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Common fixed point of four maps in S_b-metric spaces

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Abstract. In this paper is introduced a new type of generalization of metric spaces called S_b metric space. For this new kind of spaces it has been proved a common fixed point theorem for four mappings which satisfy generalized contractive condition. We also present example to confirm our theorem.

Keywords: Common fixed point, S_b-metric spaces, compatible mappings, Cauchy sequence.

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1. Introduction

The Banach contraction principle is the most celebrated fixed point theorem and has been generalized in various directions, see ([1]-[12]). Fixed point problems for contractive mappings in metric spaces with a partially order have been studied by many authors. Sedghi and Shobe [12] proved a common fixed point of four maps in complete metric spaces. Abbas et al. in [1] proved a common fixed points of four mappings satisfying a generalized weak contractive condition in the partially ordered metric spaces. Roshan et al. [8] proved a common fixed point of four maps in b-metric spaces.

The aim of this paper is to present some common fixed point results for four mappings satisfying generalized contractive condition in a S_b-metric space, where the S_b-metric is

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not necessary continuous. First we recall some notions, lemmas and examples which will be useful later.

Definition 1.1 [10] Let X be a nonempty set. A S-metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions for all $x, y, z, a \in X$.

(S1) $0 < S(x, y, z)$ for all $x, y, z \in X$ with $x \neq y \neq z \neq x$,
(S2) $S(x, y, z) = 0 \Leftrightarrow x = y = z$,
(S3) $S(x, y, z) \leqslant S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z, a \in X$.

The pair (X, S) is called a S-metric space.

***Example* 1.2** [10] Let $X = \mathbb{R}^2$ and d be an ordinary metric on X. Therefore $S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ for all $x, y, z \in \mathbb{R}^2$, is a S-metric on X.

Lemma 1.3 [9] In a S-metric space we have $S(x, x, y) = S(y, y, x)$.

Definition 1.4 [11] Let (X, S) be a S-metric space. For $r > 0$ and $x \in X$ we define the open ball $B_S(x, r)$ and closed ball $B_S\ [x, r]$ with center x and radius r as follows respectively:

$$B_s(x, r) = \{y \in X : S(y, y, x) < r\},$$
$$B_s[x, r] = \{y \in X : S(y, y, x) \leqslant r\}.$$

Definition 1.5 [11] Let (X, S) be a S-metric space and $A \subseteq X$.

(1) If for every $x \in X$ there exists $r > 0$ such that $B_s(x, r) \subseteq A$, then the subset A is called open subset of X.
(2) Subset A of X is said to be S-bounded if there exists $r > 0$ such that $S(x, x, y) < r$ for all $x, y \in A$.
(3) A sequence $\{x_n\}$ in X convergents to x if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for each $n \geqslant n_0$, $S(x_n, x_n, x) < \varepsilon$ and we denote by $\lim\limits_{n \longrightarrow \infty} x_n = x$.
(4) Sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for each $n, m \geqslant n_0$, $S(x_n, x_n, x_m) < \varepsilon$.
(5) The S-metric space (X, S) is said to be complete if every Cauchy sequence is convergent.
(6) Let τ be the of all $A \subseteq X$ witch $x \in A$ if and only if there exists $r > 0$ such that $B_s(x, r) \subseteq A$. Then τ is a topology on X.

Lemma 1.6 [11] Let (X, S) be a S-metric space. If there exist sequence $\{x_n\}, \{y_n\}$ such that $\lim\limits_{n \longrightarrow \infty} x_n = x$ and $\lim\limits_{n \longrightarrow \infty} y_n = y$, then $\lim\limits_{n \longrightarrow \infty} S(x_n, x_n, y_n) = S(x, x, y)$.

Following the results of Czerwik [3] and Bakhtin [2] in the next definition we introduced the notion of S_b-metric space, as a generalization of $S-$ metric space in which the triangular inequality has been replaced by weaker one.

Definition 1.7 Let X be a nonempty set and $b \geqslant 1$ be a given real number. Suppose that a mapping $S : X^3 \to [0, \infty)$ satisfies :

(S_b1) $0 < S(x, y, z)$ for all $x, y, z \in X$ with $x \neq y \neq z \neq x$,
(S_b2) $S(x, y, z) = 0 \Leftrightarrow x = y = z$,
(S_b3) $S(x, y, z) \leqslant b(S(x, x, a) + S(y, y, a) + S(z, z, a))$ for all $x, y, z, a \in X$

Then S is called a S_b-metric and the pair (X, S) is called a S_b-metric space.

Remark 1 *It should be noted that, the class of S_b-metric spaces is effectively larger than that of S-metric spaces. Indeed each S-metric space is a S_b-metric space with $b = 1$.*

Following example shows that a S_b-metric on X need not be a S-metric on X.

***Example* 1.8** Let (X, S) be a S-metric space, and $S_*(x, y, z) = S(x, y, z)^p$, where $p > 1$ is a real number. Note that S_* is a S_b-metric with $b = 2^{2(p-1)}$. Obviously, S_* satisfies condition $(S_b1), (S_b2)$ of Definition 1.7, so it suffice to show (S_b3) holds. If $1 < p < \infty$, then the covexity of the function $f(x) = x^p$, $(x > 0)$ implies that $(a+b)^p \leqslant 2^{p-1}(a^p + b^p)$. Thus, for each $x, y, z, a \in X$, we obtain

$$\begin{aligned}
S_*(x, y, z) &= S(x, y, z)^p \\
&\leqslant ([S(x, x, a) + S(y, y, a)] + S(z, z, a))^p \\
&\leqslant 2^{p-1}([S(x, x, a) + S(y, y, a)]^p + S(z, z, a)^p) \\
&\leqslant 2^{p-1}(2^{p-1}(S(x, x, a)^p + S(y, y, a)^p) + S(z, z, a)^p) \\
&\leqslant 2^{(p-1)}(2^{p-1}(S(x, x, a)^p + S(y, y, a)^p) + 2^{p-1}S(z, z, a)^p) \\
&= 2^{2(p-1)}(S(x, x, a)^p + S(y, y, a)^p + S(z, z, a)^p) \\
&= 2^{2(p-1)}(S_*(x, x, a) + S_*(y, y, a) + S_*(z, z, a))
\end{aligned}$$

so, S_* is a S_b-metric with $b = 2^{2(p-1)}$.

Also in the above example, (X, S_*) is not necessarily a S-metric space. For example, let $X = \mathbb{R}$ and $S_*(x, y, z) = (|y + z - 2x| + |y - z|)^2$ is a S_b-metric on $\mathbb{R}$, with $p = 2$, $b = 2^{2(2-1)} = 4$, for all $x, y, z \in \mathbb{R}$. But it is not a S-metric on $\mathbb{R}$.

To see this, let $x = 3, y = 5, z = 7, a = \frac{7}{2}$. Hence, we get

$$\begin{aligned}
S_*(3, 5, 7) &= (|5 + 7 - 6| + |5 - 7|)^2 = 8^2 = 64 \\
S_*(3, 3, \frac{7}{2}) &= (\left|3 + \frac{7}{2} - 6\right| + \left|3 - \frac{7}{2}\right|)^2 = 1^2 = 1 \\
S_*(5, 5, \frac{7}{2}) &= (\left|5 + \frac{7}{2} - 10\right| + \left|5 - \frac{7}{2}\right|)^2 = 3^2 = 9 \\
S_*(7, 7, \frac{7}{2}) &= (\left|7 + \frac{7}{2} - 14\right| + \left|7 - \frac{7}{2}\right|)^2 = 7^2 = 49.
\end{aligned}$$

Therefore, $S_*(3, 5, 7) = 64 \geqslant 59 = S_*(3, 3, \frac{7}{2}) + S_*(5, 5, \frac{7}{2}) + S_*(7, 7, \frac{7}{2})$.

Now we present some definitions and propositions in S_b-metric space.

Definition 1.9 Let (X, S) be a S_b-metric space. Then, for $x \in X$, $r > 0$ we defined the open ball $B_S(x, r)$ and closed ball $B_S[x, r]$ with center x and radius r as follows respectively:

$$\begin{aligned}
B_S(x, r) &= \{y \in X : S(y, y, x) < r\}, \\
B_S[x, r] &= \{y \in X : S(y, y, x) \leqslant r\}.
\end{aligned}$$

***Example* 1.10** Let $X = \mathbb{R}$. Denote $S(x,y,z) = (|y+z-2x| + |y-z|)^2$ is a S_b-metric on $\mathbb{R}$ with $b = 2^{2(2-1)} = 4$, for all $x, y, z \in \mathbb{R}$. Thus

$$\begin{aligned} B_S(1,2) &= \{y \in \mathbb{R} : S(y,y,1) < 2\} \\ &= \{y \in \mathbb{R} : |y-1| < \frac{\sqrt{2}}{2}\} \\ &= \{y \in \mathbb{R} : 1 - \frac{\sqrt{2}}{2} < y < 1 + \frac{\sqrt{2}}{2}\} \\ &= (1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}). \end{aligned}$$

Lemma 1.11 In a S_b-metric space, we have

$$S(x,x,y) \leqslant bS(y,y,x)$$

and

$$S(y,y,x) \leqslant bS(x,x,y)$$

Proof. By third condition of S_b-metric, we have

$$S(x,x,y) \leqslant b(2S(x,x,x) + S(y,y,x)) = bS(y,y,x)$$

and similarly

$$S(y,y,x) \leqslant b(2S(y,y,y) + S(x,x,y)) = bS(x,x,y).$$

■

Lemma 1.12 Let (X,S) be a S_b-metric space. Then

$$S(x,x,z) \leqslant 2bS(x,x,z) + b^2S(y,y,z).$$

Proof. By third condition of S_b-metric and lemma (1.3), we have

$$\begin{aligned} S(x,x,z) &\leqslant b(S(x,x,y) + S(x,x,y) + S(z,z,y)) \\ &\leqslant b(2S(x,x,y) + bS(y,y,z)) \\ &= 2bS(x,x,y) + b^2S(y,y,z). \end{aligned}$$

■

The notions of convergence and Cauchy sequence is introducing as in the case of $S-$metric spaces.

Definition 1.13 Let (X,S) be a S_b-metric space. A sequence $\{x_n\}$ in X is said to be :
(1) S_b-Cauchy sequence if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \varepsilon$ for each $m, n \geqslant n_0$.
(2) S_b-convergent to a point $x \in X$ if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that $S(x_n, x_n, x) < \varepsilon$ or $S(x, x, x_n) < \varepsilon$ for all $n \geqslant n_0$ and we denote by

$\lim_{n\longrightarrow\infty} x_n = x$.

Definition 1.14 A S_b-metric space (X,S) is called complete if every S_b-Cauchy sequence is S_b-convergent in X.

Definition 1.15 Let (X,S) and (X',S') be S_b-metric spaces, and let $f:(X,S)\to(X',S')$ be a function. Then f is said to be continuous at a point $a\in X$ if and only if for every sequence x_n in X, $S(x_n,x_n,a)\to 0$ implies $S'(f(x_n),f(x_n),f(a))\to 0$. A function f is continuous at X if and only if it is continuous at all $a\in X$.

The term of compatible mappings is introduced analogously as in the case of $S-$metric spaces.

Definition 1.16 Let (X,S) be a S_b-metric space. A pair $\{f,g\}$ is said to be compatible if and only if $\lim_{n\longrightarrow\infty} S(fgx_n,fgx_n,gfx_n)=0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\longrightarrow\infty} fx_n=\lim_{n\longrightarrow\infty} gx_n=t$ for some $t\in X$.

Lemma 1.17 Let (X,S) be a S_b-metric space with $b\geqslant 1$, and suppose that $\{x_n\}$ is a S_b-convergent to x, then we have

$$\frac{1}{b^2}S(x,x,y)\leqslant \liminf_{n\longrightarrow\infty} S(x_n,x_n,y) \leqslant \limsup_{n\longrightarrow\infty} S(x_n,x_n,y)\leqslant b^2S(x,x,y).$$

In particular, if $x=y$, then we have $\lim_{n\longrightarrow\infty} S(x_n,x_n,y)=0$.

Proof. By (S_b3) and Lemma 1.12, we have

$$S(x_n,x_n,y)\leqslant 2bS(x_n,x_n,x)+b^2S(x,x,y),$$

and

$$\frac{1}{b^2}S(x,x,y)\leqslant 2S(x_n,x_n,x)+S(x_n,x_n,y).$$

Taking the upper limit as $n\to\infty$ in the first inequality and the lower limit as $n\to\infty$ in the second inequality we obtain the desired result. ■

Lemma 1.18 Let (X,S) be a S_b-metric space. If there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\longrightarrow\infty} S(x_n,x_n,y_n)=0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\longrightarrow\infty} x_n=t$ for some $t\in X$ then $\lim_{n\longrightarrow\infty} y_n=t$.

***Proof*.** By a triangle inequality in a S_b-metric space, we have

$$S(y_n,y_n,t)\leqslant b(2S(y_n,y_n,x_n)+bS(x_n,x_n,t)).$$

Now, by taking the upper limit when $n\to\infty$ in the above inequality we get

$$\limsup_{n\longrightarrow\infty} S(y_n,y_n,t)\leqslant b^2(\limsup_{n\longrightarrow\infty} 2S(x_n,x_n,y_n)+\limsup_{n\longrightarrow\infty} S(x_n,x_n,t))=0.$$

Hence $\lim_{n\longrightarrow\infty} y_n=t$.

2. Main results

Our first results is the following common fixed point theorem.

Theorem 2.1 Suppose that f, g, M and T are self mappings on a complete S_b-metric space (X, S) such that $f(X) \subseteq T(X)$, $g(X) \subseteq M(X)$. If

$$S(fx, fx, gy) \leqslant \frac{q}{b^4} \max\{S(Mx, Mx, Ty), S(fx, fx, Mx), S(gy, gy, Ty), \qquad (1)$$

$$\frac{1}{2}(S(Mx, Mx, gy) + S(fx, fx, Ty))\}$$

holds for each $x, y \in X$ with $0 < q < 1$ and $b \geqslant \frac{3}{2}$, then f, g, M and T have a unique common fixed point in X provided that M and T are continuous and pairs $\{f, M\}$ and $\{g, T\}$ are compatible.

Proof. Let $x_0 \in X$. As $f(X) \subseteq T(X)$, there exists $x_1 \in X$ such that $fx_0 = Tx_1$. Since $gx_1 \in M(X)$, we can choose $x_2 \in X$ such that $gx_1 = Mx_2$. In general, x_{2n+1} and x_{2n+2} are chosen in X such that $fx_{2n} = Tx_{2n+1}$ and $gx_{2n+1} = Mx_{2n+2}$. Define a sequence y_n in X such that $y_{2n} = fx_{2n} = Tx_{2n+1}$, and $y_{2n+1} = gx_{2n+1} = Mx_{2n+2}$, for all $n \geqslant 0$. Now, we show that y_n is a Cauchy sequence. Consider

$$\begin{aligned}
S(y_{2n}, y_{2n}, y_{2n+1}) &= S(fx_{2n}, fx_{2n}, gx_{2n+1}) \\
&\leqslant \frac{q}{b^4} \max\big\{S(Mx_{2n}, Mx_{2n}, Tx_{2n+1}), S(fx_{2n}, fx_{2n}, Mx_{2n}), \\
&\quad S(gx_{2n+1}, gx_{2n+1}, Tx_{2n+1}), \\
&\quad \frac{1}{2}(S(Mx_{2n}, Mx_{2n}, gx_{2n+1}) + S(fx_{2n}, fx_{2n}, Tx_{2n+1}))\big\} \\
&= \frac{q}{b^4} \max\big\{S(y_{2n-1}, y_{2n-1}, y_{2n}), S(y_{2n}, y_{2n}, y_{2n-1}), \\
&\quad S(y_{2n+1}, y_{2n+1}, y_{2n}), \\
&\quad \frac{1}{2}(S(y_{2n-1}, y_{2n-1}, y_{2n+1}) + S(y_{2n}, y_{2n}, y_{2n}))\big\} \\
&\leqslant \frac{q}{b^4} \max\big\{S(y_{2n-1}, y_{2n-1}, y_{2n}), bS(y_{2n-1}, y_{2n-1}, y_{2n}), \\
&\quad S(y_{2n+1}, y_{2n+1}, y_{2n}), \frac{S(y_{2n-1}, y_{2n-1}, y_{2n+1})}{2}\big\} \\
&\leqslant \frac{q}{b^4} \max\big\{S(y_{2n-1}, y_{2n-1}, y_{2n}), bS(y_{2n-1}, y_{2n-1}, y_{2n}), S(y_{2n+1}, y_{2n+1}, y_{2n}), \\
&\quad \frac{b}{2}\big(S(y_{2n-1}, y_{2n-1}, y_{2n}) + S(y_{2n-1}, y_{2n-1}, y_{2n}) + S(y_{2n+1}, y_{2n+1}, y_{2n})\big)\big\} \\
&\leqslant \frac{q}{b^4} \max\big\{S(y_{2n-1}, y_{2n-1}, y_{2n}), bS(y_{2n-1}, y_{2n-1}, y_{2n}), bS(y_{2n}, y_{2n}, y_{2n+1}), \\
&\quad \frac{b}{2}\big(2S(y_{2n-1}, y_{2n-1}, y_{2n}) + bS(y_{2n+1}, y_{2n+1}, y_{2n})\big)\big\}.
\end{aligned}$$

Now, since

$$\begin{aligned} S(y_{2n-1}, y_{2n-1}, y_{2n}) &\leqslant bS(y_{2n-1}, y_{2n-1}, y_{2n}) \\ &\leqslant bS(y_{2n-1}, y_{2n-1}, y_{2n}) + \frac{b^2}{2} S(y_{2n}, y_{2n}, y_{2n+1}) \end{aligned}$$

we have

$$\begin{aligned} S(y_{2n}, y_{2n}, y_{2n+1}) \leqslant \max \big\{ &bS(y_{2n}, y_{2n}, y_{2n+1}), \\ &bS(y_{2n-1}, y_{2n-1}, y_{2n}) + \frac{b^2}{2} S(y_{2n}, y_{2n}, y_{2n+1}) \big\}. \end{aligned}$$

If $\max = bS(y_{2n}, y_{2n}, y_{2n+1})$ we obtain

$$S(y_{2n}, y_{2n}, y_{2n+1}) \leqslant \frac{q}{b^3} S(y_{2n}, y_{2n}, y_{2n+1}) < S(y_{2n}, y_{2n}, y_{2n+1}).$$

Contradiction. So, $\max = bS(y_{2n-1}, y_{2n-1}, y_{2n}) + \frac{b^2}{2} S(y_{2n}, y_{2n}, y_{2n+1})$ and we have

$$S(y_{2n}, y_{2n}, y_{2n+1}) \leqslant \frac{q}{b^4} \big(bS(y_{2n-1}, y_{2n-1}, y_{2n}) + \frac{b^2}{2} S(y_{2n}, y_{2n}, y_{2n+1}) \big)$$

i.e.,

$$S(y_{2n}, y_{2n}, y_{2n+1}) \leqslant \frac{2q}{2b^3 - qb} S(y_{2n-1}, y_{2n-1}, y_{2n}).$$

Let $\lambda = \frac{2q}{2b^3 - qb}$. Since $b \geqslant \frac{3}{2}$ we have that $0 < \lambda < 1$. Now,

$$\begin{aligned} S(y_{2n}, y_{2n}, y_{2n+1}) &\leqslant \lambda S(y_{2n-1}, y_{2n-1}, y_{2n}) \leqslant \lambda^2 S(y_{2n-2}, y_{2n-2}, y_{2n-1}) \\ &\leqslant \ldots \leqslant \lambda^n S(y_0, y_0, y_1). \end{aligned}$$

Hence, for all $n \geqslant 2$, we obtain

$$S(y_{n-1}, y_{n-1}, y_n) \leqslant ... \leqslant \lambda^{n-1} S(y_0, y_0, y_1). \tag{2}$$

Using Lemma 1.11 and (S_b3), and (2) for all $n > m$, we have

$$\begin{aligned}
S(y_m, y_m, y_n) &\leqslant b(2S(y_m, y_m, y_{m+1}) + S(y_n, y_n, y_{m+1}))\\
&\leqslant 2bS(y_m, y_m, y_{m+1}) + b^2 S(y_{m+1}, y_{m+1}, y_n)\\
&\leqslant 2bS(y_m, y_m, y_{m+1}) + 2b^3 S(y_{m+1}, y_{m+1}, y_{m+2})\\
&\quad + b^4 S(y_{m+2}, y_{m+2}, y_n) \leqslant \dots\\
&\leqslant 2b(S(y_m, y_m, y_{m+1}) + b^2 S(y_{m+1}, y_{m+1}, y_{m+2})\\
&\quad + \dots + b^{2(n-m-1)} S(y_{n-1}, y_{n-1}, y_n))\\
&\leqslant 2b(\lambda^m + b^2\lambda^{m+1} + \dots + b^{2(n-m-1)}\lambda^{n-1}) S(y_0, y_0, y_1)\\
&\leqslant 2bS(y_0, y_0, y_1)(\lambda^m + b^2\lambda^{m+1} + \dots)\\
&\leqslant \frac{2b\lambda^m}{1 - b^2\lambda} S(y_0, y_0, y_1).
\end{aligned}$$

On taking limit as $m, n \to \infty$, we have $S(y_m, y_m, y_n) \to 0$ as $b^2\lambda < 1$. Therefore $\{y_n\}$ is a Cauchy sequence. Since X is a complete S_b-metric space, there is some y in X such that

$$\lim_{n\longrightarrow\infty} fx_{2n} = \lim_{n\longrightarrow\infty} Tx_{2n+1} = \lim_{n\longrightarrow\infty} gx_{2n+1} = \lim_{n\longrightarrow\infty} Mx_{2n+2} = y.$$

We show that y is a common fixed point of f, g, M and T. Since M is continuous, therefore

$$\lim_{n\longrightarrow\infty} M^2 x_{2n+2} = My \qquad \textit{and} \qquad \lim_{n\longrightarrow\infty} Mfx_{2n} = My.$$

Since a pair $\{f, M\}$ is compatible, $\lim_{n\longrightarrow\infty} S(fMx_{2n}, fMx_{2n}, Mfx_{2n}) = 0$. So by Lemma 1.18, we have $\lim_{n\longrightarrow\infty} fMx_{2n} = My$. Putting $x = Mx_{2n}$ and $y = x_{2n+1}$ in (1) we obtain

$$\begin{aligned}
S(fMx_{2n}, fMx_{2n}, gx_{2n+1}) \leqslant \frac{q}{b^4} \max\big\{&S(M^2x_{2n}, M^2x_{2n}, Tx_{2n+1}),\\
&S(fMx_{2n}, fMx_{2n}, M^2x_{2n}), S(gx_{2n+1}, gx_{2n+1}, Tx_{2n+1}),\\
&\frac{1}{2}\big(S(M^2x_{2n}, M^2x_{2n}, gx_{2n+1}) + S(fMx_{2n}, fMx_{2n}, Tx_{2n+1})\big)\big\}.
\end{aligned} \tag{3}$$

Taking the upper limit as $n \to \infty$ in (3) and using Lemma 1.17, we get

$$\begin{aligned}\frac{S(My,My,y)}{b^2} &\leqslant \limsup_{n\longrightarrow\infty} S(fMx_{2n},fMx_{2n},gx_{2n+1})\\ &\leqslant \frac{q}{b^4}\max\big\{\limsup_{n\longrightarrow\infty} S(M^2x_{2n},M^2x_{2n},Tx_{2n+1}),\\ &\limsup_{n\longrightarrow\infty} S(fMx_{2n},fMx_{2n},M^2x_{2n}),\\ &\limsup_{n\longrightarrow\infty} S(gx_{2n+1},gx_{2n+1},Tx_{2n+1}),\\ &\frac{1}{2}\big(\limsup_{n\longrightarrow\infty} S(M^2x_{2n},M^2x_{2n},gx_{2n+1})\\ &+\limsup_{n\longrightarrow\infty} S(fMx_{2n},fMx_{2n},Tx_{2n+1})\big)\big\}\\ &\leqslant \frac{q}{b^4}\max\big\{b^2S(My,My,y),0,0,\frac{b^2}{2}(S(My,My,y)+S(My,My,y))\big\}\\ &= \frac{q}{b^4}b^2S(My,My,y) = \frac{q}{b^2}S(My,My,y).\end{aligned}$$

Consequently, $S(My,My,y) \leqslant qS(My,My,y)$. As $0<q<1$, so $My=y$. Using continuity of T, we obtain $\lim_{n\longrightarrow\infty} T^2x_{2n+1} = Ty$ and $\lim_{n\longrightarrow\infty} Tgx_{2n+1} = Ty$. Since g and T are compatible, $\lim_{n\longrightarrow\infty} S(gTx_n,gTx_n,Tgx_n)=0$. So, by Lemma 1.18, we have $\lim_{n\longrightarrow\infty} gTx_{2n} = Ty$. Putting $x=x_{2n}$ and $y=Tx_{2n+1}$ in (1), we obtain

$$\begin{aligned}S(fx_{2n},fx_{2n},gTx_{2n+1}) \leqslant \frac{q}{b^4}\max\big\{&S(Mx_{2n},Mx_{2n},T^2x_{2n+1}),\\ &S(fx_{2n},fx_{2n},Mx_{2n}),S(gTx_{2n+1},gTx_{2n+1},T^2x_{2n+1}),\\ &\frac{1}{2}\big(S(Mx_{2n},Mx_{2n},gTx_{2n+1})+S(fx_{2n},fx_{2n},T^2x_{2n+1})\big)\big\}.\end{aligned} \tag{4}$$

Taking upper limit as $n \to \infty$ in (4) and using Lemma 1.17, we obtain

$$\begin{aligned}\frac{S(y,y,Ty)}{b^2} &\leqslant \limsup_{n\longrightarrow\infty} S(fx_{2n},fx_{2n},gTx_{2n+1})\\ &\leqslant \frac{q}{b^4}\max\{b^2(S(y,y,Ty),0,0,\frac{b^2}{2}S(y,y,Ty)+S(y,y,Ty))\}\\ &= \frac{qS(y,y,Ty)}{b^2},\end{aligned}$$

which implies that $Ty=y$. Also, we can apply condition (1) to obtain

$$\begin{aligned}S(fy,fy,gx_{2n+1}) \leqslant \frac{q}{b^4}\max\big\{&S(My,My,Tx_{2n+1}),S(fy,fy,My),\\ &S(gx_{2n+1},gx_{2n+1},Tx_{2n+1}),\frac{1}{2}\big(S(My,My,gx_{2n+1})\\ &+S(fy,fy,Tx_{2n+1})\big)\big\}.\end{aligned} \tag{5}$$

Taking upper limit $n \to \infty$ in (5), and using $My = Ty = y$, we have

$$\begin{aligned}\frac{S(fy, fy, y)}{b^2} &\leqslant \frac{q}{b^4} \max\{b^2 S(My, My, y), b^2 S(fy, fy, My), b^2 S(y, y, y),\\ &\frac{b^2}{2}(S(My, My, y) + S(fy, fy, y))\\ &= \frac{q}{b^2} S(fy, fy, y),\end{aligned}$$

which implies that $S(fy, fy, y) = 0$ and $fy = y$ as $0 < q < 1$. Finally, from condition (1), and the fact $My = Ty = fy = y$, we have

$$\begin{aligned}S(y, y, gy) &= S(fy, fy, gy)\\ &\leqslant \frac{q}{b^4} \max\{S(My, My, Ty), S(fy, fy, My), S(gy, gy, Ty),\\ &\frac{1}{2}(S(My, My, gy) + S(fy, fy, Ty))\\ &\leqslant \frac{q}{b^3} S(y, y, gy)\\ &\leqslant qS(y, y, gy),\end{aligned}$$

which implies that $S(y, y, gy) = 0$ and $gy = y$. Hence $My = Ty = fy = gy = y$. If there exists another common fixed point x in X for f, g, M and T, then

$$\begin{aligned}S(x, x, y) &= S(fx, fx, gy)\\ &\leqslant \frac{q}{b^4} \max\{S(Mx, Mx, Ty), S(fx, fx, Mx), S(gy, gy, Ty),\\ &\frac{1}{2}(S(Mx, Mx, gy) + S(fx, fx, Ty))\\ &= \frac{q}{b^4} \max\{S(x, x, y), S(x, x, x), S(y, y, y), \frac{1}{2}(S(x, x, y) + S(x, x, y))\}\\ &= \frac{q}{b^4} S(x, x, y)\\ &\leqslant qS(x, x, y),\end{aligned}$$

which further implies that $S(x, x, y) = 0$ and hence, $x = y$. Thus, y is a unique common fixed point of f, g, M and T. ■

***Example* 2.2** Let $X = [0, 1]$ be endowed with S_b-metric $S_*(x, y, z) = (|y + z - 2x| + |y - z|)^2$, where $b = 4$. Define f, g, M and T on X by $f(x) = (\frac{x}{4})^8$, $g(x) = (\frac{x}{8})^4$, $M(x) = (\frac{x}{4})^4$, $T(x) = (\frac{x}{8})^2$.
Obviously, $f(X) \subseteq T(X)$ and $g(X) \subseteq M(X)$. Furthermore, the pairs $\{f, M\}$ and $\{g, T\}$

are compatible. For each $x,\ y \in X$, we have

$$\begin{aligned} S(fx,fx,gy) &= (|gy-fx|+|fx-gy|)^2 \\ &= (2|fx-gy|)^2 \\ &= 4((\frac{x}{4})^8-(\frac{y}{8})^4)^2 \\ &= 4((\frac{x}{4})^4+(\frac{y}{8})^2)^2.((\frac{x}{4})^4-(\frac{y}{8})^2)^2 \\ &\leqslant (\frac{1}{4^4}+\frac{1}{8^2})^2 S(Mx,Mx,Ty) \\ &= \frac{\frac{25}{4^4}}{4^4} S(Mx,Mx,Ty), \end{aligned}$$

where $\frac{25}{4^4} \leqslant q \leqslant 1$ and $b=4$. Thus, f, g, M and T satisfy all condition of Theorem 2.1. Moreover 0 is the unique common fixed point of f, g, M and T.

Corollary 2.3 Let (X,S) be a complete S_b-metric space and $f,g : X \to X$ two mappings such that

$$S(fx,fx,gy) \leqslant \frac{q}{b^4}\max\{S(x,x,y),S(fx,fx,x),S(gy,gy,y),\frac{1}{2}(S(x,x,gy)+S(fx,fx,y))\},$$

holds for all $x,y \in X$ with $0<q<1$ and $b \geqslant \frac{3}{2}$. Then, there exists a unique point $y \in X$ such that $fy=gy=y$.

Proof. If we take $M=T=I_X$(identity mapping on X), then theorem (2.1) gives that f and g have a unique common fixed point. ■

Proof. If we take f and g as identity maps on X, then Theorem 2.1 gives that M and T have a unique common fixed point. ■

Corollary 2.4 Let (X,S) be a complete S_b-metric space and $f : X \to X$ mapping such that

$$S(fx,fx,fy) \leqslant \frac{q}{b^4}\max\{S(x,x,y),S(fx,fx,x),S(fy,fy,y),\frac{1}{2}(S(x,x,fy)+S(fx,fx,y))\},$$

holds for all $x,y \in X$ with $0<q<1$ and $b \geqslant \frac{3}{2}$. Then f has a unique fixed point in X.

Proof. Take M and T as identity maps on X and $f=g$ and then apply Theorem 2.1. ■

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