\alpha y\beta z) < [t_1, t_2]$. So, $x, y, z \in \tilde{U}(\tilde{\eta} : [t_1, t_2])$, but $x\alpha y\beta z \notin \tilde{U}(\tilde{\eta} : [t_1, t_2])$, which is a contradiction. Hence $\tilde{\eta}(x\alpha y\beta z) \geqslant \min^i\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\}$. Therefore, $\tilde{\eta}$ is an i.v fuzzy weak bi-ideal of M. ■

Theorem 3.11 Let $\tilde{\eta} = [\eta^-, \eta^+]$ be an i.v fuzzy subset of M, then $\tilde{\eta}$ is an i.v fuzzy weak bi-ideal of near-ring M if and only if η^-, η^+ are fuzzy weak bi-ideals of M.

Proof. Assume that $\tilde{\eta}$ is an i.v fuzzy weak bi-ideals of near-ring M. For any $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Now,

$$\begin{aligned}
[\eta^-(x - y), \eta^+(x - y)] &= \tilde{\eta}(x - y)\\
&\geqslant \min^i\{\tilde{\eta}(x), \tilde{\eta}(y)\}\\
&= \min^i\{[\eta^-(x), \eta^+(x)], [\eta^-(y), \eta^+(y)]\}\\
&= \min^i\{[\eta^-(x), \eta^-(y)], \min^i[\eta^+(x), \eta^+(y)]\}.
\end{aligned}$$

It follows that $\eta^-(x - y) \geqslant \min\{\eta^-(x), \eta^-(y)\}$ and $\eta^+(x - y) \geqslant \min\{\eta^+(x), \eta^+(y)\}$.

$$\begin{aligned}
[\eta^-(x\alpha y\beta z), \eta^+(x\alpha y\beta z)] &= \tilde{\eta}(x\alpha y\beta z)\\
&\geqslant \min^i\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\}\\
&= \min^i\{[\eta^-(x), \eta^+(x)], [\eta^-(y), \eta^+(y)], [\eta^-(z), \eta^+(z)]\}\\
&= \min^i\{[\eta^-(x), \eta^-(y), \eta^-(z)], \min^i[\eta^+(x), \eta^+(y), \eta^+(z)]\}.
\end{aligned}$$

It follows that $\eta^-(x\alpha y\beta z) \geqslant \min\{\eta^-(x), \eta^-(y), \eta^-(z)\}$ and $\eta^+(x\alpha y\beta z) \geqslant \min\{\eta^+(x), \eta^+(y), \eta^+(z)\}$. Conversely, assume that η^-, η^+ are fuzzy weak bi-ideals of near-ring M. Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

$$\begin{aligned}
\eta^-(x - y) &= [\eta^-(x - y), \eta^+(x - y)]\\
&\geqslant \min\{[\eta^-(x), \eta^-(y)], \min[\eta^+(x), \eta^+(y)]\}\\
&= \min^i\{[\eta^-(x), \eta^+(x)], [\eta^-(y), \eta^+(y)]\}\\
&= \min^i\{\tilde{\eta}(x), \tilde{\eta}(y)\}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\eta}(x\alpha y\beta z) &= [\eta^-(x\alpha y\beta z), \eta^+(x\alpha y\beta z)]\\
&\geqslant \min[\{\eta^-(x), \eta^-(y), \eta^-(z)\}, \min\{\eta^+(x), \eta^+(y), \eta^+(z)\}]\\
&= \min^i\{[\eta^-(x), \eta^+(x)], [\eta^-(y), \eta^+(y)], [\eta^-(z), \eta^+(z)]\}\\
&= \min^i\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\}
\end{aligned}$$

Therefore $\tilde{\eta}$ is an i.v fuzzy weak bi-ideals of near-ring M. ■

Theorem 3.12 Let I be a weak bi-ideal of near-ring M then for any $[t_1, t_2] \in D[0,1]$ with $[t_1, t_2] \neq [0,0]$, there exists an i.v fuzzy weak bi-ideal $\tilde{\eta}$ of M such that $\tilde{U}(\tilde{\eta} : [t_1, t_2]) = I$.

Proof. Let I be a weak bi-ideal of M. Let $\tilde{\eta}$ be an i.v fuzzy subset of M defined by

$$\tilde{\eta}(x) = \begin{cases} [t_1, t_2] & \text{if } x \in I \\ \tilde{0} & \text{otherwise} \end{cases}$$

Then $\tilde{U}(\tilde{\eta} : [t_1, t_2]) = I$. Assume that $\tilde{\eta}(x-y) < \min^i\{\tilde{\eta}(x), \tilde{\eta}(y)\}$. This implies that $\tilde{\eta}(x-y) = \tilde{0}$ and $\min^i\{\tilde{\eta}(x), \tilde{\eta}(y)\} = [t_1, t_2]$ so $x, y \in I$ and $\alpha, \beta \in \Gamma$ but $x - y \notin I$, which is a contradiction. Thus, $\tilde{\eta}(x-y) \geqslant \min^i\{\tilde{\eta}(x), \tilde{\eta}(y)\}$. Suppose that $\tilde{\eta}(x\alpha y\beta z) < \min^i\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\}$. Then $\tilde{\eta}(x\alpha y\beta z) = \tilde{0}$ and $\min^i\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\} = [t_1, t_2]$ so $x, y, z \in I$ but $x\alpha y\beta z \notin I$ which is a contradiction. Hence $\tilde{\eta}(x\alpha y\beta z) \geqslant \min^i\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\}$. ■

Theorem 3.13 Let H be a nonempty subset of M and $\tilde{\eta}$ be an i.v fuzzy subset of M defined by

$$\tilde{\eta}(x) = \begin{cases} \tilde{s} & \text{if } x \in H \\ \tilde{t} & \text{otherwise} \end{cases}$$

for some $x \in M$, $\tilde{s}, \tilde{t} \in D[0,1]$ and $\tilde{s} > \tilde{t}$. Then H is a weak bi-ideal of M if and only if $\tilde{\eta}$ is an i.v fuzzy weak bi-ideal of H.

Proof. Assume that H is a weak bi-ideal of M. Let $x, y \in M$. We consider four Cases:
(1) $x \in H$ and $y \in H$.
(2) $x \in H$ and $y \notin H$.
(3) $x \notin H$ and $y \in H$.
(4) $x \notin H$ and $y \notin H$.
Case (1): If $x \in H$ and $y \in H$. Then $\eta(x) = \tilde{s} = \eta(y)$. Since H is a weak bi-ideal of M, then $x - y \in H$. Thus, $\tilde{\eta}(x-y) = \tilde{s} = \min^i\{\tilde{s}, \tilde{s}\} = \min^i\{\tilde{\eta}(x), \tilde{\eta}(y)\}$.
Case (2): If $x \in H$ and $y \notin H$. Then $\tilde{\eta}(x) = \tilde{s}$ and $\tilde{\eta}(y) = \tilde{t}$. So, $\min^i\{\tilde{\eta}(x), \tilde{\eta}(y)\} = \tilde{t}$. Now, $\tilde{\eta}(x-y) = \tilde{s}$ or $\tilde{t}$ according as $x - y \in H$ or $x - y \notin H$. By assumption, $\tilde{s} > \tilde{t}$, we have $\tilde{\eta}(x-y) \geqslant \min^i\{\tilde{\eta}(x), \tilde{\eta}(y)\}$. Similarly, we prove Case (3).
Case (4): $x, y \notin H$, we have, $\tilde{\eta}(x) = \tilde{t} = \tilde{\eta}(y)$. So, $\min^i\{\tilde{\eta}(x), \tilde{\eta}(y)\} = \tilde{t}$. Next, $\tilde{\eta}(x-y) = \tilde{s}$ or $\tilde{t}$, according as $x - y \in H$ or $x - y \notin H$. So, $\tilde{\eta}(x-y) \geqslant \min^i\{\tilde{\eta}(x), \tilde{\eta}(y)\}$. Now let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. We have the following eight Cases.
(1) $x \in H, y \in H$ and $z \in H$.
(2) $x \notin H, y \in H$ and $z \in H$.
(3) $x \in H, y \notin H$ and $z \in H$.
(4) $x \in H, y \in H$ and $z \notin H$.
(5) $x \notin H, y \notin H$ and $z \in H$.
(6) $x \in H, y \notin H$ and $z \notin H$.
(7) $x \notin H, y \in H$ and $z \notin H$.
(8) $x \notin H, y \notin H$ and $z \notin H$.
These cases can be proved by arguments similar to the fuzzy cases above. Hence,
$\tilde{\eta}(x\alpha y\beta z) \geqslant \min^i\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\}$. Hence $\tilde{\eta}$ is an i.v fuzzy weak bi-ideal of M. Conversely, assume that $\tilde{\eta}$ is an i.v fuzzy weak bi-ideal of M. Let $x, y, z \in H$ and $\alpha, \beta \in \Gamma$ be such that $\tilde{\eta}(x) = \tilde{\eta}(y) = \tilde{\eta}(z) = \tilde{s}$. Since $\tilde{\eta}$ is an i.v fuzzy weak bi-ideal of M, we

have $\tilde{\eta}(x-y) \geqslant \min^i\{\tilde{\eta}(x), \tilde{\eta}(y)\} = \tilde{s}$ and $\tilde{\eta}(x\alpha y\beta z) \geqslant \min^i\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\} = \tilde{s}$. So, $x-y, x\alpha y\beta z \in H$. Hence H is a weak bi-ideal of M. ■

Theorem 3.14 A nonempty subset H of M is a weak bi-ideal of M if and only if the characteristic function f_H is an i.v fuzzy weak bi-ideal of M.

Proof. The proof is straightforward. ■

Theorem 3.15 Let $\tilde{\eta}$ be an i.v fuzzy weak bi-ideal of M then the set $M_{\tilde{\eta}} = \{x \in M | \ \tilde{\eta}(x) = \tilde{\eta}(0)\}$ is weak bi-ideal of M.

Proof. Let $\tilde{\eta}$ be i.v fuzzy weak bi-ideal of M. Let $x, y \in M_{\tilde{\eta}}$. Then $\tilde{\eta}(x) = \tilde{\eta}(0), \tilde{\eta}(y) = \tilde{\eta}(0)$ and $\tilde{\eta}(x-y) \geqslant \min^i\{\tilde{\eta}(x), \tilde{\eta}(y)\} = \min^i\{\tilde{\eta}(0), \tilde{\eta}(0)\} = \tilde{\eta}(0)$. So $\tilde{\eta}(x-y) = \tilde{\eta}(0)$. Thus $x-y \in M_{\tilde{\eta}}$. For every $x, y, z \in M_{\tilde{\eta}}$ and $\alpha, \beta \in \Gamma$ we have $\tilde{\eta}(x\alpha y\beta z) \geqslant \min^i\{\tilde{\eta}(x), \tilde{\eta}(y), \tilde{\eta}(z)\} = \min^i\{\tilde{\eta}(0), \tilde{\eta}(0), \tilde{\eta}(0)\} = \tilde{\eta}(0)$. Thus $x\alpha y\beta z \in M_{\tilde{\eta}}$. Hence $M_{\tilde{\eta}}$ is a weak bi-ideal of M. ■

4. Homomorphism of interval valued fuzzy weak bi-ideals of Γ-near-rings

In this section, we characterize i.v fuzzy weak bi-ideals of Γ-near-rings using homomorphism.

Definition 4.1 [9] Let f be a mapping from a set M to a set S. Let $\tilde{\eta}$ and $\tilde{\delta}$ be i.v fuzzy subsets of M and S respectively. Then $f(\tilde{\eta})$, the image of $\tilde{\eta}$ under f is an i.v fuzzy subset of S defined by

$$f(\tilde{\eta})(y) = \begin{cases} \sup^i_{x \in f^{-1}(y)} \tilde{\eta}(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and the pre-image of $\tilde{\eta}$ under f is an i.v fuzzy subset of M defined by
$f^{-1}(\tilde{\delta}(x)) = \tilde{\delta}(f(x))$, for all $x \in M$ and $f^{-1}(y) = \{x \in M | f(x) = y\}$.

Definition 4.2 [9] Let M and S be Γ-near-rings. A map $\theta : M \to S$ is called a (Γ-near-ring)homomorphism if $\theta(x+y) = \theta(x) + \theta(y)$ and $\theta(x\alpha y) = \theta(x)\alpha\theta(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Theorem 4.3 Let $f : M \to S$ be a homomorphism between Γ-near-rings M and S. If $\tilde{\delta}$ is an i.v fuzzy weak bi-ideal of S, then $f^{-1}(\tilde{\delta})$ is an i.v fuzzy weak bi-ideal of M.

Proof. Let $\tilde{\delta}$ is an i.v fuzzy weak bi-ideal of S. Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned} f^{-1}(\tilde{\delta})(x-y) &= \tilde{\delta}(f(x-y)) \\ &= \tilde{\delta}(f(x) - f(y)) \\ &\geqslant \min^i\{\tilde{\delta}(f(x)), \tilde{\delta}(f(y))\} \\ &= \min^i\{f^{-1}(\tilde{\delta}(x)), f^{-1}(\tilde{\delta}(y))\} \end{aligned}$$

and

$$\begin{aligned} f^{-1}(\tilde{\delta})(x\alpha y\beta z) &= \tilde{\delta}(f(x\alpha y\beta z)) \\ &= \tilde{\delta}(f(x)\alpha f(y)\beta f(z)) \\ &\geqslant \min^i\{\tilde{\delta}(f(x)), \tilde{\delta}(f(y)), \tilde{\delta}(f(z))\} \\ &= \min^i\{f^{-1}(\tilde{\delta}(x)), f^{-1}(\tilde{\delta}(y)), f^{-1}(\tilde{\delta}(z))\}. \end{aligned}$$

Therefore $f^{-1}(\tilde{\delta})$ is an i.v fuzzy weak bi-ideal of M. ■

We can also state the converse of the Theorem 4.3 by strengthening the condition on f as follows.

Theorem 4.4 Let $f : M \to S$ be an onto homomorphism of Γ- near-rings M and S. Let $\tilde{\delta}$ be an i.v fuzzy subset of S. If $f^{-1}(\tilde{\delta})$ is an i.v fuzzy weak bi-ideal of M, then $\tilde{\delta}$ is an i.v fuzzy weak bi-ideal of S.

Proof. Let $x, y, z \in S$. Then $f(a) = x, f(b) = y$ and $f(c) = z$ for some $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. It follows that

$$\begin{aligned} \tilde{\delta}(x-y) &= \tilde{\delta}(f(a)-f(b)) \\ &= \tilde{\delta}(f(a-b)) \\ &= f^{-1}(\tilde{\delta})(a-b) \\ &\geqslant \min^i\{f^{-1}(\tilde{\delta})(a), f^{-1}(\tilde{\delta})(b)\} \\ &= \min^i\{\tilde{\delta}(f(a)), \tilde{\delta}(f(b))\} \\ &= \min^i\{\tilde{\delta}(x), \tilde{\delta}(y)\}. \end{aligned}$$

and

$$\begin{aligned} \tilde{\delta}(x\alpha y\beta z) &= \tilde{\delta}(f(a)\alpha f(b)\beta f(c)) \\ &= \tilde{\delta}(f(a\alpha b\beta c)) \\ &= f^{-1}(\tilde{\delta})(a\alpha b\beta c) \\ &\geqslant \min^i\{f^{-1}(\tilde{\delta})(a), f^{-1}(\tilde{\delta})(b), f^{-1}(\tilde{\delta})(c)\} \\ &= \min^i\{\tilde{\delta}(f(a)), \tilde{\delta}(f(b)), \tilde{\delta}(f(c))\} \\ &= \min^i\{\tilde{\delta}(x), \tilde{\delta}(y), \tilde{\delta}(z)\}. \end{aligned}$$

Hence $\tilde{\delta}$ is an i.v fuzzy weak bi-ideal of S. ■

Theorem 4.5 Let $f : M \to S$ be an onto Γ-near-ring homomorphism. If $\tilde{\eta}$ is an i.v fuzzy weak bi-ideal of M, then $f(\tilde{\eta})$ is an i.v fuzzy weak bi-ideal of S.

Proof. Let $\tilde{\eta}$ be an i.v fuzzy weak bi-ideal of M. Since $f(\tilde{\eta})(x') = \sup^i_{f(x)=x'}(\tilde{\eta}(x))$ for $x' \in S$ and hence $f(\tilde{\eta})$ is nonempty. Let $x', y' \in S$ and $\alpha, \beta \in \Gamma$. Then we have $\{x|x \in f^{-1}(x'-y')\} \supseteq \{x-y|x \in f^{-1}(x') \text{ and } y \in f^{-1}(y')\}$ and $\{x|x \in f^{-1}(x'y')\} \supseteq$

$\{x\alpha y | x \in f^{-1}(x') \text{ and } y \in f^{-1}(y')\}$.

$$\begin{aligned} f(\tilde{\eta})(x'-y') &= \sup^i_{f(z)=x'-y'}\{\tilde{\eta}(z)\} \\ &\geqslant \sup^i_{f(x)=x',f(y)=y'}\{\tilde{\eta}(x-y)\} \\ &\geqslant \sup^i_{f(x)=x',f(y)=y'}\{\min^i\{\tilde{\eta}(x),\tilde{\eta}(y)\}\} \\ &= \min^i\{\sup^i_{f(x)=x'}\{\tilde{\eta}(x)\},\sup^i_{f(y)=y'}\{\tilde{\eta}(y)\}\} \\ &= \min^i\{f(\tilde{\eta})(x'),f(\tilde{\eta})(y')\}. \end{aligned}$$

Next,

$$\begin{aligned} f(\tilde{\eta})(x'\alpha y'\beta z') &= \sup^i_{f(h)=x'\alpha y'\beta z'}\{\tilde{\eta}(h)\} \\ &\geqslant \sup^i_{f(x)=x',f(y)=y',f(z)=z'}\{\tilde{\eta}(x\alpha y\beta z)\} \\ &\geqslant \sup^i_{f(x)=x',f(y)=y',f(z)=z'}\{\min^i\{\tilde{\eta}(x),\tilde{\eta}(y),\tilde{\eta}(z)\}\} \\ &= \min^i\{\sup^i_{f(x)=x'}\{\tilde{\eta}(x)\},\sup^i_{f(y)=y'}\{\tilde{\eta}(y)\},\sup^i_{f(z)=z'}\{\tilde{\eta}(z)\}\} \\ &= \min^i\{f(\tilde{\eta})(x'),f(\tilde{\eta})(y'),f(\tilde{\eta})(z')\}. \end{aligned}$$

Therefore $f(\tilde{\eta})$ is an i.v fuzzy weak bi-ideal of S. ■

5. Anti-homomorphism of interval valued fuzzy weak bi-ideals of Γ-near-rings

In this section, we characterize i.v fuzzy weak bi-ideals of Γ-near-rings using anti-homomorphism.

Definition 5.1 [11] Let M and S be Γ-near-rings. A map $\theta : M \to S$ is called a (Γ-near-ring)anti-homomorphism if $\theta(x+y) = \theta(y) + \theta(x)$ and $\theta(x\alpha y) = \theta(y)\alpha\theta(x)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Theorem 5.2 Let $f : M \to S$ be a anti-homomorphism between Γ-near-rings M and S. If $\tilde{\delta}$ is an i.v fuzzy weak bi-ideal of S, then $f^{-1}(\tilde{\delta})$ is an i.v fuzzy weak bi-ideal of M.

Proof. Let $\tilde{\delta}$ be an i.v fuzzy weak bi-ideal of S. Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned} f^{-1}(\tilde{\delta})(x-y) &= \tilde{\delta}(f(x-y)) \\ &= \tilde{\delta}(f(y)-f(x)) \\ &\geqslant \min^i\{\tilde{\delta}(f(y)),\tilde{\delta}(f(x))\} \\ &= \min^i\{\tilde{\delta}(f(x)),\tilde{\delta}(f(y))\} \\ &= \min^i\{f^{-1}(\tilde{\delta}(x)),f^{-1}(\tilde{\delta}(y))\}. \end{aligned}$$

and

$$\begin{aligned} f^{-1}(\tilde{\delta})(x\alpha y\beta z) &= \tilde{\delta}(f(x\alpha y\beta z)) \\ &= \tilde{\delta}(f(z)\alpha f(y)\beta f(x)) \\ &\geqslant \min{}^i\{\tilde{\delta}(f(z)), \tilde{\delta}(f(y)), \tilde{\delta}(f(x))\} \\ &= \min{}^i\{\tilde{\delta}(f(x)), \tilde{\delta}(f(y)), \tilde{\delta}(f(z))\} \\ &= \min{}^i\{f^{-1}(\tilde{\delta}(x)), f^{-1}(\tilde{\delta}(y)), f^{-1}(\tilde{\delta}(z))\}. \end{aligned}$$

Therefore $f^{-1}(\tilde{\delta})$ is an i.v fuzzy weak bi-ideal of M. ■

We can also state the converse of the Theorem 5.2 by strengthening the condition on f as follows.

Theorem 5.3 Let $f : M \to S$ be an onto anti-homomorphism of Γ-near-rings M and S. Let $\tilde{\delta}$ be an i.v fuzzy subset of S. If $f^{-1}(\tilde{\delta})$ is an i.v fuzzy weak bi-ideal of M, then $\tilde{\delta}$ is an i.v fuzzy weak bi-ideal of S.

Proof. Let $x, y, z \in S$. Then $f(a) = x, f(b) = y$ and $f(c) = z$ for some $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. It follows that

$$\begin{aligned} \tilde{\delta}(x - y) &= \tilde{\delta}(f(a) - f(b)) \\ &= \tilde{\delta}(f(b - a)) \\ &= f^{-1}(\tilde{\delta})(b - a) \\ &\geqslant \min{}^i\{f^{-1}(\tilde{\delta})(b), f^{-1}(\tilde{\delta})(a)\} \\ &= \min{}^i\{\tilde{\delta}(f(a)), \tilde{\delta}(f(b))\} \\ &= \min{}^i\{\tilde{\delta}(x), \tilde{\delta}(y)\} \end{aligned}$$

and

$$\begin{aligned} \tilde{\delta}(x\alpha y\beta z) &= \tilde{\delta}(f(a)\alpha f(b)\beta f(c)) \\ &= \tilde{\delta}(f(c\alpha b\beta a)) \\ &= f^{-1}(\tilde{\delta})(c\alpha b\beta a) \\ &\geqslant \min{}^i\{f^{-1}(\tilde{\delta})(c), f^{-1}(\tilde{\delta})(b), f^{-1}(\tilde{\delta})(a)\} \\ &= \min{}^i\{f^{-1}(\tilde{\delta})(a), f^{-1}(\tilde{\delta})(b), f^{-1}(\tilde{\delta})(c)\} \\ &= \min{}^i\{\tilde{\delta}(f(a)), \tilde{\delta}(f(b)), \tilde{\delta}(f(c))\} \\ &= \min{}^i\{\tilde{\delta}(x), \tilde{\delta}(y), \tilde{\delta}(z)\}. \end{aligned}$$

Hence $\tilde{\delta}$ is an i.v fuzzy weak bi-ideal of S. ■

Theorem 5.4 Let $f : M \to S$ be an onto Γ-near-ring anti-homomorphism. If $\tilde{\eta}$ is an i.v fuzzy weak bi-ideal of M, then $f(\tilde{\eta})$ is an i.v fuzzy weak bi-ideal of S.

Proof. Let $\tilde{\eta}$ be an i.v fuzzy weak bi-ideal of M. Since

$f(\tilde{\eta})(x') = \sup^i_{f(x)=x'}(\tilde{\eta}(x))$, for $x' \in S$ and hence $f(\tilde{\eta})$ is nonempty. Let $x', y' \in S$ and $\alpha, \beta \in \Gamma$. Then we have $\{x|x \in f^{-1}(x'-y')\} \supseteq \{x-y|x \in f^{-1}(x') \text{ and } y \in f^{-1}(y')\}$ and $\{x|x \in f^{-1}(x'y')\} \supseteq \{x\alpha y|x \in f^{-1}(x') \text{ and } y \in f^{-1}(y')\}$

$$\begin{aligned} f(\tilde{\eta})(x'-y') &= \sup^i_{f(z)=x'-y'}\{\tilde{\eta}(z)\} \\ &\geqslant \sup^i_{f(x)=x',f(y)=y'}\{\tilde{\eta}(x-y)\} \\ &\geqslant \sup^i_{f(x)=x',f(y)=y'}\{\min^i\{\tilde{\eta}(x),\tilde{\eta}(y)\}\} \\ &= \min^i\{\sup_{f(x)=x'}\{\tilde{\eta}(x)\}, \sup^i_{f(y)=y'}\{\tilde{\eta}(y)\}\} \\ &= \min^i\{f(\tilde{\eta})(x'), f(\tilde{\eta})(y')\}. \end{aligned}$$

Next,

$$\begin{aligned} f(\tilde{\eta})(x'\alpha y'\beta z') &= \sup^i_{f(h)=x'\alpha y'\beta z'}\{\tilde{\eta}(h)\} \\ &\geqslant \sup^i_{f(x)=x',f(y)=y',f(z)=z'}\{\tilde{\eta}(x\alpha y\beta z)\} \\ &\geqslant \sup^i_{f(x)=x',f(y)=y',f(z)=z'}\{\min^i\{\tilde{\eta}(x),\tilde{\eta}(y),\tilde{\eta}(z)\}\} \\ &= \min^i\{\sup^i_{f(x)=x'}\{\tilde{\eta}(x)\}, \sup^i_{f(y)=y'}\{\tilde{\eta}(y)\}, \sup^i_{f(z)=z'}\{\tilde{\eta}(z)\}\} \\ &= \min^i\{f(\tilde{\eta})(x'), f(\tilde{\eta})(y'), f(\tilde{\eta})(z')\}. \end{aligned}$$

Therefore $f(\tilde{\eta})$ is an i.v fuzzy weak bi-ideal of S. ■

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