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Characterization of δ-double derivations on rings and algebras

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Abstract. The main purpose of this article is to offer some characterizations of δ-double derivations on rings and algebras. To reach this goal, we prove the following theorem:

Let $n > 1$ be an integer and let $\mathcal{R}$ be an $n!$-torsion free ring with the identity element $\mathbf{1}$. Suppose that there exist two additive mappings $d, \delta : \mathcal{R} \to \mathcal{R}$ such that

$$d(x^n) = \sum_{j=1}^{n} x^{n-j} d(x) x^{j-1} + \sum_{k=0}^{n-2} \sum_{i=0}^{n-2-k} x^k \delta(x) x^i \delta(x) x^{n-2-k-i}$$

is fulfilled for all $x \in \mathcal{R}$. If $\delta(\mathbf{1}) = 0$, then d is a Jordan δ-double derivation. In particular, if $\mathcal{R}$ is a semiprime algebra and further, $\delta^2(x^2) = \delta^2(x)x + x\delta^2(x) + 2(\delta(x))^2$ holds for all $x \in \mathcal{R}$, then $d - \frac{1}{2}\delta^2$ is an ordinary derivation on $\mathcal{R}$.

Keywords: δ-Double derivation, Jordan δ-double derivation, n-torsion free semiprime ring.

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1. Introduction and preliminaries

Throughout the paper, $\mathcal{R}$ will represent an associative ring with the identity element $\mathbf{1}$. We consider $x^0 = \mathbf{1}$ for all $x \in \mathcal{R}$. The center of $\mathcal{R}$ is

$$Z(\mathcal{R}) = \{x \in \mathcal{R} \mid xy = yx \ for \ all \ y \in \mathcal{R}\}.$$

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Given an integer $n \geqslant 2$, a ring $\mathcal{R}$ is said to be n-torsion free, if for $x \in \mathcal{R}$, $nx = 0$ implies $x = 0$. We denote the commutator $xy - yx$ by $[x, y]$ for all $x, y \in \mathcal{R}$. Recall that a ring $\mathcal{R}$ is prime if for $x, y \in \mathcal{R}$, $x\mathcal{R}y = \{0\}$ implies $x = 0$ or $y = 0$, and is semiprime in case $x\mathcal{R}x = \{0\}$ implies $x = 0$.
As well, the above-mentioned statements are considered for algebras. An additive mapping $d : \mathcal{R} \to \mathcal{R}$, where $\mathcal{R}$ is an arbitrary ring, is called a derivation (resp. Jordan derivation) if $d(xy) = d(x)y + xd(y)$ (resp. $d(x^2) = d(x)x + xd(x)$) holds for all $x, y \in \mathcal{R}$. One can easily prove that every derivation is a Jordan derivation, but the converse is not true, in general . An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is called a left derivation (resp. Jordan left derivation) if $d(xy) = xd(y) + yd(x)$ (resp. $d(x^2) = 2xd(x)$) holds for all $x, y \in \mathcal{R}$. A classical result of Herstein [7] asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein's result can be found in [3]. Cusack [5] generalized Herstein's result to 2-torsion free semiprime rings (see also [2] for an alternative proof). A series of results related to derivations on prime and semiprime rings can be found in [1–4, 8, 11–13].

M. Mirzavaziri and E. O. Tehrani [9] introduced the concept of a (δ, ε)-double derivation. Let $\delta, \varepsilon : \mathcal{R} \to \mathcal{R}$ be additive mappings. An additive mapping $D : \mathcal{R} \to \mathcal{R}$ is a (δ, ε)-double derivation if $D(xy) = D(x)y + xD(y) + \delta(x)\varepsilon(y) + \varepsilon(x)\delta(y)$ is fulfilled for all $x, y \in \mathcal{R}$. By a δ-double derivation we mean a (δ, δ)-double derivation, i.e.

$$D(xy) = D(x)y + xD(y) + 2\delta(x)\delta(y),$$

for all $x, y \in \mathcal{R}$. Let $\mathcal{A}$ be an algebra and let $D : \mathcal{A} \to \mathcal{A}$ be a linear (δ, δ)-double derivation. If $d = \frac{1}{2}D$, then $d(ab) = d(a)b + ad(b) + \delta(a)\delta(b)$ holds for all $a, b \in \mathcal{A}$. In this study, we consider the additive mapping d as a (δ, δ)-double derivation on a ring $\mathcal{R}$. Indeed, an additive mapping $d : \mathcal{R} \to \mathcal{R}$ is called a (δ, δ)-double derivation if $d(xy) = d(x)y + xd(y) + \delta(x)\delta(y)$ holds for all $x, y \in \mathcal{R}$. It is clear that if $\delta(x)\delta(y) = 0$ for all $x, y \in \mathcal{R}$, then d is an ordinary derivation. Here, we want to characterize such δ-double derivations. Similar to Jordan derivations, an additive mapping d is called a Jordan δ-double derivation if $d(x^2) = d(x)x + xd(x) + (\delta(x))^2$ holds for all $x \in \mathcal{R}$. Let $n > 1$ be an integer and let $d, \delta : \mathcal{R} \to \mathcal{R}$ be two additive maps satisfying

$$d(x^n) = \sum_{j=1}^{n} x^{n-j} d(x) x^{j-1} + \sum_{k=0}^{n-2} \sum_{i=0}^{n-2-k} x^k \delta(x) x^i \delta(x) x^{n-2-k-i},$$

for all $x \in \mathcal{R}$. If $\mathcal{R}$ is an $n!$-torsion free ring with the identity element $\mathbf{1}$ and $\delta(\mathbf{1}) = 0$, then d is a Jordan δ-double derivation. In particular, if $\mathcal{R}$ is a semiprime algebra and further,

$$\delta^2(x^2) = \delta^2(x)x + x\delta^2(x) + 2(\delta(x))^2,$$

for all $x \in \mathcal{R}$, then $d - \frac{1}{2}\delta^2$ is an ordinary derivation on $\mathcal{R}$. After defining a left δ-double derivation, we present a characterization of such mappings on algebras.

At the end of the paper, by getting idea from a work of Vukman [10], we offer another characterization of δ-double derivations on Banach algebras as follows. Let $\mathcal{A}$ be a Banach algebra with the identity element $\mathbf{1}$ and $\delta, d : \mathcal{A} \to \mathcal{A}$ be two additive maps satisfying $d(a) = -ad(a^{-1})a - a\delta(a^{-1})\delta(a)$ for each invertible element $a \in \mathcal{A}$. If $\delta(a) = -a\delta(a^{-1})a$ holds for every invertible element a, then d is a Jordan δ-double derivation. In particular, if $\mathcal{A}$ is semiprime and $\Big(\delta(a)\Big)^2 = \frac{1}{2}\Big(\delta^2(a^2) - \delta^2(a)a - a\delta^2(a)\Big)$ holds for all $a \in \mathcal{A}$, then $d - \frac{1}{2}\delta^2$ is a derivation on $\mathcal{A}$.

2. Main results

We begin with the following definition.

Definition 2.1 Let $\mathcal{R}$ be a ring and let $\delta : \mathcal{R} \to \mathcal{R}$ be an additive mapping. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is called a δ-double derivation if $d(xy) = d(x)y + xd(y) + \delta(x)\delta(y)$ for all $x, y \in \mathcal{R}$. The additive mapping d is said to be a Jordan δ-double derivation if $d(x^2) = d(x)x + xd(x) + (\delta(x))^2$ for all $x \in \mathcal{R}$.

The first main theorem reads as follows:

Theorem 2.2 Let $n > 1$ be an integer and $\mathcal{R}$ be an $n!$-torsion free ring with the identity element $\mathbf{1}$. Suppose that $d, \delta : \mathcal{R} \to \mathcal{R}$ are two additive maps satisfying $d(x^n) = \sum_{j=1}^{n} x^{n-j}d(x)x^{j-1} + \sum_{k=0}^{n-2} \sum_{i=0}^{n-2-k} x^k\delta(x)x^i\delta(x)x^{n-2-k-i}$ for all $x \in \mathcal{R}$. If $\delta(\mathbf{1}) = 0$, then d is a Jordan δ-double derivation. In particular, if $\mathcal{R}$ is a semiprime algebra and further, $\delta^2(x^2) = \delta^2(x)(x) + x\delta^2(x) + 2(\delta(x))^2$ holds for all $x \in \mathcal{R}$, then $d - \frac{1}{2}\delta^2$ is a derivation on $\mathcal{R}$.

Proof. Let y be an element of $Z(\mathcal{R})$ such that both $d(y)$ and $\delta(y)$ are zero. Based on the above hypothesis, we have

$$d(x^n) = \sum_{j=1}^{n} x^{n-j}d(x)x^{j-1} + \sum_{k=0}^{n-2} \sum_{i=0}^{n-2-k} x^k\delta(x)x^i\delta(x)x^{n-2-k-i} \tag{1}$$

for all $x \in \mathcal{R}$. Putting $x + y$ instead of x in equation (1), we have

$$\begin{aligned} d\Big(\sum_{i=0}^{n} \binom{n}{i} x^{n-i}y^i\Big) &= \sum_{j=1}^{n}(x+y)^{n-j}d(x)(x+y)^{j-1} \\ &+ \sum_{k=0}^{n-2}\sum_{i=0}^{n-2-k} (x+y)^k\delta(x)(x+y)^i\delta(x)(x+y)^{n-2-k-i} \\ &= \sum_{k_1=0}^{n-1} \binom{n-1}{k_1} x^{n-1-k_1}y^{k_1}d(x) \\ &+ \sum_{k_1=0}^{n-2} \binom{n-2}{k_1} x^{n-2-k_1}y^{k_1}d(x)(x+y) \\ &+ \sum_{k_1=0}^{n-3} \binom{n-3}{k_1} x^{n-3-k_1}y^{k_1}d(x)(x+y)^2 \to \end{aligned}$$

$$+\ldots+(x+y)^2 d(x)\sum_{k_1=0}^{n-3}\binom{n-3}{k_1}x^{n-3-k_1}y^{k_1}$$

$$+(x+y)d(x)\sum_{k_1=0}^{n-2}\binom{n-2}{k_1}x^{n-2-k_1}y^{k_1}$$

$$+d(x)\sum_{k_1=0}^{n-1}\binom{n-1}{k_1}x^{n-1-k_1}y^{k_1}$$

$$+\sum_{i=0}^{n-2}\delta(x)(x+y)^i\delta(x)(x+y)^{n-2-i}$$

$$+\sum_{i=0}^{n-3}(x+y)\delta(x)(x+y)^i\delta(x)(x+y)^{n-3-i}$$

$$+\sum_{i=0}^{n-4}(x+y)^2\delta(x)(x+y)^i\delta(x)(x+y)^{n-4-i}$$

$$+\ldots+(x+y)^{n-2}(\delta(x))^2=\sum_{k_1=0}^{n-1}\binom{n-1}{k_1}x^{n-1-k_1}y^{k_1}d(x)$$

$$+\sum_{k_1=0}^{n-2}\binom{n-2}{k_1}x^{n-2-k_1}y^{k_1}d(x)(x+y)$$

$$+\sum_{k_1=0}^{n-3}\binom{n-3}{k_1}x^{n-3-k_1}y^{k_1}d(x)(x+y)^2$$

$$+\ldots+(x+y)^2 d(x)\sum_{k_1=0}^{n-3}\binom{n-3}{k_1}x^{n-3-k_1}y^{k_1}$$

$$+(x+y)d(x)\sum_{k_1=0}^{n-2}\binom{n-2}{k_1}x^{n-2-k_1}y^{k_1}$$

$$+d(x)\sum_{k_1=0}^{n-1}\binom{n-1}{k_1}x^{n-1-k_1}y^{k_1}+\Big[(\delta(x))^2\sum_{k_1=0}^{n-2}\binom{n-2}{k_1}x^{n-2-k_1}y^{k_1}$$

$$+\delta(x)(x+y)\delta(x)\sum_{k_1=0}^{n-3}\binom{n-3}{k_1}x^{n-3-k_1}y^{k_1}$$

$$+\ldots+\delta(x)\sum_{k_1=0}^{n-2}\binom{n-2}{k_1}y^{k_1}x^{n-2-k_1}\delta(x)\Big]$$

$$+\Big[(x+y)(\delta(x))^2\sum_{k_2=0}^{n-3}\binom{n-3}{k_2}x^{n-3-k_2}y^{k_2}$$

$$+(x+y)\delta(x)(x+y)\delta(x)\sum_{k_2=0}^{n-4}\binom{n-4}{k_2}x^{n-4-k_2}y^{k_2}\rightarrow$$

$$+ \ldots + (x+y)\delta(x) \sum_{k_2=0}^{n-3} \binom{n-3}{k_2} x^{n-3-k_2} y^{k_2} \delta(x)\Big]$$

$$+ \Big[(x+y)^2(\delta(x))^2 \sum_{k_3=0}^{n-4} \binom{n-4}{k_3} x^{n-4-k_3} y^{k_3}$$

$$+ (x+y)^2\delta(x)(x+y)\delta(x) \sum_{k_3=0}^{n-5} \binom{n-5}{k_3} x^{n-5-k_3} y^{k_3}$$

$$+ \ldots + (x+y)^2\delta(x) \sum_{k_3=0}^{n-4} \binom{n-4}{k_3} x^{n-4-k_3} y^{k_3} \delta(x)\Big]$$

$$+ \ldots + \sum_{k_{n-1}=0}^{n-2} \binom{n-2}{k_{n-1}} x^{n-2-k_{n-1}} y^{k_{n-1}} (\delta(x))^2.$$

Using (1) and collecting together terms of the above-mentioned relations involving the same number of factors of y, it can be obtained that

$$\sum_{i=1}^{n-1} \gamma_i(x,y) = 0, \qquad x \in \mathcal{R}, \tag{2}$$

where

$$\gamma_i(x,y) = \binom{n}{i} d(x^{n-i}y^i) - \sum_{l=1}^{n-i} \binom{n}{i} x^{n-i-l} y^i d(x) x^{l-1}$$
$$- \sum_{p=0}^{n-2-i} \sum_{q=0}^{n-2-i-p} \binom{n}{i} y^i x^p \delta(x) x^q \delta(x) x^{n-2-i-p-q}$$

Having replaced y, $2y$, $3y$,..., $(n-1)y$ instead of y in (2), we obtain a system of $n-1$ homogeneous equations as follows:

$$\begin{cases} \sum_{i=1}^{n-1} \gamma_i(x,y) = 0 \\ \sum_{i=1}^{n-1} \gamma_i(x,2y) = 0 \\ \sum_{i=1}^{n-1} \gamma_i(x,3y) = 0 \\ \quad\vdots \\ \sum_{i=1}^{n-1} \gamma_i(x,(n-1)y) = 0 \end{cases}$$

It is observed that the coefficient matrix of the above system is:

$$X = \begin{bmatrix} \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \dots & \binom{n}{n-1} \\ 2\binom{n}{1} & 2^2\binom{n}{2} & 2^3\binom{n}{3} & \dots & 2^{n-1}\binom{n}{n-1} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ (n-1)\binom{n}{1} & (n-1)^2\binom{n}{2} & (n-1)^3\binom{n}{3} & \dots & (n-1)^{n-1}\binom{n}{n-1} \end{bmatrix}$$

It is evident that

$$det\ X = \Big(\prod_{k=1}^{n-1}\tbinom{n}{k}\Big)(n-1)! \prod_{1\leqslant i<j\leqslant n-1}(i-j).$$

Since $det\ X \neq 0$, the above-mentioned system has only a trivial solution. In particular, $\gamma_{n-2}(x,y) = 0$. Indeed,

$$0 = \binom{n}{n-2}d(x^2y^{n-2}) - \sum_{l=1}^{2}\binom{n}{n-2}x^{2-l}y^{n-2}d(x)x^{l-1} - \sum_{p=0}^{0}\sum_{q=0}^{0}\binom{n}{n-2}y^{n-2}x^0\delta(x)x^0\delta(x)x^0$$
$$= \binom{n}{n-2}d(x^2y^{n-2}) - \binom{n}{n-2}xy^{n-2}d(x) - \binom{n}{n-2}y^{n-2}d(x)x - \binom{n}{n-2}(\delta(x))^2 \qquad (*).$$

Since $\delta(\mathbf{1}) = 0$, we have $d(\mathbf{1}) = nd(\mathbf{1}) + 0 = nd(\mathbf{1})$ and it demonstrates that $d(\mathbf{1}) = 0$. Substituting $\mathbf{1}$ instead of y in (*), we achieve

$$\binom{n}{n-2}d(x^2) - \binom{n}{n-2}xd(x) - \binom{n}{n-2}d(x)x - \binom{n}{n-2}(\delta(x))^2 = 0 \qquad (3)$$

for all $x \in \mathcal{R}$. Since $\mathcal{R}$ is an $n!$-torsion free ring, it follows from equation (3) that

$$d(x^2) = xd(x) + d(x)x + (\delta(x))^2, \qquad x \in \mathcal{R}. \qquad (4)$$

In other words, d is a Jordan δ-double derivation. Now, assume that $\mathcal{R}$ is a semiprime algebra and further, $\delta^2(x^2) = \delta^2(x)(x) + x\delta^2(x) + 2(\delta(x))^2$ for all $x \in \mathcal{R}$. This equation along with (4) imply that $d(x^2) = xd(x) + d(x)x + \frac{1}{2}(\delta^2(x^2) - \delta^2(x)x - x\delta^2(x))$. Hence, $(d-\frac{1}{2}\delta^2)(x^2) = x(d-\frac{1}{2}\delta^2)(x)+(d-\frac{1}{2}\delta^2)(x)x$. It means that $d-\frac{1}{2}\delta^2$ is a Jordan derivation. It follows from Theorem 1 of [2] that $d - \frac{1}{2}\delta^2$ is an ordinary derivation on $\mathcal{R}$. Thereby, our claim is achieved. ∎

Using the above theorem, we obtain the following corollary:

Corollary 2.3 Let $n > 1$ be an integer and $\mathcal{A}$ be a semiprime algebra with the identity element $\mathbf{1}$. Suppose that $d, \delta : \mathcal{A} \to \mathcal{A}$ are two additive mappings such that $d(a^n) = \sum_{j=1}^{n} a^{n-j}d(a)a^{j-1} + \sum_{k=0}^{n-2}\sum_{i=0}^{n-2-k} a^k\delta(a)a^i\delta(a)a^{n-2-k-i}$ for all $a \in \mathcal{A}$. If δ is a derivation, then d is a δ-double derivation.

Proof. Previous theorem along with the assumption that δ is a derivation imply that

$\Delta = d - \frac{1}{2}\delta^2$ is a derivation. Therefore, we have

$$\begin{aligned} d(ab) &= \Delta(ab) + \frac{1}{2}\delta^2(ab) = \Delta(a)b + a\Delta(b) + \frac{1}{2}\Big(\delta^2(a)b + a\delta^2(b) + 2\delta(a)\delta(b)\Big) \\ &= d(a)b + ad(b) + \delta(a)\delta(b) \end{aligned}$$

for all $a, b \in \mathcal{A}$. It means that d is a δ-double derivation. ■

Definition 2.4 Let $\mathcal{R}$ be a ring and let $\delta : \mathcal{R} \to \mathcal{R}$ be an additive mapping. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is called a left δ-double derivation if $d(xy) = xd(y)+yd(x)+\delta(x)\delta(y)$ holds for all $x, y \in \mathcal{R}$. In addition, the additive mapping d is said to be a Jordan left δ-double derivation if $d(x^2) = 2xd(x) + (\delta(x))^2$ is fulfilled for all $x \in \mathcal{R}$.

Below, we provide a characterization of Jordan left δ-double derivations.

Theorem 2.5 Let $n > 1$ be an integer and $\mathcal{R}$ be an $n!$-torsion free ring with the identity element $\mathbf{1}$. Suppose that $d, \delta : \mathcal{R} \to \mathcal{R}$ are two additive maps satisfying

$$d(x^n) = nx^{n-1}d(x) + \binom{n}{2}x^{n-2}(\delta(x))^2$$

for all $x \in \mathcal{R}$. If $\delta(\mathbf{1}) = 0$, then $d(x^2) = 2xd(x)+(\delta(x))^2$. In particular, if $\mathcal{R}$ is a semiprime algebra and further, $\delta^2(x^2) = 2\Big(x\delta^2(x) + (\delta(x))^2\Big)$ holds for all $x \in \mathcal{R}$, then $d - \frac{1}{2}\delta^2$ is a derivation mapping $\mathcal{R}$ into $Z(\mathcal{R})$.

Proof. Similar to the presented argument in Theorem 2.2, let y be an element of $Z(\mathcal{R})$ such that both $d(y)$ and $\delta(y)$ are zero. According to the aforementioned assumption, we have

$$d(x^n) = nx^{n-1}d(x) + \binom{n}{2}x^{n-2}(\delta(x))^2 \tag{5}$$

for all $x \in \mathcal{R}$. Having put $x + y$ instead of x in the above equation, we have

$$\begin{aligned} d\Big(\sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^i\Big) &= n(x+y)^{n-1}d(x) + \binom{n}{2}(x+y)^{n-2}(\delta(x))^2 \\ &= n\sum_{i=0}^{n-1}\binom{n-1}{i}x^{n-1-i}y^i d(x) + \binom{n}{2}\sum_{i=0}^{n-2}\binom{n-2}{i}x^{n-2-i}y^i(\delta(x))^2 \end{aligned}$$

Therefore, we have

$$\begin{aligned} &d(x^n) + \binom{n}{1}d(x^{n-1}y) + \binom{n}{2}d(x^{n-2}y^2) + ... + \binom{n}{n-1}d(xy^{n-1}) \\ &= nx^{n-1}d(x) + n\binom{n-1}{1}x^{n-2}yd(x) + n\binom{n-1}{2}x^{n-3}y^2d(x) + ... + ny^{n-1}d(x) \\ &+ \binom{n}{2}x^{n-2}(\delta(x))^2 + \binom{n}{2}\binom{n-2}{1}x^{n-3}y(\delta(x))^2 + ... + \binom{n}{2}y^{n-2}(\delta(x))^2 \end{aligned}$$

Using (5) and collecting together terms of above-mentioned relations involving the same number of factors of y, we obtain

$$\sum_{i=1}^{n-1} \lambda_i(x,y) = 0, \qquad x \in \mathcal{R}, \tag{6}$$

where

$$\lambda_i(x,y) = \binom{n}{i} d(x^{n-i}y^i) - n\binom{n-1}{i} x^{n-1-i}y^i d(x) - \binom{n}{2}\binom{n-2}{i} x^{n-2-i}y^i(\delta(x))^2.$$

Having replaced y, $2y$, $3y$, . . . , $(n-1)y$ instead of y in (6), we obtain a system of $n-1$ homogeneous equations as follows:

$$\begin{cases} \sum_{i=1}^{n-1} \lambda_i(x,y) = 0 \\ \sum_{i=1}^{n-1} \lambda_i(x,2y) = 0 \\ \sum_{i=1}^{n-1} \lambda_i(x,3y) = 0 \\ \quad\vdots \\ \sum_{i=1}^{n-1} \lambda_i(x,(n-1)y) = 0 \end{cases}$$

It is evident that the coefficient matrix of the above system is:

$$Y = \begin{bmatrix} \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \cdots & \binom{n}{n-1} \\ 2\binom{n}{1} & 2^2\binom{n}{2} & 2^3\binom{n}{3} & \cdots & 2^{n-1}\binom{n}{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (n-1)\binom{n}{1} & (n-1)^2\binom{n}{2} & (n-1)^3\binom{n}{3} & \cdots & (n-1)^{n-1}\binom{n}{n-1} \end{bmatrix}$$

Obviously,

$$det\ Y = \Big(\prod_{k=1}^{n-1}\tbinom{n}{k}\Big)(n-1)! \prod_{1\leqslant i<j\leqslant n-1}(i-j).$$

Since $det\ Y \neq 0$, the above-mentioned system has only a trivial solution. In particular, $\lambda_{n-2}(x,y) = 0$, i.e.

$$\binom{n}{n-2} d(x^2y^{n-2}) - 2\binom{n}{n-2} xy^{n-2}d(x) - \binom{n}{n-2} y^{n-2}(\delta(x))^2 = 0.$$

Since $\mathcal{R}$ is an $n!$-torsion free ring, we have

$$d(x^2y^{n-2}) - 2xy^{n-2}d(x) - y^{n-2}(\delta(x))^2 = 0. \tag{7}$$

Putting $x = \mathbf{1}$ in equation (5) and using the hypothesis that $\delta(\mathbf{1}) = 0$, we achieve

$d(\mathbf{1}) = 0$. Thus, we can put $\mathbf{1}$ instead of y in (7) to obtain

$$d(x^2) = 2xd(x) + (\delta(x))^2, \tag{8}$$

for all $x \in \mathcal{A}$. It means that d is a Jordan left δ-double derivation. Now, assume that $\mathcal{R}$ is a semiprime algebra and further, $\delta^2(x^2) = 2\Big(x\delta^2(x) + (\delta(x))^2\Big)$ holds for all $x \in \mathcal{R}$. From this equation and equation (8), we arrive at

$$d(x^2) = 2xd(x) + \frac{1}{2}\delta^2(x^2) - x\delta^2(x) \tag{9}$$

Therefore, $(d - \frac{1}{2}\delta^2)(x^2) = 2x(d - \frac{1}{2}\delta^2)(x)$, and it means that $\Delta = d - \frac{1}{2}\delta^2$ is a Jordan left derivation. At this moment, Theorem 2 of [10] is exactly what we need to complete the proof. ■

We are now ready to establish another characterization of δ-double derivations on algebras.

Corollary 2.6 Let $n > 1$ be an integer and $\mathcal{A}$ be a semiprime algebra with the identity element $\mathbf{1}$. Suppose that $d, \delta : \mathcal{A} \to \mathcal{A}$ are two additive maps satisfying

$$d(a^n) = na^{n-1}d(a) + \binom{n}{2}a^{n-2}(\delta(a))^2$$

for all $a \in \mathcal{A}$. If δ is a left derivation, then d is a δ-double derivation mapping $\mathcal{A}$ into $Z(\mathcal{A})$.

Proof. It follows from Theorem 2 of [10] that δ is a derivation mapping $\mathcal{A}$ into $Z(\mathcal{A})$. Theorem 2.5 of the current study implies that $\Delta(a) = d(a) - \frac{1}{2}\delta^2(a) \in Z(\mathcal{A})$ for all $a \in \mathcal{A}$, and consequently, $d(\mathcal{A}) \subseteq Z(\mathcal{A})$. A straightforward verification shows that d is a δ-double derivation. ■

The following theorem has been motivated by a work of Vukman [10].

Theorem 2.7 Let $\mathcal{A}$ be a Banach algebra with the identity element $\mathbf{1}$ and let

$$d, \delta : \mathcal{A} \to \mathcal{A},$$

be two additive maps satisfying

$$d(a) = -ad(a^{-1})a - a\delta(a^{-1})\delta(a) \tag{10}$$

for all invertible elements $a \in \mathcal{A}$. If $\delta(a) = -a\delta(a^{-1})a$ for all invertible elements a, then d is a Jordan δ-double derivation. In particular, if $\mathcal{A}$ is semiprime and further, $(\delta(x))^2 = \frac{1}{2}\Big(\delta^2(x^2) - \delta^2(x)x - x\delta^2(x)\Big)$ holds for all $x \in \mathcal{A}$, then $d - \frac{1}{2}\delta^2$ is a derivation.

Proof. Let x be an arbitrary element of $\mathcal{A}$ and let n be a positive number so that $\|\frac{x}{n-1}\| < 1$. It is evident that $\|\frac{x}{n}\| < 1$, too. If we consider $a = n\mathbf{1} + x$, then we have $\frac{a}{n} = \mathbf{1} + \frac{x}{n} = \mathbf{1} - \frac{-x}{n}$. Since $\|\frac{-x}{n}\| < 1$, it follows from Theorem 1.4.2 of [6] that $\mathbf{1} - \frac{-x}{n}$ is invertible and consequently, a is invertible. Similarly, we can show that $\mathbf{1} - a$ is also an invertible element of $\mathcal{A}$. In the following, we use the well-known Hua identity

$$a^2 = a - \Big(a^{-1} + (\mathbf{1} - a)^{-1}\Big)^{-1}.$$

Applying equation (10), we have

$$\begin{aligned}
d(a^2) &= d(a) - d\Big((a^{-1} + (\mathbf{1} - a)^{-1})^{-1}\Big) \\
&= d(a) + (a^{-1} + (\mathbf{1} - a)^{-1})^{-1} d(a^{-1} + (\mathbf{1} - a)^{-1})(a^{-1} + (\mathbf{1} - a)^{-1})^{-1} \\
&+ (a^{-1} + (\mathbf{1} - a)^{-1})^{-1} \delta(a^{-1} + (\mathbf{1} - a)^{-1}) \delta((a^{-1} + (\mathbf{1} - a)^{-1})^{-1}) \\
&= d(a) + a(\mathbf{1} - a)(-a^{-1} d(a) a^{-1} - a^{-1}\delta(a)\delta(a^{-1}))a(\mathbf{1} - a) \\
&+ a(\mathbf{1} - a)(-(\mathbf{1} - a)^{-1} d(\mathbf{1} - a)(\mathbf{1} - a)^{-1}) - \Big((\mathbf{1} - a)^{-1}\delta(\mathbf{1} - a)\delta((\mathbf{1} - a)^{-1}) \\
&\times a(\mathbf{1} - a)\Big) + \Big(a(\mathbf{1} - a)(-a^{-1}\delta(a)a^{-1})(\mathbf{1} - a)^{-1}\delta(\mathbf{1} - a)(\mathbf{1} - a)^{-1} \\
&\times (-(a^{-1} + (\mathbf{1} - a)^{-1})^{-1})\delta(a^{-1})\Big) + (\mathbf{1} - a)^{-1}(a^{-1} + (\mathbf{1} - a)^{-1})^{-1} \\
&= d(a) - a(\mathbf{1} - a)a^{-1} d(a) a^{-1} a(\mathbf{1} - a) - a(\mathbf{1} - a)a^{-1}\delta(a)\delta(a^{-1})a(\mathbf{1} - a) \\
&+ a(\mathbf{1} - a)(\mathbf{1} - a)^{-1} d(a)(\mathbf{1} - a)^{-1} a(\mathbf{1} - a) + \Big(a(\mathbf{1} - a)(\mathbf{1} - a)^{-1}\delta(a) \\
&\times \delta((\mathbf{1} - a)^{-1})a(\mathbf{1} - a)\Big) + a(\mathbf{1} - a)a^{-1}\delta(a)a^{-1}(a\delta(a) + \delta(a)a - \delta(a)) \\
&- a(\mathbf{1} - a)(\mathbf{1} - a)^{-1}\delta(a)(\mathbf{1} - a)^{-1}(a\delta(a) + \delta(a)a - \delta(a)) \\
&= d(a) - (\mathbf{1} - a)d(a)(\mathbf{1} - a) - (\mathbf{1} - a)\delta(a)\delta(a^{-1})a(\mathbf{1} - a) + ad(a)a \\
&+ a\delta(a)\delta((\mathbf{1} - a)^{-1})a(\mathbf{1} - a) + (\mathbf{1} - a)(\delta(a))^2 + (\mathbf{1} - a)\delta(a)a^{-1}\delta(a)a \\
&- (\mathbf{1} - a)\delta(a)a^{-1}\delta(a) - a\delta(a)(\mathbf{1} - a)^{-1}a\delta(a) - a\delta(a)(\mathbf{1} - a)^{-1}\delta(a)a \\
&+ a\delta(a)(\mathbf{1} - a)^{-1}\delta(a) \\
&= d(a)a + ad(a) - \delta(a)\delta(a^{-1})a + \delta(a)\delta(a^{-1})a^2 + a\delta(a)\delta(a^{-1})a \\
&- a\delta(a)\delta(a^{-1})a^2 + a\delta(a)(\mathbf{1} - a)^{-1}\delta(a)(\mathbf{1} - a)^{-1}a(\mathbf{1} - a) + (\delta(a))^2 \\
&- a(\delta(a))^2 + \delta(a)a^{-1}\delta(a)a - a\delta(a)a^{-1}\delta(a)a - \delta(a)a^{-1}\delta(a) + \big(a\delta(a)a^{-1} \\
&\times \delta(a)\big) - a\delta(a)(\mathbf{1} - a)^{-1}a\delta(a) - a\delta(a)(\mathbf{1} - a)^{-1}\delta(a)a + a\delta(a)(\mathbf{1} - a)^{-1}\delta(a) \\
&= d(a)a + ad(a) + (\delta(a))^2 - a(\delta(a))^2 - a\delta(a)(\mathbf{1} - a)^{-1}a\delta(a) \\
&+ a\delta(a)(\mathbf{1} - a)^{-1}\delta(a) \qquad (see \;\; \delta(a) = -a\delta(a^{-1})a) \\
&= d(a)a + ad(a) + (\delta(a))^2 - a(\delta(a))^2 - a\delta(a)((\mathbf{1} - a)^{-1} - \mathbf{1})\delta(a) \\
&+ a\delta(a)(\mathbf{1} - a)^{-1}\delta(a) = d(a)a + ad(a) + (\delta(a))^2.
\end{aligned}$$

Since $\delta(\mathbf{1}) = -\mathbf{1}\delta(\mathbf{1}^{-1})\mathbf{1}$, $\delta(\mathbf{1}) = 0$ and it implies that $d(\mathbf{1}) = 0$. We know that $d(a^2) = d(a)a + ad(a) + (\delta(a))^2$. Having put $a = n\mathbf{1} + x$ in the previous equation, we have

$d(n^2 + 2nx + x^2) = d(x)(n\mathbf{1} + x) + (n\mathbf{1} + x)d(x) + (\delta(x))^2$. Therefore,

$$d(x^2) = d(x)x + xd(x) + (\delta(x))^2,$$

for all $x \in \mathcal{A}$, i.e. d is a Jordan δ-double derivation. Now, assume that $(\delta(x))^2 = \frac{1}{2}\Big(\delta^2(x^2) - \delta^2(x)x - x\delta^2(x)\Big)$ for all $x \in \mathcal{A}$. Hence, $d(x^2) = xd(x) + d(x)x + \frac{1}{2}\Big(\delta^2(x^2) - \delta^2(x)x - x\delta^2(x)\Big)$; equivalently we have, $(d - \frac{1}{2}\delta^2)(x^2) = x(d - \frac{1}{2}\delta^2)(x) + (d - \frac{1}{2}\delta^2)(x)x$. It means that $d - \frac{1}{2}\delta^2$ is a Jordan derivation. Now, Theorem 1 of [2] completes our proof. ■

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