Journal of
Linear and Topological Algebra
Vol. 05*, No.* 02*,* 2016*,* 105- 109

On baer type criterion for C-dense, C-closed and quasi injectivity

H. Barzegar[a*], H. Arianpoor[a]

[a] *Department of Mathematics, Tafresh University, Tafresh 3951879611, Iran.*

Received 13 February 2016; Revised 21 May 2016; Accepted 10 September 2016.

Abstract. For the subclasses $\mathcal{M}_1$ and $\mathcal{M}_2$ of monomorphisms in a concrete category $\mathcal{C}$, if $\mathcal{M}_2 \subseteq \mathcal{M}_1$, then $\mathcal{M}_1$-injectivity implies $\mathcal{M}_2$-injectivity. The Baer type criterion is about the converse of this fact. In this paper, we apply injectivity to the classes of *C-dense, C-closed* monomorphisms. The concept of quasi injectivity is also introduced here to investigte the Baer type criterion for these notions.

Keywords: C-Dense injective, C-closed injective, quasi injective.

2010 AMS Subject Classification: 08B30, 18G05.

1. Introduction

The general theory of algebras and categories borrow techniques, ideas, and inspiration from older and more specialized branches of mathematics such as groups, rings, and modules. In this direction, **Injectivity** is one of the most central and important concepts which category theory inherited from homological and commutative algebra. The behaviour of this notion plays a central role in categorical model theory, notably in the characterization theorem for accessible categories with products, as the small-injectivity classes of locally presentable categories [1]. However, the study of injectivity with respect to different classes of monomorphisms is crucial in almost all categories.
Throughout this paper $\mathcal{C}$ will denote a given concrete category (in which the objects are sets endowed with some additional structures and the morphisms are structure-preserving mappings) containing a zero object. The reader is refereed to [2] and [6] for some required

*Corresponding author.
E-mail address: h56bar@tafreshu.ac.ir (H. Barzegar).

Print ISSN: 2252-0201
Online ISSN: 2345-5934

categorical arguments.
An object A is said to be a subobject of B and denoted by $\langle A, f\rangle$ (or $A \leqslant B$) if there exists a monomorphism $f : A \to B$. So B is called an extension of A.

$\mathcal{M}$-morphisms are devoted to a subclass $\mathcal{M}$ of monomorphisms in a category $\mathcal{C}$. An object A of $\mathcal{C}$ is said to be $\mathcal{M}$-*injective* if for any $\mathcal{M}$-morphism $g : B \to C$, the morphism $f : B \to A$ can be lifted to a morphism $\bar{f} : C \to A$ of $\mathcal{C}$, i.e. the following diagram is commutative:

$$\begin{array}{ccc} B & \xrightarrow{g} & C \\ f \downarrow & \swarrow \bar{f} & \\ A & & \end{array}$$

When $\mathcal{M}$ is the class of all monomorphisms, we get the usual notion "injectivity". It is clear that for a subclass $\mathcal{M}$ of monomorphisms, every injective object is an $\mathcal{M}$-injective object.

Definition 1.1 An $\mathcal{M}$-subobject $\langle A, \tau\rangle$ of an object B is an $\mathcal{M}$-*retract* of B if there exists a morphism $f : B \to A$ such that $f\tau = id_A$. We say that f is a retraction of τ.

One line of study in this context is to investigate the relation between $\mathcal{M}_1$-injectivity and injectivity associated to another subclass $\mathcal{M}_2$ of monomorphisms, the result of which may be called the **Baer type criterion**. Note that if $\mathcal{M}_2 \subseteq \mathcal{M}_1$, then clearly $\mathcal{M}_1$-injectivity implies $\mathcal{M}_2$-injectivity. The Baer type problem is considered as the converse of this fact using injectivity with respect to the classes of, the so-called, *C-dense, C-closed* monomorphisms. Also we will define the notion of quasi-injectivity and work on the Baer type criterion for these concepts.

2. *C*-dense *C*-closed injective objects

In this section, we present a kind of Baer type criterion for injectivity by using the notion of closure operator. First recall the definition of a categorical closure operator from [7]. We denote by $Sub(B)$, the lattice of all subobjects of an object B.

Definition 2.1 A family $C = (C_B)_{B\in\mathcal{C}}$, with $C_B : Sub(B) \to Sub(B)$, taking any subobject $A \leqslant B$ to a subobject $C_B(A)$ is called a *closure operator* on $\mathcal{C}$ if it satisfies the following:
(c_1) (*Extension*) $A \leqslant C_B(A)$,
(c_2) (*Monotonicity*) $A_1 \leqslant A_2 \leqslant B$ implies $C_B(A_1) \leqslant C_B(A_2)$,
(c_3) (*Continuity*) $f(C_B(A)) \leqslant C_C(f(A))$, for all morphisms $f : B \to C$.

A closure operator C is said to be *weakly hereditary* if $C_{C_B(A)}(A) = C_B(A)$, for any subobject $A \leqslant B$.

Now, one has the usual two classes of monomorphisms related to every closure operator $C = (C_B)_{B\in\mathcal{C}}$ as follows:

Definition 2.2 Let A be a subobject of B. We say that A is *C-closed* in B if $C_B(A) = A$, and it is *C-dense* in B if $C_B(A) = B$.

In the sequel we obtain two main results concerning the relation between injectivity and C-dense and C-closed injectivity.

Theorem 2.3 Let a closure operator C be weakly hereditary and for every object B and every subobject A of B, there exists $n \in \mathbb{N}$ such that $C_B^{n+1}(A) = C_B^n(A)$. An object E is injective if and only if it is C-closed injective as well as C-dense injective.

Proof. Suppose E is a C-closed and C-dense injective object. Let $\tau : A \to B$ be a subobject of B and $f : A \to E$ a morphism. Using hypothesis, there exists $n \in \mathbb{N}$ such that $C_B^{n+1}(A) = C_B^n(A)$. Since C is weakly hereditary, $C_{C_B(A)}(A) = C_B(A)$. Then A is a C-dense subobject of $C_B(A)$. Similarly, for every $1 \leqslant i \leqslant n-1$, $C_B^i(A)$ is a C-dense subobject of $C_B^{i+1}(A)$. Thus for each $1 \leqslant i \leqslant n$, there is a morphism $f_i : C_B^i(A) \to E$ which extends f. The object $C_B^n(A)$ is C-closed in B because of $C_B^{n+1}(A) = C_B^n(A)$. Therefore, there exists $\bar{f} : B \to E$ such that $\bar{f}\mid_{C_B^n(A)} = f_n$ and hence $\bar{f}\tau = f$. The converse is clear. ■

Theorem 2.4 Let $\mathcal{C}$ be a category with coproducts and intersections, and a closure operator C be weakly hereditary. Then an object A is injective if and only if it is C-dense injective as well as C-closed injective.

Proof. Assume that A is a C-closed and C-dense injective object, and $\tau : B \to D$, denoted by $\langle B, \tau \rangle$, is a subobject of an object D and $f : B \to A$ a morphism. Consider Σ to be the set of all $(E, f_{_E})$ such that $\langle B, i_{_E} \rangle$ is a subobject of E and $\langle E, j_{_E} \rangle$ is a subobject of D such that for a morphism $f_{_E} : E \to A$, $f_{_E} i_{_E} = f$. The set Σ is a nonempty partially ordered set by the order $(E_1, f_{_{E_1}}) \leqslant (E_2, f_{_{E_2}})$ if and only if $\langle E_1, i \rangle$ is a subobject of E_2 such that $f_{_{E_2}} i = f_{_{E_1}}$. Let $\{(E_i, f_{_{E_i}})\}_{i \in I}$ be a chain in Σ. By the universal property of coproducts, there is a morphism $f_{_E} : E = \coprod E_i \to A$ such that $f_{_E} p_{_{E_i}} = f_{_{E_i}}$, in which every $p_{_{E_i}} : E_i \to E$ is an injection morphism of the coproduct. Consider an object N to be an intersection of D and E. For each E_i of the chain we have the following diagram:

$$\begin{array}{ccc} E_i & \xrightarrow{\gamma_{_{E_i}}} & N \\ p_{_{E_i}} \downarrow & \swarrow \gamma & \\ E & & \end{array}$$

Since $\mathcal{C}$ has a zero object, $p_{_{E_i}}$'s are monomorphisms and hence so are $\gamma_{_{E_i}}$'s. The object N is a subobject of E and D, and for each $i \in I$, $f_{_E} \gamma \gamma_{_{E_i}} = f_{_E} p_{_{E_i}} = f_{_{E_i}}$. Moreover, $f_{_E} \gamma \gamma_{_{E_i}} i_{_{E_i}} = f_{_{E_i}} i_{_{E_i}} = f$. Thus $N \in \Sigma$, which is an upper bound for the chain. Using Zorn's lemma, take $(M, \bar{f})$ to be a maximal element of Σ. Let $M \neq D$. If $M \neq C_D(M)$, since the closure operator C is weakly hereditary, M is C-dense in $C_D(M)$. So $\bar{f}$ can be extended to $C_D(M)$, which contradicts the maximality of $\bar{f}$. Thus $M = C_D(M)$, which means M is a C-closed subobject of D. Consequently, $(M, \bar{f})$ can be extended to (D, h), which is again a contradiction by the maximality of $\bar{f}$. This implies that $M = D$ and the proof is complete. ■

3. Quasi injective objects

In this section we consider *quasi injectivity* to study some Baer type criteria for injectivity. Every injective object is quasi injective, but the converse is not generally true. Here we give an equivalent condition over which all quasi injective objects are injective. To this end, first note the following two lemmas.

Lemma 3.1 Let 0 be a zero object and A be an object in $\mathcal{C}$. For every extension E of A, $A \times 0$ is a retract of $A \times E$.

Proof. Consider the following two product diagrams,

$$
\begin{array}{ccccc}
 & & A\times E & & \\
 & \swarrow_{\pi_A} & \uparrow & \searrow^{\pi_E} & \\
A & & | & & E \\
 & \nwarrow^{\pi^A} & | & \nearrow_{\pi^E} & \\
 & & A\times 0 & &
\end{array}
\quad \textit{and} \quad
\begin{array}{ccccc}
 & & A\times 0 & & \\
 & \swarrow_{\pi^A} & \uparrow & \searrow^{z} & \\
A & & | & & 0 \\
 & \nwarrow^{\pi_A} & | & \nearrow_{z} & \\
 & & A\times E & &
\end{array}
$$

where π_A, π^A and π_E are injection morphisms, zs are zero morphisms and π^E is a composition of two zero morphisms $A \times 0 \to 0 \to E$. By the universal property of products there are unique morphisms $\tau_A : A\times 0 \to A\times E$ and $\rho_A : A\times E \to A\times 0$ which commute diagrams. It is enough to show that $\rho_A \tau_A = id_{A\times 0}$.

We have $\pi^A \rho_A \tau_A = \pi_A \tau_A = \pi^A = \pi^A id_{A\times 0}$ and $z\rho_A \tau_A = z = zid_{A\times 0}$. Now by [[6]. Th.3.5], $\rho_A \tau_A = id_{A\times 0}$.

■

Lemma 3.2 Let 0 be a zero object in $\mathcal{C}$. For every object A of $\mathcal{C}$, A and $0 \times A$ are isomorphic.

Definition 3.3 An object A in a category $\mathcal{C}$ is said to be *quasi injective* if it is injective with respect to all subobjects of A.

A concrete category $\mathcal{C}$ has *enough injective* provided that, each of its objects has an injective extension and it is called completely injective(completely quasi injective), if all objects in $\mathcal{C}$ are injective(quasi injective).

Theorem 3.4 Let a category $\mathcal{C}$ have enough injective. The product of any two quasi injective objects is quasi injective if and only if each quasi injective object is injective.

Proof. ($\Rightarrow$) Assume that A is a quasi injective object in $\mathcal{C}$, E is an injective extension of A, and 0 is a zero object in $\mathcal{C}$. By Lemma 3.2, A and $0\times A$ are isomorphic (so are E and $0\times E$). Thus $0\times E$ is injective and it is enough to show that $0\times A$ is a retract of $0\times E$. In the sense of the universal property of products, we obtain morphisms $0\times A \overset{\tau_1}{\to} 0\times E \overset{\tau_2}{\to} A\times E$ which are monomorphisms. By Lemma 3.1, $0 \times A$ is a retract of $E \times A$ and since $E \times A$ is isomorphic to $A \times E$, there is a morphism $\tau_A : 0 \times A \to A \times E$ which has a retraction p_A (i.e. $p_A\tau_A = id_{0\times A}$). Since every injective object is quasi injective, $A \times E$ is quasi injective by the assumption. It follows that there exists a morphism $g : A \times E \to A \times E$ such that $g\tau_2\tau_1 = \tau_A$. Thus $p_A g\tau_2\tau_1 = p_A\tau_A = id_{0\times A}$ so that $0 \times A$ is a retract of the injective object $0 \times E$, showing that A is injective.
($\Leftarrow$) It follows from the fact that the product of injective objects is injective and each injective object is quasi injective. ■

Corollary 3.5 Let a category $\mathcal{C}$ have enough injectives. Then $\mathcal{C}$ is completely quasi injective if and only if it is completely injective.

Remark 1 *In the category $\mathcal{A}ct_S$ of right S-acts over a semigroup S, the above result is as a generalization of Theorem 3 in [3] because of being so enough injectives (see [5] for more details).*

The Corollary 3.5 is also true for the category $\mathcal{M}od_R$ of right modules over a ring R with identity.

References

[1] J. Adamek, and J. Rosicky, Locally Presentable and Accessible Categories. Cambridge University Press, Cambridge, 1994.
[2] J. Adamek, H. Herrlich and G.E. Strecker, Abstract and Concrete Categories. John Wiley and Sons, Inc., 1990.
[3] J. Ahsan, Monoids characterized by their quasi-injective S-systems. Semigroup Forum 36 (3) (1987), 285-292.
[4] H. Barzegar, M.M. Ebrahimi and M. Mahmoudi, Essentiality and Injectivity. Appl. Categor. Struct. 18 (1) (2010), 73-83.
[5] P. Berthiaume, The injective envelope of S-Sets. Canad. Math. Bull. 10 (2) (1967), 261-273.
[6] T.S. Blyth, Categories. Essex CM202JE, England, 1986.
[7] D. Dikranjan and W. Tholen, Categorical structure of closure operators, with applications to topology, algebra, and discrete mathematics. Mathematics and Its Applications, Kluwer Academic Publ., 1995.
[8] W. Tholen, Injective objects and cogenerating sets. J. Algebra, 73 (1) (1981), 139-155.