Journal of
Linear and Topological Algebra
Vol. 07*, No.* 04*,* 2018*,* 273- 280

Algebraic distance in algebraic cone metric spaces and its properties

K. Fallahi[a,*], G. Soleimani Rad[a]

[a] *Department of Mathematics, Payame Noor University, P.O. Box 19395-4697, Tehran, Iran.*

Received 30 July 2018; Revised 20 September 2018; Accepted 22 September 2018.

Communicated by Mohammad Sadegh Asgari

Abstract. In this paper, we prove some properties of algebraic cone metric spaces and introduce the notion of algebraic distance in an algebraic cone metric space. As an application, we obtain some famous fixed point results in the framework of this algebraic distance

Keywords: Algebraic cone metric space, algebraic distance, algebraic interior.

2010 AMS Subject Classification: 46A19, 47L07, 47H10, 46B99.

1. Introduction and preliminaries

Ordered normed spaces and cones have many applications in optimization theory. Hence, fixed point theory in K-metric and K-normed spaces was developed in the mid-20th century ([2, 13]). Huang and Zhang [5] reintroduced such spaces under the name of cone metric spaces by considering an ordered normed space for the real numbers and proved some fixed point results (also, see [6, 12]). Moreover, topological vector space-valued cone metric space (tvs-cone metric space) introduced by Du [4] as a generalization of the Banach-valued cone metric space. In 1996, Kada et al. [8] defined the concept of w-distance in metric spaces.

Let (X, d) be a metric space. Then a function $p : X \times X \to [0, \infty)$ is called a w-distance on X if the following conditions are satisfied:

a) $p(x, z) \leqslant p(x, y) + p(y, z)$ for any $x, y, z \in X$;

---

*Corresponding author.
E-mail address: fallahi1361@gmail.com (K. Fallahi); gh.soleimani2008@gmail.com (G. Soleimani Rad).

Print ISSN: 2252-0201
Online ISSN: 2345-5934

http://jlta.iauctb.ac.ir

b) for each $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semi-continuous $\big($i.e., if $x \in X$ and $y_n \to y$ in X, then $p(x, y) \leqslant \liminf_n p(x, y_n)\big)$;

c) for each $\epsilon > 0$, there exists $\delta > 0$ such that $p(x, z) \leqslant \delta$ and $p(z, y) \leqslant \delta$ imply $d(x, y) \leqslant \epsilon$ for all $x, y, z \in X$.

In the sequel, Cho et al. [1] defined the concept of the c-distance in a cone metric space, which is a generalization of the w-distance and proved some fixed point theorems under c-distance (also, see [3, 10] and references therein).

In 2014, Niknam et al. [9] defined the concept of algebraic cone metric space and proved Banach contraction principle in this space. Consistent with the content of this paper, the following definitions will be needed in the sequel.

Let Y be a real vector space and P be a convex subset of Y. A point $x \in P$ is said to be an algebraic interior point of P if for each $y \in Y$ there exists $\epsilon > 0$ such that $x + ty \in P$, for all $t \in [0, \epsilon]$. This definition is equivalent to the statement:

A point x is called an algebraic interior point of the convex set $P \subseteq Y$ if $x \in P$ and for each $y \in Y$ there exists $\epsilon > 0$ such that $[x, x + \epsilon y] \subset P$, where $[x, x + \epsilon y] = \big\{\lambda x + (1 - \lambda)(x + \epsilon y) : \forall \lambda \in [0, 1]\big\}$. The set of all algebraic interior points of P is called algebraic interior and is denoted by *aint* P. Also, P is called algebraically open if $P = aint\ P$.

Let Y be vector space with the zero vector θ. A proper nonempty and convex subset P of Y is called an algebraic cone if $P + P \subseteq P$, $\lambda P \subseteq P$ for $\lambda \geqslant 0$ and $P \cap (-P) = \{\theta\}$. Given a algebraic cone $P \subseteq Y$, a partial ordering $\preceq_a$ with respect to P is defined by $x \preceq_a y \Leftrightarrow y - x \in P$. We shall write $x \prec_a y$ to mean $x \preceq_a y$ and $x \neq y$. Also, we write $x \ll_a y$ if and only if $y - x \in aint\ P$, where $aint\ P$ is the algebraic interior of P. Also, P is said to be Archimedean if for each $x, y \in P$ there exists $n \in \mathbb{N}$ such that $x \preceq_a ny$. For example, $P = \big\{(x, y) \in \mathbb{R}^2 : x, y \geqslant 0\big\}$ is an algebraic cone with the Archimedean property in the real vector space $\mathbb{R}^2$. In the sequel we assume that (Y, P) has the Archimedean property.

***Example* 1.1** $P = \big\{f \in C_{\mathbb{R}}[a, b] : f(x) \geqslant 0, \forall x \in [a, b]\big\}$ is an algebraic cone with the Archimedean property in the real vector space $C_{\mathbb{R}}[a, b]$. But $P = \big\{f \in C_{\mathbb{R}}(0, 1) : f(x) \geqslant 0, \forall x \in (0, \infty)\big\}$ in the real vector space $C_{\mathbb{R}}(0, \infty)$ that does not have the Archimedean property.

Lemma 1.2 [9] Let Y be a real vector space and P be an algebraic cone in Y with non-empty algebraic interior.

(i) $P + aint\ P \subset aint\ P$;

(ii) $\lambda aint\ P \subset aint\ P$, for each $\lambda > 0$.

2. Main results

Definition 2.1 [9] Let X be a nonempty set and (Y, P) be an algebraic cone space with $aint\ P \neq \emptyset$. Suppose that a vector valued function $d_a : X \times X \to Y$ satisfies the following conditions:

(ACM1) $\theta \preceq_a d_a(x, y)$ for all $x, y \in X$ and $d_a(x, y) = \theta$ if and only if $x = y$;

(ACM2) $d_a(x, y) = d_a(y, x)$ for all $x, y \in X$;

(ACM3) $d_a(x, z) \preceq_a d_a(x, y) + d_a(y, z)$ for all $x, y, z \in X$.

Then d_a is called an algebraic cone metric and (X, d_a) is called an algebraic cone metric space.

***Example* 2.2** Let X be a vector space over F ($\mathbb{R}$ or $\mathbb{C}$) and $||\cdot||_a : X \to Y$ be a mapping that satisfies the following conditions:

(ACN1) $\theta \ll_a ||x||$ for all $x \in X \setminus \{\theta_X\}$ and $||x||_a = \theta$ if and only if $x = \theta_X$, where θ_X is the zero vector in X;
(ACN2) $||\alpha x||_a = |a|||x||_a$ for all $x \in X$ and $\alpha \in F$;
(ACN3) $||x + y||_a \preceq_a ||x||_a + ||y||_a$.

Then $|| \cdot ||_a$ is called an algebraic cone norm on X and $(X, || \cdot ||_a)$ is called an algebraic cone normed space [11]. Now, if define $d_a(x, y) = ||x - y||_a$, then d_a is called an algebraic cone metric space obtained from algebraic cone norm.

Definition 2.3 Let (X, d_a) be an algebraic cone metric space, $\{x_n\}$ a sequence in X and $x \in X$. Then the following statements hold:

- **(i)** $\{x_n\}$ converges to x if, for every $c \in Y$ with $c \in aint\ P$ there exists an $n_0 \in \mathbb{N}$ such that $d_a(x_n, x) \ll_a c$ for all $n > n_0$. We denote this by $d_a - \lim_{n\to\infty} x_n = x$ or $x_n \xrightarrow{d_a} x$ as $n \to \infty$;
- **(ii)** $\{x_n\}$ is called a Cauchy sequence if, for every $c \in Y$ with $c \in aint\ P$ there exists an $n_0 \in \mathbb{N}$ such that $d_a(x_n, x_m) \ll_a c$ for all $m, n > n_0$;
- **(iii)** (X, d_a) is complete algebraic cone metric space if every Cauchy sequence in X is convergent in X.

Lemma 2.4 Let X be a nonempty set, (Y, P) be an algebraic cone space with $aint\ P \neq \emptyset$ and (X, d_a) is call be an algebraic cone metric space. Then, for all $u, v, w, c \in Y$, the following assertions are true:

(p_1) If $u \preceq_a v$ and $v \ll_a w$, then $u \ll_a w$.
(p_2) If $u \ll_a v$ and $v \preceq_a w$, then $u \ll_a w$.
(p_3) If $u \preceq_a v$ and $v \preceq_a w$, then $u \preceq_a w$. Also, if $u \ll_a v$ and $v \ll_a w$, then $u \ll_a w$.
(p_4) If $\theta \preceq_a u \ll_a c$ for each $c \in aint\ P$, then $u = \theta$.
(p_5) If $u \preceq_a \lambda u$ where $u \in P$ and $0 < \lambda < 1$, then $u = \theta$.
(p_6) Let $b_n \xrightarrow{a} \theta$ in Y, $\theta \preceq_a b_n$ and $c \in aint\ P$. Then there exists positive integer n_0 such that $b_n \ll_a c$ for each $n > n_0$.
(p_7) If $\theta \preceq_a u \preceq_a v$ and $k \in \mathbb{R}^+$, then $\theta \preceq_a ku \preceq_a kv$.
(p_8) If $\theta \preceq_a u_n \preceq_a v_n$ for all $n \in \mathbb{N}$ and $u_n \xrightarrow{a} u$, $v_n \xrightarrow{a} v$ as $n \to \infty$, then $\theta \preceq_a u \preceq_a v$.
(p_9) $x_n \xrightarrow{d_a} x$ and $x_n \xrightarrow{d_a} y$ implies that $x = y$.
(p_{10}) Let $\theta \ll_a c$. If $\theta \preceq_a d_a(x_n, x) \preceq_a b_n$ and $b_n \xrightarrow{a} \theta$, then eventually $d_a(x_n, x) \ll_a c$, where x_n, x are sequence and given point in X.
(p_{11}) If $u \preceq_a v + c$ for every $c \in aint\ P$, then $u \preceq_a v$.

Proof.

(p_1) If $u \preceq_a v$ and $v \ll_a w$, then $v - u \in P$ and $w - v \in aint\ P$. Thus, by Lemma 1.2.(**i**), we have

$$w - u = (w - v) + (v - u) \in aint\ P + P \subset aint\ P.$$

Consequently, $w - u \in aint\ P$; that is, $u \ll_a w$.

(p_2) If $u \ll_a v$ and $v \preceq_a w$, then $v - u \in aint\ P$ and $w - v \in P$. Thus, by Lemma

1.2.(**i**), we have

$$w-u=(w-v)+(v-u)\in P+aint\ P\subset aint\ P.$$

Consequently, $w-u\in aint\ P$; that is, $u\ll_a w$.

(p_3) If $u\preceq_a v$ and $v\preceq_a w$, then $v-u\in P$ and $w-v\in P$. Thus, by definition of cone, we have

$$w-u=(w-v)+(v-u)\in P+P\subseteq P.$$

Consequently, $w-u\in P$; that is, $u\preceq_a w$. Similarly, by using definition $aint\ P$ and $\ll_a$, it follows from $u\ll_a v$ and $v\ll_a w$ that $u\ll_a w$.

(p_4) The proof is the same as in the Banach case or *tvs*-cone metric space.

(p_5) The condition $u\preceq_a \lambda u$ means that $\lambda u-u\in P$; that is, $-(1-\lambda)u\in P$. On the other hand, from $a\in P$ and $1-\lambda>0$, we have $(1-\lambda)u\in P$. Thus, we have $(1-\lambda)u\in P\cap(-P)=\{\theta\}$. Consequently, $u=\theta$.

(p_6) Using Lemma 1.2, the proof is similar to the version of *tvs*-cone metric spaces.

(p_7) If $\theta\preceq_a u\preceq_a v$, then $v-u\preceq_a P$. Since k is a nonnegative real number, by using the definition of algebraic cone, $k(v-u)\in P$; that is, $kv-ku\in P$. Thus, $\theta\preceq_a ku\preceq_a kv$.

(p_8) Using the definition of convergence and $\preceq_a$, the proof is straightforward.

(p_9) From $x_n\xrightarrow{d_a}x$ and $x_n\xrightarrow{d_a}y$, we have $d_a(x_n,x)\ll_a \frac{c}{2}$ and $d_a(x_n,x)\ll_a\frac{c}{2}$ for each $n>n_0$ and $c\in aint\ P$. Thus, by using (**ACM3**), $d_a(x,y)\preceq_a d_a(x,x_n)+d_a(x_n,y)\ll_a\frac{c}{2}+\frac{c}{2}$. It follows that $d_a(x,y)\ll_a c$ (by (p_1)) for arbitrary $c\in aint\ P$. Using (p_4), we have $d_a(x,y)=\theta$; that is, $x=y$.

(p_{10}) It follows from (p_1), (p_6) and convergence of a sequence in an algebraic cone metric space that the assertion of (p_{10}) is true.

(p_{11}) The proof is similar to the proof of (p_1), (p_2) and (p_3).

■

Remark 1 *Huang and Zhang [5] proved that if P is a normal cone then $x_n\in X$ converges to $x\in X$ if and only if $d(x_n,x)\to\theta$, as $n\to\infty$, and that $x_n\in X$ is a Cauchy sequence if and only if $d(x_n,x_m)\to\theta$, as $n,m\to\infty$. It follows from (p_6) and (p_{10}) that the sequence $\{x_n\}$ converges to $x\in X$ in an algebraic cone metric space if $d_a(x_n,x)\xrightarrow{a}\theta$, as $n\to\infty$ and $\{x_n\}$ is a Cauchy sequence if $d_a(x_n,x_m)\xrightarrow{a}\theta$, as $n\to\infty$. In the case when the cone is not necessarily normal (such as tvs-cone metric spaces [4] or algebraic cone metric spaces), we have only one half of the statements of Lemmas 1 and 4 from [5]. Also, in this case, the fact that $d_a(x_n,y_n)\xrightarrow{a}d_a(x,y)$ if $x_n\xrightarrow{d_a}x$ and $y_n\xrightarrow{d_a}y$ is not applicable.*

Theorem 2.5 Let (X,d_a) be an algebraic cone metric space. Then the family $\{N_a(x,c): x\in X,\theta\ll_a c\}$, where $N_a(x,c)=\{y\in X: d_a(y,x)\ll_a c\}$, is a subbasis for topology on X (see [9]). We denote this algebraic cone topology by τ_a, and note that τ_a is a Hausdorff topology.

Proof. For the proof of the last statement, suppose that $N_a(x,c)\cap N_a(y,c)\neq\emptyset$ for each $\theta\ll_a c$. Then there exists $z\in X$ such that $d_a(z,x)\ll_a\frac{c}{2}$ and $d_a(z,y)\ll_a\frac{c}{2}$. Hence, $d_a(x,y)\preceq_a d_a(x,z)+d_a(z,y)\ll_a\frac{c}{2}+\frac{c}{2}=c$. Clearly, for each n, we have $\frac{c}{n}\in aint\ P$, so $\frac{c}{n}-d_a(x,y)\in aint\ P\subseteq P$. Now, $\theta-d_a(x,y)\in P$; that is, $d_a(x,y)\in P\cap(-P)=\{\theta\}$, and we have $d_a(x,y)=\theta$. Thus, $x=y$. ■

Now, we define algebraic distance and introduce some its properties.

Definition 2.6 Let (X, d_a) be an algebraic cone metric space. A function $q_a : X \times X \to Y$ is called a c-algebraic distance (or briefly, algebraic distance) on X if the following are satisfied:

- $\mathbf{q_1}$) $\theta \preceq_a q_a(x, y)$ for all $x, y \in X$;
- $\mathbf{q_2}$) $q_a(x, z) \preceq_a q_a(x, y) + q_a(y, z)$ for all $x, y, z \in X$;
- $\mathbf{q_3}$) for $x \in X$, if $q_a(x, y_n) \preceq_a u$ for some $u = u_x$ and all $n \geqslant 1$, then $q_a(x, y) \preceq_a u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;
- $\mathbf{q_4}$) for all $c \in Y$ with $\theta \ll_a c$, there exists $e \in Y$ with $\theta \ll_a e$ such that $q_a(z, x) \ll_a e$ and $q_a(z, y) \ll_a e$ imply $d_a(x, y) \ll_a c$.

***Example* 2.7** Let (Y, P) be an algebraic cone space with *aint* $P \neq \emptyset$ and (X, d_a) be an algebraic cone metric space such that the metric $d_a(\cdot, \cdot)$ is a continuous function in second variable. Then, $q_a(x, y) = d_a(x, y)$ is an algebraic distance. In fact, ($\mathbf{q_1}$) and ($\mathbf{q_2}$) are immediate. But, property ($\mathbf{q_3}$) is nontrivial and it follows from $q_a(x, y_n) = d_a(x, y_n) \preceq u$, passing to the limit when $n \to \infty$ and using continuity of d_a. Let $c \in Y$ with $c \in$ *aint* P be given and put $e = \frac{c}{2}$. Suppose that $q_a(z, x) \ll_a e$ and $q_a(z, y) \ll_a e$. Then $d_a(x, y) = q_a(x, y) \preceq q_a(x, z) + q_a(z, y) \ll_a e + e = c$. Using (p_1), this shows that $d_a(x, y) \ll_a c$ and thus q_a satisfies ($\mathbf{q_4}$). Hence, q_a is an algebraic distance.

***Example* 2.8** Let $Y = \mathbb{R}$ and $P = \{x \in Y : x \geqslant 0\}$. Let $X = [0, \infty)$ and define a mapping $d : X \times X \to Y$ by $d_a(x, y) = |x - y|_a$ for all $x, y \in X$. Then (X, d_a) is an algebraic cone metric space. Define a mapping $q_a : X \times X \to Y$ by $q_a(x, y) = y$ for all $x, y \in X$. Then, q is an algebraic distance. In fact, $(\mathbf{q_1}) - (\mathbf{q_3})$ are immediate. From $d_a(x, y) = |x - y|_a \leqslant_a x + y = q_a(z, x) + q_a(z, y)$, it follows that ($\mathbf{q_4}$) holds. Hence q_a is an algebraic distance.

In Examples 2.7 and 2.8, we introduce two known algebraic distances in an algebraic cone metric space. There exist some of other examples about distance in [1, 10] that reader can consider them in algebraic version. Also, similar to Example 3 of Dordević [3], one can consider algebraic distances which are not c-distances in cone metric spaces of [1, 10].

***Remark* 2** *From Examples 2.7 and 2.8, we have two important results:*

- *i) For an algebraic distance q_a, $q_a(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.*
- *ii) For an algebraic distance q_a, $q_a(x, y) = q_a(y, x)$ does not necessarily hold for all $x, y \in X$.*

We will recall a sequence $\{u_n\}$ in algebraic cone P is a c-sequence if for every $c \in$ *aint* P there exists $n_0 \in \mathbb{N}$ such that $u_n \ll_a c$ for $n \geqslant n_0$. It is easy to prove that if $\{u_n\}$ and $\{v_n\}$ are c-sequences in Y and $\alpha, \beta > 0$, then $\{\alpha u_n + \beta v_n\}$ is a c-sequence. Note that in the case cone P is normal, a sequence in Y is a c-sequence if and only if it is a θ-sequence. However, similar to *tvs*-cone metric spaces, when the cone is not normal, a c-sequence need not be a θ-sequence in algebraic cone metric spaces.

Lemma 2.9 Let (X, d_a) be an algebraic cone metric space and q_a be an algebraic distance on X. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$, and $\{u_n\}$ and $\{v_n\}$ be two c-sequences in algebraic cone P. Then the following hold:

- qp_1) f $q_a(x_n, y) \preceq_a u_n$ and $q_a(x_n, z) \preceq_a v_n$ for $n \in \mathbb{N}$, then $y = z$. Specifically, if $q_a(x, y) = \theta$ and $q_a(x, z) = \theta$, then $y = z$.
- qp_2) If $q_a(x_n, y_n) \preceq_a u_n$ and $q_a(x_n, z) \preceq_a v_n$ for $n \in \mathbb{N}$, then $\{y_n\}$ converges to z.

qp_3) If $q_a(x_n, x_m) \preceq_a u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.
qp_4) If $q_a(y, x_n) \preceq_a u_n$ for $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence in X.

Proof. The proof is straightforward and left to reader. ■

Now, as an application, we prove some fixed point theorems.

Theorem 2.10 Let (X, d_a) be a complete algebraic cone metric space and let q_a be an algebraic distance on X. Also, let $f : X \to X$ be a continuous self-map. Suppose that there exist mappings $\alpha, \beta, \gamma : X \to [0, 1)$ such that the following conditions hold:

$\mathbf{t_1}$) $\alpha(fx) \leqslant \alpha(x)$, $\beta(fx) \leqslant \beta(x)$, $\gamma(fx) \leqslant \gamma(x)$ and $(\alpha+\beta+\gamma)(x) < 1$ for all $x \in X$;
$\mathbf{t_2}$) for all $x, y \in X$,

$$q_a(fx, fy) \preceq_a \alpha(x)q_a(x, y) + \beta(x)q_a(x, fx) + \gamma(x)q_a(y, fy). \tag{1}$$

Then f has a fixed point in X. If $fx^* = x^*$, then $q_a(x^*, x^*) = \theta$.

Proof. Let $x_0 \in X$ and $fx_0 = x_0$. Then x_0 is a fixed point of f and the proof is finished. Suppose that $fx_0 \neq x_0$. Then we construct the sequence $\{x_n\}$ in X such that $x_n = f^n x_0 = fx_{n-1}$. In order to prove that it is a Cauchy sequence, put $x = x_{n-1}$ and $y = x_n$ in (1) and use $(\mathbf{t_1})$. We have

$$\begin{aligned} q_a(x_n, x_{n+1}) &= q_a(fx_{n-1}, fx_n) \\ &\preceq_a \alpha(x_{n-1})q_a(x_{n-1}, x_n) + \beta(x_{n-1})q_a(x_{n-1}, fx_{n-1}) + \gamma(x_{n-1})q_a(x_n, fx_n) \\ &= \alpha(fx_{n-2})q_a(x_{n-1}, x_n) + \beta(fx_{n-2})q_a(x_{n-1}, x_n) + \gamma(fx_{n-2})q_a(x_n, x_{n+1}) \\ &\preceq_a \big(\alpha(x_{n-2}) + \beta(x_{n-2})\big)q_a(x_{n-1}, x_n) + \gamma(x_{n-2})q_a(x_n, x_{n+1}) \\ &\ \ \vdots \\ &\preceq_a \big(\alpha(x_0) + \beta(x_0)\big)q_a(x_{n-1}, x_n) + \gamma(x_0)q_a(x_n, x_{n+1}), \end{aligned}$$

which implies that

$$q_a(x_n, x_{n+1}) \preceq_a \frac{\alpha(x_0) + \beta(x_0)}{1 - \gamma(x_0)} q_a(x_{n-1}, x_n)$$

for all $n \in \mathbb{N}$. Repeating this process, we get

$$q_a(x_n, x_{n+1}) \preceq_a \delta^n q_a(x_0, x_1) \tag{2}$$

for all $n \in \mathbb{N}$, where $0 \leqslant \delta = \frac{\alpha(x_0)+\beta(x_0)}{1-\gamma(x_0)} < 1$ (by $(\mathbf{t_1})$). Let $m > n$. In the usual way, it follows from (2) that

$$\begin{aligned} q_a(x_n, x_m) &\preceq_a q_a(x_n, x_{n+1}) + \cdots + q_a(x_{m-1}, x_m) \\ &\preceq_a \frac{\delta^n}{1 - \delta} q_a(x_0, x_1) = v_n, \end{aligned}$$

where $\{v_n\}$ is a c-sequence. Lemma 2.9.(qp_3) implies that $\{x_n\}$ is a Cauchy sequence in algebraic cone metric space X. Since X is complete, $x_n \xrightarrow{d_a} z \in X$ as $n \to \infty$. Continuity of f implies that $x_{n+1} = fx_n \xrightarrow{d_a} fz$, and since the limit of a sequence in an algebraic cone metric space is unique, we get $fz = z$; that is, z is a fixed point of f. Now, we suppose that $fx^* = x^*$. It follows from (1) that

$$q_a(x^*, x^*) = q_a(fx^*, fx^*) \preceq_a (\alpha(x^*) + \beta(x^*) + \gamma(x^*))q_a(x^*, x^*),$$

which is, by property (p_5) and $(\mathbf{t_1})$, possible only if $q_a(x^*, x^*) = \theta$. ■

Question 2.11 Can you the continuity condition of mapping f replace by another condition?

Let $\alpha(x) = \alpha$, $\beta(x) = \beta$ and $\gamma(x) = \gamma$ in Theorem 2.10, we have the following theorem.

Theorem 2.12 Let (X, d_a) be a complete algebraic cone metric space and let q_a be an algebraic distance on X. Suppose that a continuous self-map $f : X \to X$ satisfies the following condition

$$q_a(fx, fy) \preceq_a \alpha q_a(x, y) + \beta q_a(x, fx) + \gamma q_a(y, fy) \tag{3}$$

for all $x, y \in X$, where α, β and γ are nonnegative constants such that $\alpha + \beta + \gamma < 1$. Then f has a fixed point in X. If $fx^* = x^*$, then $q_a(x^*, x^*) = \theta$.

Sometimes the constant numbers α, β, γ which satisfy Theorem 2.12 is difficult to find. Thus, it is the better to define such control functions $\alpha(x), \beta(x), \gamma(x)$ as another auxiliary tool of the algebraic cone metric.

Definition 2.13 [7]If a map $f : X \to X$ satisfies $Fix(f) = Fix(f^n)$ for each $n \in \mathbb{N}$, where $Fix(f)$ stands for the set of fixed points of f, then f is said to have property (P).

Theorem 2.14 Let (X, d_a) be a complete algebraic cone metric space and $q_a : X \times X \to Y$ be an algebraic distance on X. Suppose that a continuous self-map $f : X \to X$ satisfies the contractive condition

$$q_a(fx, f^2x) \preceq_a \alpha q_a(x, fx), \tag{4}$$

for each $x \in X$, where $\alpha \in (0, 1)$. Then

i) f has a fixed point and if $fx^* = x^*$, then $q_a(x^*, x^*) = \theta$;
ii) f has property (P).

Proof. The proof of (i) is similar to tvs-cone metric version and left to the reader. We only prove (ii). Obviously, $Fix(f) \subseteq Fix(f^n)$ for each $n \in \mathbb{N}$. Let $z \in Fix(f^n)$; that is, $f^n z = z$. Using (4), we have

$$\begin{aligned} q_a(z, fz) &= q_a(f^n z, ff^n z) = q_a(ff^{n-1}z, f^2 f^{n-1}z) \\ &\preceq_a \alpha q_a(f^{n-1}z, ff^{n-1}z) = \alpha q_a(f^{n-1}z, f^n z) = \alpha q_a(ff^{n-2}z, f^2 f^{n-2}z) \\ &\preceq \alpha^2 q_a(f^{n-2}z, ff^{n-2}z) = \alpha^2 q_a(f^{n-2}z, f^{n-1}z) \preceq_a \cdots \preceq_a \alpha^n q_a(z, fz). \end{aligned}$$

It follows from (p_5) that $q_a(z, fz) = \theta$. Moreover, $q_a(f^m z, f^{m+1}z) \preceq_a q_a(z, fz)$ for each integer number $1 \leqslant m \leqslant n$. Therefore, $q_a(f^m z, f^{m+1}z) = \theta$. Now, by applying $(\mathbf{q_2})$, we obtain $q_a(z, fz) = q_a(z, f^2 z) = \cdots = q_a(z, f^n z) = \theta$. Using Lemma 2.9.$(qp_1)$, we have $z = fz$; that is, $z \in Fix(f)$. Therefore, $Fix(f^n) \subseteq Fix(f)$. This completes the proof. ■

3. Conclusion

In this paper we proved other properties of algebraic cone metric spaces, defined algebraic distance in an algebraic cone metric space and studied some of its elementary properties. Also, we obtained some well-known fixed point results under algebraic distance in an algebraic cone metric space. As a new work, one can be discussed on the validity these results in algebraic cone b-metric spaces introduced by Rahimi et al. [11].

Acknowledgements

The authors are grateful to the associate editor and referees for their accurate reading and their helpful suggestions.

References

[1] Y. J. Cho, R. Saadati, S. H. Wang, Common fixed point theorems on generalized distance in ordered cone metric spaces, Comput. Math. Appl. 61 (2011), 1254-1260.
[2] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, 1985.
[3] M. Dordević, D. Dorić, Z. Kadelburg, S. Radenović, D. Spasić, Fixed point results under c-distance in tvs-cone metric spaces, Fixed Point Theory Appl. (2011), 2011:29.
[4] W. S. Du, A note on cone metric fixed point theory and its equivalence, Nonlinear Anal. 72 (5) (2010), 2259-2261.
[5] L. G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007), 1467-1475.
[6] S. Janković, Z. Kadelburg, S. Radenović, On cone metric spaces, a survey, Nonlinear Anal. 74 (2011), 2591-2601.
[7] G. S. Jeong, B. E. Rhoades, Maps for which $F(T) = F(T^n)$, Fixed Point Theory Appl. 6 (2005), 87-131.
[8] O. Kada, T. Suzuki, W. Takahashi, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, Math. Japon. 44 (1996), 381-391.
[9] A. Niknam, S. Shamsi Gamchi, M. Janfada, Some results on TVS-cone normed spaces and algebraic cone metric spaces, Iranian J. Math. Sci. Inf. 9 (1) (2014), 71-80.
[10] H. Rahimi, G. Soleimani Rad, Common fixed-point theorems and c-distance in ordered cone metric spaces, Ukrain. Math. J. 65 (12) (2014), 1845-1861.
[11] H. Rahimi, G. Soleimani Rad, S. Radenović, Algebraic cone b-metric spaces and its equivalence, Miskolc. Math. Notes. 17 (1) (2016), 553-560.
[12] S. Rezapour, R. Hamlbarani, Some note on the paper cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 345 (2008), 719-724.
[13] P. P. Zabrejko, K-metric and K-normed linear spaces: survey, Collect. Math. 48 (1997), 825-859.