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Preorder relators and generalized topologies

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Abstract. In this paper we investigate generalized topologies generated by a subbase of preorder relators and consider its application in the concept of the complement. We introduce the notion of principal generalized topologies obtained from the new type of open sets and study some of their important properties.

Keywords: Generalized topology, preoerder relator, principal generalized topology.

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1. Introduction

The concept of generalized topology was introduced by Á. Császár [3, 4]. It is devoted to the study of kinds of subsets of a topological space are often generalized the notion of open sets. He and many other authers have extensively studied on important properties and various results in framework of the type of generalizations. Here we recall some definitions and the most essential concepts needed in our work.

Let X be a nonempty set and the collection μ be a subset of the power set $\mathcal{P}(X)$. Then μ is called a *generalized topology* (briefly GT) on X if it contains $\emptyset$ and any union of elements of μ belongs to μ. Therefore every topology is a generalized topology. We call the pair (X, μ) a generalized topological space on X. The family of all generalized topologies defined on X denote by $\mathcal{G}T(X)$. The elements of μ are called μ-*open* sets and the complements are called μ-*closed* sets.

It is quite natural that a considerable part of the properties of generalized topologies can be deduced from suitable more general definitions. The family of all generalized

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topologies on a set, partially ordered by set inclusion is a bounded lattice, neither distributive nor complemented which discussed in [1]. According to [10], Steiner have been studied the structure of the lattice of principal general topologies, employing the notion of ultratopology. Every preorder relation is induced by a principal general topology. In fact, there is a canonical isomorphism between the lattice of principal topologies and the lattice of preorder relations on a set [7, 8, 10]. The aim of the present paper is to show that all generalized topologies derived from the corresponding preorder relations and to get some unknown results by special choice of the generalized topology, the socalled *principal generalized topology*. Principal generalized topologies are then characterized by properties of generalized open sets, analogous to some topological concepts of ultratopologies introduced in [5, 10]. By this purpose, we obtain generalizations of results in [1, 6].

2. Generalized Topologies Generated by Preorder Relators

In this section, we start with recalling some basic facts concerning (preorder) relators on a set. Let X be a nonempty set and $X^2 = X \times X$. A subset R of X^2 is called a relation on X. In particular, $\Delta_X = \{(x,x) |\ x \in X\}$, the diameter of X is called the identity relation on X. For any $x \in X$ and subset $A \subset X$, the sets $R(x) = \{y \in X |\ (x,y) \in R\}$ and $R(A) = \bigcup_{a \in A} R(a)$ are said to be the image of x and A under R, respectively. The inverse relation R^{-1} can be defined as $R^{-1}(x) = \{y \in X |\ x \in R(y)\}$ for all $x \in X$. We may briefly write R^2 instead of $R \circ R$, the composition relation defined by $(R \circ R)(x) = R(R(x))$ for any $x \in X$. A relation R on X is reflexive, symmetric and trasitive if $\Delta_X \subset R, \quad R^{-1} \subset R$ and $R^2 \subset R$. A reflexive and transitive relation may be called a preorder relation on X. A family $\mathcal{R}$ of relations on X is called a relator on X. Moreover, the pair $X(\mathcal{R}) = (X, \mathcal{R})$ is said to be a relator space. Relator spaces are natural generalization of ordered sets. Therefore, all generalizations of the usual topological structures can be derived from relators according to [2, 9]. A relator $\mathcal{R}$ on X may be called reflexive, symmetric and transitive if each of its members has the corresponding property. Thus, we may also naturally speak of preorder relators.

Definition 2.1 Let $A \subset X$. The relation $R_A = A^2 \cup A^c \times X$, where $A^c = X \setminus A$, is called the Pervin relation on X generated by A.

These relations are particular cases of the relations $R_{(A,\ B)} = A \times B \cup A^c \times X$, with $A \subset B \subset X$, introduced first by Á.Császár [2]. It is clear that for any $\emptyset \neq B \subset X$, we have $R_A(B) = \bigcup_{x \in B} R_A(x)$ is equal to A, if $B \subset A$ and is X, otherwise.

Lemma 2.2 For $A \subset X$, R_A is a preorder relation on X and $R_A^{-1} = R_{A^c}$.

Proof. Since $\Delta_X \subset R_A$, R_A is reflexive. Moreover, $R_A(x) = A$ if $x \in A$ and $R_A(x) = X$ if $x \in A^c$. We can easily see that $R_A^2(x) = R_A(R_A(x))$ is A, if $x \in A$ and X, if $x \in A^c$. Therefore $R_A^2 = R_A$ and R_A is transitive. Similarly, $R_A^{-1}(x) = X$, if $x \in A$ and A^c, if $x \in A^c$. Hence $R_A^{-1} = R_{A^c}$. ■

Lemma 2.3 Let μ be a GT on X. Then the family $\mathcal{R}_\mu = \{R_A |\ A \in \mu\}$, generated by μ is a preorder relator on X.

Proof. It follows from Lemma (2.2). ■

Theorem 2.4 Let $\mu \in \mathcal{GT}(X)$ and $\mathcal{R}_\mu$ be a preorder relator on X. A collection $\mathcal{T}_{\mathcal{R}_\mu}$ of subsets of $X(\mathcal{R}_\mu)$ is defined by:

$$V \in \mathcal{T}_{\mathcal{R}_\mu}, \quad \text{iff} \quad \forall\, x \in V,\ \exists\, R_A \in \mathcal{R}_\mu,\ R_A(x) \subset V \ \text{ for some } A \subset X.$$

Then $\mathcal{T}_{\mathcal{R}_\mu}$ is a generalized topology generated by $\mathcal{R}_\mu$ on X.

Proof. We must to show that $\mathcal{T}_{\mathcal{R}_\mu}$ is closed under arbitrary unions. Assume that $\mathcal{B} \subset \mathcal{T}_{\mathcal{R}_\mu}$ and let $V = \bigcup \mathcal{B}$. We claim that $V \in \mathcal{T}_{\mathcal{R}_\mu}$. For each $x \in V$, there exists $B \in \mathcal{B}$ with $x \in B$. Since $B \in \mathcal{T}_{\mathcal{R}_\mu}$ by definition there exists $R_A \in \mathcal{R}$, for some $A \subset X$ such that $R_A(x) \subset B$ and since $B \subset V$, it follows that $R_A(x) \subset V$. Hence $V \in \mathcal{T}_{\mathcal{R}_\mu}$. ■

***Example* 2.5** Let $X = \{a, b, c\}$. Consider the preorder relator $\mathcal{R} = \{R_1, R_2\}$ on X, for which $R_1 = \{(a, a), (b, b), (c, c), (a, c), (a, b)\}$ and $R_1 = \{(a, a), (b, b), (c, c), (b, c)\}$. Then $\mathcal{T}_\mathcal{R} = \{\emptyset, \{a\}, \{b, c\}, X\}$ is a GT on X.

Theorem 2.6 If μ is a proper in $\mathcal{GT}(X)$ then $\mathcal{R}_\mu$ induces a subbase for a generalized topology $\mathcal{T}_{\mathcal{R}_\mu}$ on X.

Proof. It is enough to show that if $V \in \mathcal{T}_{\mathcal{R}_\mu}$, with $\emptyset \neq V \subsetneq X$ then there exists $\mathcal{B} \subset \mu$ such that $V = \bigcup \mathcal{B}$. Suppose $V \in \mathcal{T}_{\mathcal{R}_\mu}$, then by using definition of open sets in $\mathcal{T}_{\mathcal{R}_\mu}$ for each $x \in V$, we have $A_x \in \mu$ such that $R_{A_x} \subset V$. Since $V \neq X$ and $R_{A_x}(x) = A_x$, it follows that $x \in A_x \subset V$. Therefore, $V = \bigcup_{x \in A_x} A_x$. Setting $\mathcal{B} = \{A_x |\ x \in V\}$, the proof is completed. ■

Theorem 2.7 If $\emptyset \neq \mu \subsetneq \mathcal{P}(X)$, then the following statements are equivalent:

(i) μ is a GT on X.
(ii) $\mu = \mathcal{T}_{\mathcal{R}_\mu}$.
(iii) $\mu = \mathcal{T}_\mathcal{R}$ for some preorder relator $\mathcal{R}$ on X.

Proof. $(i) \Rightarrow (ii)$: for any $A \in \mu$ we have $R_A(A) = A$ and thus $A \in \mathcal{T}_{\mathcal{R}_\mu}$ so $\mu \subset \mathcal{T}_{\mathcal{R}_\mu}$. Let $V \in \mathcal{T}_{\mathcal{R}_\mu}$, with $V \neq X$, then by Theorem (2.6), there exists $\mathcal{B} \subset \mu$ such that $V = \bigcup \mathcal{B}$. Since $\mu \in \mathcal{GT}(X)$, A is closed under arbitrary unions. Therefore, $V \in \mu$ and $\mathcal{T}_{\mathcal{R}_\mu} \subset \mu$. Thus (ii) holds. The implications $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$ are immediate from Theorem (2.4) and Theorem (2.6). ■

Corollary 2.8 There is a (not necessarily one to one)correspondence between proper GT's on X and preorder relators $\mathcal{R}$ on X.

3. Complemented Generalized Topologies

Let $\mu, \eta \in \mathcal{GT}(X)$. We denote the intersection (the meet) of μ and η by $\mu \wedge \eta$. We have shown in [1], it is a generalized topology on X contained in both. As we know that the union of two generalized topologies need not be a generalized topology. Denote the join of μ and η by $\mu \vee \eta$. It is also the smallest generalized topology on X containing both of them [1].

Definition 3.1 Let $\mu \in \mathcal{GT}(X)$. A generalized topology η on X is said to be a *complement* of μ if $\mu \vee \eta = \mathcal{P}(X)$ and $\mu \wedge \eta = \{\emptyset\}$. We will denote it by μ^c.

***Example* 3.2** (a) Let Y be proper nonempty subset of X and $\mu = \mathcal{P}(X)$, then $\mu^c = \mathcal{P}(X \setminus Y)$ is a complement of μ.
(b) Let $X = \{1, ..., n\}$ and $\mu = \{\emptyset, X, X - \{i\} \ |i = 1, ..., n\}$ is co-singleton GT on X. Then it has no complement.

Lemma 3.3 If $\mu \in \mathcal{GT}(X)$ and μ^c exists, then for a preorder relator $\mathcal{R}_\mu$ on X we have $\mathcal{R}_\mu^{-1} = \mathcal{R}_{\mu^c}$.

Proof. It evidently follows from Lemma (2.2) and Lemma (2.3). ■

Concerning the concept of the complement in generalized topology theory and accoding to Theorem (2.11) cf.[1], we can imply that $(\mathcal{GT}(X), \wedge, \vee)$ is a lattice with the botton element $\{\emptyset\}$ and the top element $\mathcal{P}(X)$ such that satisfies the commutative, associative and attractive properties but neither distributive nor complemented.

Theorem 3.4 (Theorem 2.6, [1]) Let μ be a GT on X. Then μ^c exists iff for any $A \in \mu$, there exists $x_0 \in A$ such that $\{x_0\} \in \mu$.

Corollary 3.5 In general, for any nonempty subset L of X, $\mathcal{P}(L)$ is the unique complement of generalized topology $(\mathcal{P}(X) \setminus \mathcal{P}(L)) \cup \{\emptyset\}$. Consider μ is a generalized topology on X and the complement μ^c exists. From Theorem (2.9) in [1] we can see that $\bigwedge \mu^c = \mathcal{P}(L)$ is the smallest complement of μ, where $L = X \setminus \{x \in X| \ \{x\} \in \mu\}$ but the set $\bigvee \mu^c$, the join of all complements of μ need not be also a complement of μ.

Here, if $\mu = \mathcal{P}(L)$ then $\bigvee \mu^c = (\mathcal{P}(X) \setminus \mathcal{P}(L)) \cup \{\emptyset\}$ as the maximum complement of μ (i.e. the largest element of the set of all complements of μ under the relation $\subseteq$).

Example **3.6** Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$ is a GT on X. Then $\lambda_1 = \{\emptyset, \{a\}, \{a, b\}\}$ and $\lambda_1 = \{\emptyset, \{a\}, \{a, c\}\}$ are two complements of μ. But $\lambda_1 \vee \lambda_2 = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$ is not a complement of μ.

Theorem 3.7 Let $\mu \in \mathcal{GT}(X)$ be a proper GT and has nonunique complements. Then μ satisfies T_1-axiom iff μ has not the maximum complement in $\mathcal{GT}(X)$.

Proof. ($\Rightarrow$) Assume $\mu \in \mathcal{GT}(X)$ with $\mu \neq \mathcal{P}(X)$ and μ^c exists but not unique. Since μ is T_1, then there exists a point $x \in X$ such that $\{x\}$ is not in μ (if not, being closedness of μ under unions, it is equal to $\mathcal{P}(X)$, a contradiction). So $\{x\}$ has to be in μ^c. Hence μ^c contains all x where $\{x\}$ does not belong to μ. Since μ^c is a GT, according to Theorem (2.7) (ii) we have $\mu^c = \mathcal{T}_{\mathcal{R}_{\mu^c}}$ and using Lemma (3.3), $\mu^c = \mathcal{T}_{\mathcal{R}_\mu^{-1}} \subsetneq \mathcal{P}(X)$. It means that μ^c properly contains $\mathcal{P}(L)$ where $L = X \setminus \{x \in X| \ \{x\} \in \mu\}$ and $\mathcal{P}(L)$ is the smallest complement of μ. But $\mu \vee \mu^c = \mathcal{P}(X)$, it implies that $\mu \subsetneq (\mathcal{P}(X) \setminus \mathcal{P}(L)) \cup \{\emptyset\}$. Thus μ has not the maximum complement.
($\Leftarrow$) Suppose μ has not the largest complement and μ is not T_1. Then there exists a point $x \in X$ such that $\{x\}$ is open and so $\{x\} \in \mu$. Let $M = \{x \in X| \ \{x\} \in \mu\}$. Hence M is nonempty by using Theorem (3.4). If $M = X$ then $\mu = \mathcal{P}(X)$, a contradiction so M is proper subset of X. Set $L = X \setminus M \neq \emptyset$ and $\lambda = \mathcal{P}(L)$, λ is a GT on X. By using Corollary (3.5), λ is the smallest complement of μ. Since $\mu \vee \lambda = \mathcal{P}(X)$, $\mu = (\mathcal{P}(X) \setminus \mathcal{P}(L)) \cup \{\emptyset\}$ that is a contradiction. ■

4. Principal Generalized Topologies Associated to Preorder Relators

In this section, we introduce the notion of $\bigwedge_{\mathcal{R}}$-sets. It is shown that some results of the related subjects in [1, 6] can be considered as special cases of our results.

Definition 4.1 Let $\mathcal{R}$ be a preorder relator on X. Consider $\mathcal{T}_\mathcal{R}$ is a generalized topology associated to $\mathcal{R}$. The subsets $\bigwedge_{\mathcal{R}} A$ are defined as follows:

$$\textstyle\bigwedge_{\mathcal{R}} A = \bigcap_{A \subset \mathcal{U}_\mathcal{R},\ \mathcal{U}_\mathcal{R} \in \mathcal{T}_\mathcal{R}} \mathcal{U}_\mathcal{R}, \text{ and it is equal to } X, \text{ otherwise.}$$

It is obvious that $A \subseteq \bigwedge_{\mathcal{R}} A$.

Theorem 4.2 For any family of subsets of generalized topological space $(X, \mathcal{T}_{\mathcal{R}})$, the following properties hold:

(1) $A = \bigwedge_{\mathcal{R}} A$, if $A \in \mathcal{T}_{\mathcal{R}}$.
(2) $\bigwedge_{\mathcal{R}}(\bigwedge_{\mathcal{R}} A) = \bigwedge_{\mathcal{R}} A$.
(3) If $A \subset B$, then $\bigwedge_{\mathcal{R}} A \subset \bigwedge_{\mathcal{R}} B$.
(4) $\bigwedge_{\mathcal{R}}(\bigcap_{\alpha} A_{\alpha}) \subseteq \bigcap_{\alpha}(\bigwedge_{\mathcal{R}} A_{\alpha})$.
(5) $\bigwedge_{\mathcal{R}}(\bigcup_{\alpha} A_{\alpha}) = \bigcup_{\alpha}(\bigwedge_{\mathcal{R}} A_{\alpha})$.

Proof. (1) From Definition (4.1) we have $A \subseteq \bigwedge_{\mathcal{R}} A$ and since $A \in \mathcal{T}_{\mathcal{R}}$ then $\bigwedge_{\mathcal{R}} A = A$.
(2) It follows from (1) and Definition (4.1).
(3) If $\bigwedge_{\mathcal{R}} B = X$, the assertation is trivial. Let $U \in \mathcal{T}_{\mathcal{R}}$ such that $B \subset U$ and x is not in $\bigwedge_{\mathcal{R}} B$ then there exists a $V \in \mathcal{T}_{\mathcal{R}}$ containing B with x does not belong to V. Since $A \subseteq B$, $A \subseteq V$ and x is not in $\bigwedge_{\mathcal{R}} A$ hence, $\bigwedge_{\mathcal{R}} A \subseteq \bigwedge_{\mathcal{R}} B$.
(4) It is immediate from (3).
(5) By using (3), we can easily see that $\bigcup_{\alpha}(\bigwedge_{\mathcal{R}} A_{\alpha}) \subseteq \bigwedge_{\mathcal{R}}(\bigcup_{\alpha} A_{\alpha})$. If there is not any $U \in \mathcal{T}_{\mathcal{R}}$ such that for some α, $B_{\alpha} \subseteq U$ then the claim holds by Definition (4.1). Suppose there exists a $x \in X$ that x does not belong to $\bigcup_{\alpha}(\bigwedge_{\mathcal{R}} A_{\alpha})$. So for each α, there is a $U_{\alpha} \in \mathcal{T}_{\mathcal{R}}$ such that $A_{\alpha} \subseteq U_{\alpha}$ and x is not in A_{α}. Set $U = \bigcup_{\alpha} U_{\alpha}$ we have $\bigcup_{\alpha} A_{\alpha} \subseteq U$ with x does not belong to U. Then $U \in \mathcal{T}_{\mathcal{R}}$. Hence x is not in $\bigwedge_{\mathcal{R}}(\bigcup_{\alpha} A_{\alpha})$ and the proof is complete. ■

***Example* 4.3** In Example (2.5), consider $A = \{a, b\}$ and $B = \{a, c\}$. Then $\bigwedge_{\mathcal{R}} A$ and $\bigwedge_{\mathcal{R}} B$ are equal to X and $\bigwedge_{\mathcal{R}}(A \cap B) = \{a\}$. Thus $\bigwedge_{\mathcal{R}}(A \cap B) \neq \bigwedge_{\mathcal{R}} A \cap \bigwedge_{\mathcal{R}} B$.

Definition 4.4 A subset A in $\mathcal{T}_{\mathcal{R}}$ is a $\bigwedge_{\mathcal{R}}$-set if $A = \bigwedge_{\mathcal{R}} A$.

Corollary 4.5 The collection of $\bigwedge_{\mathcal{R}}$-sets is a subbase for GT $\mathcal{T}_{\mathcal{R}}$ induced by a preorder relator $\mathcal{R}$ on X.

Theorem 4.6 For a generalized topology $\mathcal{T}_{\mathcal{R}}$ on X we have

(i) $\emptyset$ and X are $\bigwedge_{\mathcal{R}}$-sets.
(ii) The arbitrary union (intersection) of $\bigwedge_{\mathcal{R}}$-sets is a $\bigwedge_{\mathcal{R}}$-set.

Proof. It is straightforward from Definition (4.4) and Theorem (4.2). ■

***Example* 4.7** Let $X = \{a, b, c\}$ and $\mathcal{T}_{\mathcal{R}} = \{\emptyset, \{a, b\}, \{a, c\}, X\}$ be a generalized topology associated to a preorder relator $\mathcal{R}$ on X. It is easy to see that $\{a\}$ is the only singleton as $\bigwedge_{\mathcal{R}}$-set.

Theorem 4.8 $\mathcal{T}_{\mathcal{R}} \in \mathcal{GT}(X)$ satisfies T_1-axiom iff any subset of X is a $\bigwedge_{\mathcal{R}}$-set.

Proof. ($\Rightarrow$) Let A be any subset of X. Since $\mathcal{T}_{\mathcal{R}}$ is T_1 and $A = \bigcup_{x \in A} x$ is union of closed sets, A is closed so A^c is open and belongs to $\mathcal{T}_{\mathcal{R}}$. Hence $\bigwedge_{\mathcal{R}}(A^c) = A^c$. Set $B = A^c$, thus the assertation is proved.
($\Leftarrow$) Let $x \in X$ and consider any subset of X is a $\bigwedge_{\mathcal{R}}$-set then $\{x\}$ is a $\bigwedge_{\mathcal{R}}$-set. Let y be any other point in X then y is not in $\bigwedge_{\mathcal{R}}\{x\} = \{x\}$. It means that there exists an open set U containing y not x. Similarly, the other case can be done. Thus $\mathcal{T}_{\mathcal{R}}$ is T_1. ■

Definition 4.9 Let $\mu \in \mathcal{GT}(X)$ and $\mathcal{R}$ be a preorder relator on X. μ is called a *principal generalized topology* (briefly, PGT) on X if $\mu = \mathcal{P}(X)$ or any arbitrary intersection of $\bigwedge_{\mathcal{R}}$-sets in μ is also a $\bigwedge_{\mathcal{R}}$-set.

From Theorem (4.2)(5) we can easily see that any PGT is a GT and therefore $P\mathcal{GT}(X)$, the set of principal generalized topologies on X, is a meet-complete sublattice of the lattice $(\mathcal{GT}(X), \wedge, \vee)$, with the least element $\{\emptyset, X\}$ and the greatest element $\mathcal{P}(X)$ but is not a complete sublattice since every principal generalized topology on X, different from $\{\emptyset, X\}$, is the join of generalized topologies of the form $\{\emptyset, E, X\}$ for nonempty proper subset $E \subset X$.

Definition 4.10 A subcollection $\mathfrak{B}$ of $\mu \in \mathcal{GT}(X)$ is called a *minimal* basis of $\bigwedge_{\mathcal{R}}$-open sets in μ at each point if for every $x \in X$ there is a $U \in \mathfrak{B}$ containing x such that every $\bigwedge_{\mathcal{R}}$-open set containing x must contain U.

Theorem 4.11 Let μ be a principal generalized topology associated to a preorder relator $\mathcal{R}$ on X. Then it has a minimal basis of $\bigwedge_{\mathcal{R}}$-open sets at each point.

Proof. According to Definition (4.9) and in view of Theorem (4.2)(4), then for each $x \in X$ we can define the minimal open set U at x as the intersection of all $\bigwedge_{\mathcal{R}}$-sets containing x. Hence, from Definition (4.10) there exists a minimal basis $\mathfrak{B}$ of $\bigwedge_{\mathcal{R}}$-open sets in μ at x such that $U \in \mathfrak{B}$. ■

Corollary 4.12 There is a one to one correspondence between $P\mathcal{GT}(X)$ and preorder relators $\mathcal{R}$ on X.

Infact, for nonempty set X, a special preorder relator defines a equivalent relation on the set of generalized topologies.

Theorem 4.13 For each equivalent class of generalized topologies assigning to a preorder relator $\mathcal{R}$ on X. the following assertions hold:

(1) There exists a unique PGT on X.
(2) PGT is the largest element in $\mathcal{GT}(x)$.
(3) PGT induces a preorder relation on X.

Proof. (1) It directly follows from Theorem (4.11), Definition (4.1) of $\bigwedge_{\mathcal{R}}$-sets and Corollary (4.12).

(2) From Corollary (4.5) and the fact that a PGT is closed under intersections of $\bigwedge_{\mathcal{R}}$-sets then for $\mu \in \mathcal{GT}(X)$, $\bigwedge \mu$ containing all of intersections of $\bigwedge_{\mathcal{R}}$-sets, is a PGT on X. Thus it is the largset generalized topology on X.

(3) According to Corollary (4.5), any PGT as a GT on X corresponds to a preorder relator $\mathcal{R}$ which it follows some preorder relation on X. ■

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