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On characterizations of weakly e-irresolute functions

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Abstract. The aim of this paper is to introduce and obtain some characterizations of weakly e-irresolute functions by means of e-open sets defined by Ekici [6]. Also, we look into further properties relationships between weak e-irresoluteness and separation axioms and completely e-closed graphs.

Keywords: e-irresolute, weakly e-irresolute, strongly e-regular space, completely e-closed graph.

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1. Introduction

In 1972, Crossley et al. [4] introduced the concept of irresolute functions in topological spaces. The class of α-irresolute functions were introduced by Maheshwari and Thakur [9]. Recently, the class of semi α-irresolute functions and almost α-irresolute functions and weakly B-irresolute functions were introduced in [3], [2] and [14], respectively. In this paper, we introduce and investigate the concept of weakly e-irresolute functions and study several characterizations and some fundamental properties of these classes of functions. Relations between this class and some other existing classes of functions ([5, 6, 10, 12, 13]) are also obtained.

Throughout this paper (X, τ) and (Y, σ) (or simply X and Y) represent nonempty topological spaces on which no separation axioms are assumed unless otherwise stated.

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Let X be a topological space and A be a subset of X. The closure of A and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively. $\mathcal{U}(x)$ denotes all open neighborhoods of the point $x \in X$. A subset A of a space X is called regular open [15] (resp. regular closed [15]) if $A = int(cl(A))$ (resp. $A = cl(int(A))$). The δ-interior [16] of a subset A of X is the union of all regular open sets of X contained in A and is denoted by $int_\delta(A)$. The subset A is called δ-open [16] if $A = int_\delta(A)$, i.e., a set is δ-open if it is the union of regular open sets. The complement of a δ-open set is called δ-closed [16].

The family of all δ-open (resp. δ-closed) sets in X is denoted by $\delta O(X)$ (resp. $\delta C(X)$). A subset A of a space X is called e-open [6] (resp. β-open [1]) if $A \subseteq int(cl_\delta(A)) \cup cl(int_\delta(A))$ (resp. $A \subseteq cl(int(cl(A)))$). The complement of an e-open (resp. β-open) set is said to be e-closed [6] (resp. β-closed [1]). The e-interior [6] of a subset A of X is the union of all e-open sets of X contained in A and is denoted by $e\text{-}int(A)$. The e-closure [6] of a subset A of X is the intersection of all e-closed sets of X containing A and is denoted by $e\text{-}cl(A)$. The family of all e-open (resp. e-closed, both e-open and e-closed) sets of X is denoted by $eO(X)$ (resp. $eC(X), eR(X)$). The family of all e-open (resp. e-closed, both e-open and e-closed) sets of X containing a point $x \in X$ is denoted by $eO(X,x)$ (resp. $eC(X,x), eR(X,x)$).

We shall use the well-known accepted language almost in the whole of the proofs of theorems in article.

2. Preliminaries

Definition 2.1 [11] A point x of X is called an e-θ-cluster points of $A \subseteq X$ if $e\text{-}cl(U) \cap A \neq \emptyset$ for every $U \in eO(X,x)$. The set of all e-θ-cluster points of A is called the e-θ-closure of A and is denoted by $e\text{-}cl_\theta(A)$. A subset A is said to be e-θ-closed if and only if $A = e\text{-}cl_\theta(A)$. The complement of an e-θ-closed set is said to be e-θ-open. The family of all e-θ-open (resp. e-θ-closed) sets in X is denoted by $e\theta O(X)$ (resp. $e\theta C(X)$).

Theorem 2.2 [6] Let X be a topological space and $A \subseteq X$. Then the followings hold:
(a) If $A \in eC(X)$, then $A = e\text{-}cl(A)$,
(b) If $A \subseteq B$, then $e\text{-}cl(A) \subseteq e\text{-}cl(B)$,
(c) $e\text{-}cl(A) \in eC(X)$,
(d) $x \in e\text{-}cl(A)$ if and only if $U \cap A \neq \emptyset$ for each $U \in eO(X,x)$,
(e) $e\text{-}cl(X \setminus A) = X \setminus e\text{-}int(A)$.

Theorem 2.3 [11] Let X be a topological space and $A \subseteq X$. Then the followings hold:
(a) $A \in eO(X)$ if and only if $e\text{-}cl(A) \in eR(X)$,
(b) $A \in eC(X)$ if and only if $e\text{-}int(A) \in eR(X)$,
(c) If $A \in eO(X)$, then $e\text{-}cl(A) = e\text{-}cl_\theta(A)$,
(d) $A \in eR(X)$ if and only if $e\theta O(X) \cap e\theta C(X)$,
(e) $x \in e\text{-}cl_\theta(A)$ if and only if $e\text{-}cl(U) \cap A \neq \emptyset$ for each $U \in eO(X,x)$,
(f) $e\text{-}int_\theta(X \setminus A) = X \setminus e\text{-}cl_\theta(A)$.

Definition 2.4 A function $f : X \to Y$ is called:
(a) weakly continuous [8] (briefly $w.c.$) if for each $x \in X$ and for each open set V of Y containing $f(x)$, there exists an open set U of X containing x such that $f[U] \subseteq cl(V)$,
(b) weakly e-continuous [12] if for each $x \in X$ and for each open set V of Y containing $f(x)$, there exists an e-open set U of X containing x such that $f[U] \subseteq cl(V)$,
(c) weakly β-continuous [13] if for each $x \in X$ and for each open set V of Y containing

$f(x)$, there exists a β-open set U of X containing x such that $f[U] \subseteq cl(V)$,

(d) e-continuous [6] if $f^{-1}[V] \in eO(X)$ for every open set V of Y,

(e) e-irresolute [7] if $f^{-1}[V] \in eO(X)$ for every e-open set V of Y,

(f) β-irresolute [10] if $f^{-1}[V] \in \beta O(X)$ for every β-open set V of Y,

(g) weakly B-irresolute [14] if for each $x \in X$ and for each b-open V of Y containing $f(x)$, there exists a b-open set U of X containing x such that $f[U] \subseteq bcl(V)$.

3. Weakly e-irresolute Functions

In this section we define the notion of weakly e-irresolute functions. Then we obtain several characterizations of them.

Definition 3.1 Let X and Y be topological spaces. A function $f : X \to Y$ is said to be weakly e-irresolute if for each x in X and for each e-open set V of Y containing $f(x)$, there exists $U \in eO(X, x)$ such that $f[U] \subseteq e\text{-}cl(V)$.

Remark 1 *We have the following diagram from Definition 2.4 and Definition 3.1. The converses of these implications are not true in general as shown by the following examples.*

$$\begin{array}{ccc} \textit{continuity} & \rightarrow & \textit{weak continuity} \\ \downarrow & & \downarrow \\ e\textit{-continuity} & \rightarrow & \textit{weak } e\textit{-continuity} \\ \uparrow & & \uparrow \\ e\textit{-irresoluteness} & \rightarrow & \textit{weak } e\textit{-irresoluteness} \end{array}$$

Example 3.2 Let $X := \{a, b, c\}$, $\tau := \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$ and $\sigma := \{\emptyset, X, \{c\}\}$. Define a function $f : (X, \tau) \to (X, \sigma)$ such that $f(x) = x$. Then f is weakly e-continuous but not weakly e-irresolute.

Example 3.3 Let $X := \{a, b, c, d, e\}$, $\tau := \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$. Define a function $f : (X, \tau) \to (X, \tau)$ such that $f = \{(a, a), (b, d), (c, d), (d, d), (e, e)\}$. Then f is weakly e-irresolute but not e-irresolute.

Remark 2 *A weakly e-irresolute function need not be a weakly B-irresolute function as shown by the following example.*

Example 3.4 Let $X := \{a, b, c\}$, $\tau := \{\emptyset, X, \{a, b\}\}$. Then $eR(X) = \mathcal{P}(X)$ and $BR(X) = \{\emptyset, \{a\}, \{b\}, \{a, c\}, \{b, c\}, X\}$. Define a function $f : (X, \tau) \to (X, \tau)$ such that $f = \{(a, b), (b, c), (c, a)\}$. Then f is weakly e-irresolute but not weakly B-irresolute.

QUESTION. Is there any weakly B-irresolute function which is not weakly e-irresolute?

Theorem 3.5 Let $f : X \to Y$ be a function. Then the following properties are equivalent:
(a) f is weakly e-irresolute;
(b) $f^{-1}[V] \subseteq e\text{-}int\left(f^{-1}[e\text{-}cl(V)]\right)$ for every $V \in eO(Y)$;
(c) $e\text{-}cl\left(f^{-1}[V]\right) \subseteq f^{-1}[e\text{-}cl(V)]$ for every $V \in eO(Y)$.

Proof. $(a) \Longrightarrow (b)$: Let $V \in eO(Y)$ and $x \in f^{-1}[V]$.

$$\left.\begin{array}{r} (V \in eO(Y))(x \in f^{-1}[V]) \Rightarrow V \in eO(Y, f(x)) \\ (a) \end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists U \in eO(X, x))(f[U] \subseteq e\text{-}cl(V))$$

$\Rightarrow (\exists U \in eO(X,x))\left(U \subseteq f^{-1}[e\text{-}cl(V)]\right)$
$\Rightarrow (\exists U \in eO(X,x))\left(x \in U = e\text{-}int(U) \subseteq e\text{-}int\left(f^{-1}[e\text{-}cl(V)]\right)\right)$
$\Rightarrow x \in e\text{-}int\left(f^{-1}[e\text{-}cl(V)]\right).$
$(b) \Longrightarrow (c)$: Let $V \in eO(Y)$ and $x \notin f^{-1}[e\text{-}cl(V)]$.

$$\begin{aligned} x \notin f^{-1}[e\text{-}cl(V)] &\Rightarrow f(x) \notin e\text{-}cl(V) \\ &\Rightarrow (\exists F \in eO(Y,f(x)))(F \cap V = \emptyset) \\ &\Rightarrow (\exists F \in eO(Y,f(x)))(F \subseteq Y \setminus V) \\ &\Rightarrow (\exists F \in eO(Y,f(x)))(e\text{-}cl(F) \subseteq e\text{-}cl(Y \setminus V) = Y \setminus V) \\ &\Rightarrow (\exists F \in eO(Y,f(x)))(e\text{-}cl(F) \cap V = \emptyset) \\ &\Rightarrow (\exists F \in eO(Y,f(x)))\left(f^{-1}[e\text{-}cl(F) \cap V] = \emptyset\right) \\ &\Rightarrow (\exists F \in eO(Y,f(x)))\left(f^{-1}[e\text{-}cl(F)] \cap f^{-1}[V] = \emptyset\right) \\ &\Rightarrow (\exists F \in eO(Y,f(x)))\left(e\text{-}int\left(f^{-1}[e\text{-}cl(F)]\right) \cap f^{-1}[V] = \emptyset\right) \\ &\overset{(b)}{\Rightarrow} \left(e\text{-}int\left(f^{-1}[e\text{-}cl(F)]\right) \in eO(X,x)\right)\left(e\text{-}int\left(f^{-1}[e\text{-}cl(F)]\right) \cap f^{-1}[V] = \emptyset\right) \\ &\Rightarrow x \notin e\text{-}cl\left(f^{-1}[V]\right). \end{aligned}$$

$(c) \Longrightarrow (a)$: Let $x \in X$ and $V \in eO(Y,f(x))$.

$$\left.\begin{array}{r} V \in eO(Y,f(x)) \Rightarrow e\text{-}cl(V) \in eR(Y,f(x)) \Rightarrow x \notin f^{-1}[e\text{-}cl(Y \setminus e\text{-}cl(V))] \\ (c) \end{array}\right\} \Rightarrow$$

$\Rightarrow x \notin e\text{-}cl\left(f^{-1}[Y \setminus e\text{-}cl(V)]\right)$
$\Rightarrow (\exists U \in eO(X,x))\left(U \cap f^{-1}[Y \setminus e\text{-}cl(V)] = \emptyset\right)$
$\Rightarrow (\exists U \in eO(X,x))(f[U] \cap (Y \setminus e\text{-}cl(V)) = \emptyset)$
$\Rightarrow (\exists U \in eO(X,x))(f[U] \subseteq e\text{-}cl(V)).$ ■

Theorem 3.6 Let $f : X \to Y$ be a function. Then the following properties are equivalent:
(a) f is weakly e-irresolute;
(b) $e\text{-}cl\left(f^{-1}[B]\right) \subseteq f^{-1}[e\text{-}cl_\theta(B)]$ for every subset B of Y;
(c) $f[e\text{-}cl(A)] \subseteq e\text{-}cl_\theta(f[A])$ for every subset A of X;
(d) $f^{-1}[F] \in eC(X)$ for every e-θ-closed set F of Y;
(e) $f^{-1}[V] \in eO(X)$ for every e-θ-open set V of Y.

Proof. $(a) \Longrightarrow (b)$: Let $B \subseteq Y$ and $x \notin f^{-1}[e\text{-}cl_\theta(B)]$.
$x \notin f^{-1}[e\text{-}cl_\theta(B)] \Rightarrow f(x) \notin e\text{-}cl_\theta(B) \Rightarrow (\exists V \in eO(Y,f(x)))(e\text{-}cl(V) \cap B = \emptyset) \ldots (1)$

$$\left.\begin{array}{r} V \in eO(Y,f(x)) \\ (a) \end{array}\right\} \Rightarrow (\exists U \in eO(X,x))(f[U] \subseteq e\text{-}cl(V)) \ldots (2)$$

$$\begin{aligned} (1),(2) &\Rightarrow (\exists U \in eO(X,x))(f[U] \cap B = \emptyset) \\ &\Rightarrow (\exists U \in eO(X,x))\left(U \cap f^{-1}[B] = \emptyset\right) \\ &\Rightarrow x \notin e\text{-}cl\left(f^{-1}[B]\right). \end{aligned}$$

$(b) \Longrightarrow (c)$: Let $A \subseteq X$.

$$\left.\begin{array}{r} A \subseteq X \Rightarrow f[A] \subseteq Y \\ (b) \end{array}\right\} \Rightarrow e\text{-}cl(A) \subseteq e\text{-}cl\left(f^{-1}[f[A]]\right) \subseteq f^{-1}[e\text{-}cl_\theta(f[A])]$$

$\Rightarrow f[e\text{-}cl(A)] \subseteq e\text{-}cl_\theta(f[A])$.
$(c) \Longrightarrow (d)$: Let $F \in e\theta C(Y)$.

$$\left.\begin{array}{r} F \in e\theta C(Y) \Rightarrow f^{-1}[F] \subseteq X \\ (c) \end{array}\right\} \Rightarrow f\left[e\text{-}cl\left(f^{-1}[F]\right)\right] \subseteq e\text{-}cl_\theta\left(f\left[f^{-1}[F]\right]\right) \subseteq e\text{-}cl_\theta(F) = F$$

$\Rightarrow e\text{-}cl\left(f^{-1}[F]\right) \subseteq f^{-1}[F]$
$\Rightarrow f^{-1}[F] \in eC(X)$.
$(d) \Longrightarrow (e)$: Clear.
$(e) \Longrightarrow (a)$: Let $x \in X$ and $V \in eO(Y,f(x))$.

$$\left.\begin{array}{r} V \in eO\left(Y, f(x)\right) \Rightarrow e\text{-}cl\left(V\right) \in e\theta O\left(Y\right) \\ (e) \end{array}\right\} \Rightarrow$$
$$\Rightarrow \left(U := f^{-1}\left[e\text{-}cl\left(V\right)\right] \in eO\left(X, x\right)\right) \left(f\left[U\right] = f\left[f^{-1}\left[e\text{-}cl\left(V\right)\right]\right] \subseteq e\text{-}cl\left(V\right)\right).$$ ■

Theorem 3.7 Let $f : X \to Y$ be a function. Then the following properties are equivalent:
(a) f is weakly e-irresolute;
(b) For each $x \in X$ and each $V \in eO(Y, f(x))$, there exists $U \in eO\left(X, x\right)$ such that $f\left[e\text{-}cl\left(U\right)\right] \subseteq e\text{-}cl\left(V\right)$;
(c) $f^{-1}\left[F\right] \in eR\left(X\right)$ for every $F \in eR\left(Y\right)$.

Proof. $(a) \Longrightarrow (b)$: Let $x \in X$ and $V \in eO\left(Y, f(x)\right)$.

$$\left.\begin{array}{r} \left.\begin{array}{r} V \in eO\left(Y, f(x)\right) \\ \text{Theorem 2.3} \end{array}\right\} \Rightarrow \quad e\text{-}cl\left(V\right) \in e\theta O\left(Y\right) \cap e\theta C\left(Y\right) \\ \text{Theorem 3.6}(d)(e) \end{array}\right\} \Rightarrow$$
$$\Rightarrow \left(U := f^{-1}\left[e\text{-}cl\left(V\right)\right] \in eO(X) \cap eC(X)\right) \left(f\left[e\text{-}cl\left(U\right)\right] \subseteq e\text{-}cl\left(V\right)\right).$$
$(b) \Longrightarrow (c)$: Let $F \in eR\left(Y\right)$ and $x \in f^{-1}\left[F\right]$.
$$\left.\begin{array}{r} (x \in f^{-1}\left[F\right])(F \in eR\left(Y\right)) \Rightarrow F \in eR(Y, f(x)) \\ (b) \end{array}\right\} \Rightarrow$$
$$\Rightarrow \left(\exists U \in eO\left(X, x\right)\right) \left(f\left[e\text{-}cl\left(U\right)\right] \subseteq e\text{-}cl\left[F\right] = F\right)$$
$$\Rightarrow \left(\exists U \in eO\left(X, x\right)\right) \left(U \subseteq e\text{-}cl\left(U\right) \subseteq f^{-1}\left[F\right]\right)$$
$$\Rightarrow x \in e\text{-}int\left(f^{-1}\left[F\right]\right)$$
Then $f^{-1}[F] \in eO(X) \ldots (1)$
$$\left.\begin{array}{r} (x \in f^{-1}\left[Y \setminus F\right])(F \in eR\left(Y\right)) \Rightarrow Y \setminus F \in eR(Y, f(x)) \\ (b) \end{array}\right\} \Rightarrow$$
$$\Rightarrow \left(\exists U \in eO(X, x)\right) \left(f\left[e\text{-}cl\left(U\right)\right] \subseteq e\text{-}cl\left[Y \setminus F\right] = Y \setminus F\right)$$
$$\Rightarrow \left(\exists U \in eO(X, x)\right) \left(U \subseteq e\text{-}cl\left(U\right) \subseteq f^{-1}\left[Y \setminus F\right]\right)$$
$$\Rightarrow x \in e\text{-}int\left(f^{-1}\left[Y \setminus F\right]\right) \in eO\left(X\right)$$
Then $f^{-1}\left[Y \setminus F\right] \in eO(X)$ and so $f^{-1}\left[F\right] \in eC(X) \ldots (2)$
$(1), (2) \Rightarrow f^{-1}\left[F\right] \in eR\left(X\right)$.
$(c) \Longrightarrow (a)$: Let $x \in X$ and $V \in eO\left(Y, f\left(x\right)\right)$.
$$\left.\begin{array}{r} V \in eO\left(Y, f\left(x\right)\right) \Rightarrow e\text{-}cl\left(V\right) \in eR\left(Y, f\left(x\right)\right) \\ (c) \end{array}\right\} \Rightarrow$$
$$\Rightarrow \left(U := f^{-1}\left[e\text{-}cl\left(V\right)\right] \in eR\left(X, x\right)\right) \left(f\left[U\right] \subseteq e\text{-}cl\left(V\right)\right).$$ ■

Theorem 3.8 Let $f : X \to Y$ be a function. Then the following properties are equivalent:
(a) f is weakly e-irresolute;
(b) $f^{-1}\left[V\right] \subseteq e\text{-}int_\theta\left(f^{-1}\left[e\text{-}cl_\theta\left(V\right)\right]\right)$ for every $V \in eO\left(Y\right)$;
(c) $e\text{-}cl_\theta\left(f^{-1}\left[V\right]\right) \subseteq f^{-1}\left[e\text{-}cl_\theta\left(V\right)\right]$ for every $V \in eO\left(Y\right)$.

Proof. $(a) \Longrightarrow (b)$: Let $V \in eO\left(Y\right)$.
$$\left.\begin{array}{r} V \in eO\left(Y\right) \Rightarrow e\text{-}cl_\theta\left(V\right) \in eR\left(Y\right) \\ (a) \end{array}\right\} \Rightarrow f^{-1}\left[e\text{-}cl_\theta\left(V\right)\right] \in eR\left(X\right)$$
$$\Rightarrow f^{-1}\left[e\text{-}cl_\theta\left(V\right)\right] \in e\theta O\left(X\right) \Rightarrow e\text{-}int_\theta\left(f^{-1}\left[e\text{-}cl_\theta\left(V\right)\right]\right) = f^{-1}\left[e\text{-}cl_\theta\left(V\right)\right] \supseteq f^{-1}[V].$$
$(b) \Longrightarrow (c)$: Let $V \in eO\left(Y\right)$.
$$\left.\begin{array}{r} V \in eO\left(Y\right) \Rightarrow Y \setminus V \in eC\left(Y\right) \Rightarrow e\text{-}int_\theta\left(Y \setminus V\right) \in eR\left(Y\right) \\ (b) \end{array}\right\} \Rightarrow$$
$$f^{-1}\left[e\text{-}int_\theta\left(Y \setminus V\right)\right] \subseteq e\text{-}int_\theta\left(f^{-1}\left[e\text{-}cl_\theta\left(e\text{-}int_\theta\left(Y \setminus V\right)\right)\right]\right) = e\text{-}int_\theta\left(f^{-1}\left[e\text{-}int_\theta\left(Y \setminus V\right)\right]\right)$$
$$\Rightarrow X \setminus e\text{-}int_\theta\left(f^{-1}\left[e\text{-}int_\theta\left(Y \setminus V\right)\right]\right) \subseteq X \setminus f^{-1}\left[e\text{-}int_\theta\left(Y \setminus V\right)\right]$$
$$\Rightarrow e\text{-}cl_\theta\left(X \setminus f^{-1}\left[e\text{-}int_\theta\left(Y \setminus V\right)\right]\right) \subseteq f^{-1}\left[Y \setminus e\text{-}int_\theta\left(Y \setminus V\right)\right]$$

$\Rightarrow e\text{-}cl_\theta\left(f^{-1}\left[Y\setminus e\text{-}int_\theta\left(Y\setminus V\right)\right]\right)\subseteq f^{-1}\left[e\text{-}cl_\theta\left(V\right)\right]$
$\Rightarrow e\text{-}cl_\theta\left(f^{-1}\left[e\text{-}cl_\theta\left(V\right)\right]\right)\subseteq f^{-1}\left[e\text{-}cl_\theta\left(V\right)\right]$
$\Rightarrow e\text{-}cl_\theta\left(f^{-1}\left[V\right]\right)\subseteq e\text{-}cl_\theta\left(f^{-1}\left[e\text{-}cl_\theta\left(V\right)\right]\right)\subseteq f^{-1}\left[e\text{-}cl_\theta\left(V\right)\right]$.
$(c)\Longrightarrow(a)$: Let $V\in eR(Y)$.
$$\left.\begin{array}{r}V\in eR\left(Y\right)\Rightarrow V\in eO\left(Y\right)\\ (c)\end{array}\right\}\Rightarrow e\text{-}cl_\theta\left(f^{-1}\left[V\right]\right)\subseteq f^{-1}\left[e\text{-}cl_\theta\left(V\right)\right]=f^{-1}\left[V\right]$$
$\Rightarrow f^{-1}\left[V\right]=e\text{-}cl_\theta\left(f^{-1}\left[V\right]\right)$
$\Rightarrow f^{-1}\left[V\right]\in e\theta C\left(X\right)\ldots(1)$
$$\left.\begin{array}{r}V\in eR\left(Y\right)\Rightarrow Y\setminus V\in eR\left(Y\right)\Rightarrow Y\setminus V\in eO\left(Y\right)\\ (c)\end{array}\right\}\Rightarrow$$
$\Rightarrow e\text{-}cl_\theta\left(f^{-1}\left[Y\setminus V\right]\right)\subseteq f^{-1}\left[e\text{-}cl_\theta\left(Y\setminus V\right)\right]=f^{-1}\left[Y\setminus V\right]$
$\Rightarrow X\setminus f^{-1}\left[Y\setminus V\right]\subseteq X\setminus e\text{-}cl_\theta\left(f^{-1}\left[Y\setminus V\right]\right)$
$\Rightarrow f^{-1}\left[V\right]\subseteq e\text{-}int_\theta\left(f^{-1}\left[V\right]\right)$
$\Rightarrow f^{-1}\left[V\right]=e\text{-}int_\theta\left(f^{-1}\left[V\right]\right)$
$\Rightarrow f^{-1}\left[V\right]\in e\theta O\left(X\right)\ldots(2)$
$(1),(2)\Rightarrow f^{-1}\left[V\right]\in eR\left(X\right)$. ■

Theorem 3.9 Let $f:X\to Y$ be a function. Then the following properties are equivalent:
(a) f is weakly e-irresolute;
(b) $e\text{-}cl_\theta\left(f^{-1}\left[B\right]\right)\subseteq f^{-1}\left[e\text{-}cl_\theta\left(B\right)\right]$ for every subset B of Y;
(c) $f\left[e\text{-}cl_\theta\left(A\right)\right]\subseteq e\text{-}cl_\theta\left(f\left[A\right]\right)$ for every subset A of X;
(d) $f^{-1}\left[F\right]$ is e-θ-closed in X for every e-θ-closed set F of Y;
(e) $f^{-1}\left[V\right]$ is e-θ-open in X for every e-θ-open set V of Y.

Proof. $(a)\Longrightarrow(b)$: Let $B\subseteq Y$ and $x\notin f^{-1}\left[e\text{-}cl_\theta\left(B\right)\right]$.
$$\left.\begin{array}{r}x\notin f^{-1}\left[e\text{-}cl_\theta\left(B\right)\right]\Rightarrow f(x)\notin e\text{-}cl_\theta\left(B\right)\Rightarrow\left(\exists V\in eO\left(Y,f\left(x\right)\right)\right)\left(e\text{-}cl\left(V\right)\cap B=\emptyset\right)\\ f\text{ is }w.e.i.\end{array}\right\}\Rightarrow$$
$\Rightarrow\left(\exists U\in eO\left(X,x\right)\right)\left(f\left[e\text{-}cl\left(U\right)\right]\cap B=\emptyset\right)$
$\Rightarrow\left(\exists U\in eO\left(X,x\right)\right)\left(e\text{-}cl\left(U\right)\cap f^{-1}\left[B\right]=\emptyset\right)$
$\Rightarrow x\notin e\text{-}cl_\theta\left(f^{-1}\left[B\right]\right)$.
$(b)\Longrightarrow(c)$: Let $A\subseteq X$.
$$\left.\begin{array}{r}A\subseteq X\Rightarrow f\left[A\right]\subseteq Y\\ (b)\end{array}\right\}\Rightarrow e\text{-}cl_\theta(A)\subseteq e\text{-}cl_\theta\left(f^{-1}\left[f[A]\right]\right)\subseteq f^{-1}\left[e\text{-}cl_\theta\left(f[A]\right)\right]$$
$\Rightarrow f\left[e\text{-}cl_\theta\left(A\right)\right]\subseteq e\text{-}cl_\theta\left(f\left[A\right]\right)$.
$(c)\Longrightarrow(d)$: Let $F\in e\theta C(Y)$.
$$\left.\begin{array}{r}F\in e\theta C(Y)\Rightarrow\left(e\text{-}cl_\theta\left(F\right)=F\right)\left(f^{-1}\left[F\right]\subseteq X\right)\\ (c)\end{array}\right\}\Rightarrow$$
$\Rightarrow f\left[e\text{-}cl_\theta\left(f^{-1}[F]\right)\right]\subseteq e\text{-}cl_\theta\left(f\left[f^{-1}[F]\right]\right)\subseteq e\text{-}cl_\theta\left(F\right)=F$
$\Rightarrow e\text{-}cl_\theta\left(f^{-1}\left[F\right]\right)\subseteq f^{-1}\left[F\right]$
$\Rightarrow f^{-1}\left[F\right]\in e\theta C(X)$.
$(d)\Longrightarrow(e)$: Let $V\in e\theta O(Y)$.
$$\left.\begin{array}{r}V\in e\theta O(Y)\Rightarrow Y\setminus V\in e\theta C(Y)\\ (d)\end{array}\right\}\Rightarrow X\setminus f^{-1}\left[V\right]=f^{-1}\left[Y\setminus V\right]\in e\theta C(X)$$
$\Rightarrow f^{-1}\left[V\right]\in e\theta O(X)$.
$(e)\Longrightarrow(a)$: Let $V\in eR(Y)$.
$$\left.\begin{array}{r}V\in eR(Y)\Rightarrow V\in e\theta O(Y)\cap e\theta C(Y)\Rightarrow\left(V\in e\theta O(Y)\right)\left(Y\setminus V\in e\theta C(Y)\right)\\ (e)\end{array}\right\}\Rightarrow$$
$\Rightarrow\left(f^{-1}\left[V\right]\in e\theta O(X)\right)\left(X\setminus f^{-1}\left[V\right]=f^{-1}\left[Y\setminus V\right]\in e\theta O(X)\right)$

$\Rightarrow \left(f^{-1}[V] \in e\theta O(X)\right)\left(f^{-1}[V] \in e\theta C(X)\right) \overset{\text{Theorem 2.3}(d)}{\Rightarrow} f^{-1}[V] \in eR(X)$. ∎

4. Some Fundamental Properties

Definition 4.1 A topological space X is said to be strongly e-regular if for each point $x \in X$ and each e-open set U of X containing x, there exists $V \in eO(X,x)$ such that $V \subseteq e\text{-}cl(V) \subseteq U$.

Theorem 4.2 Let X and Y be topological spaces and $f : X \to Y$ be a function. If Y is strongly e-regular and $f : X \to Y$ is weakly e-irresolute, then the function f is e-irresolute.

Proof. $V \in eO(Y)$ and $x \in f^{-1}[V]$.

$$\left.\begin{array}{r}(V \in eO(Y))(x \in f^{-1}[V]) \Rightarrow V \in eO(Y, f(x)) \\ Y \text{ is strongly } e\text{-regular}\end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{r}(\exists F \in eO(Y, f(x)))(F \subseteq e\text{-}cl(F) \subseteq V) \\ f \text{ is } w.e.i.\end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists U \in eO(X,x))(f[U] \subseteq e\text{-}cl(F) \subseteq V)$

$\Rightarrow (\exists U \in eO(X,x))\left(U \subseteq f^{-1}[f[U]] \subseteq f^{-1}[e\text{-}cl(F)] \subseteq f^{-1}[V]\right)$

$\Rightarrow x \in e\text{-}int\left(f^{-1}[V]\right)$

Then $f^{-1}[V] \in eO(X)$. ∎

Definition 4.3 A space X is said to be e-T_2 [7] if for each pair of distinct points x and y in X, there exist $A \in eO(X,x)$ and $B \in eO(X,y)$ such that $A \cap B = \emptyset$.

Lemma 4.4 [11] A topological space X is e-T_2 if and only if for each pair of distinct points x and y of X, there exist $U \in eO(X,x)$ and $V \in eO(X,y)$ such that $e\text{-}cl(U) \cap e\text{-}cl(V) = \emptyset$.

Theorem 4.5 Let X and Y be topological spaces and $f : X \to Y$ be a function. If Y is e-T_2 and $f : X \to Y$ is weakly e-irresolute injection, then X is e-T_2.

Proof. Let $x, y \in X$ and $x \neq y$.

$$\left.\begin{array}{r}(x,y \in X)(x \neq y) \\ f \text{ is injective}\end{array}\right\} \Rightarrow \left.\begin{array}{r}f(x) \neq f(y) \\ \text{Lemma 4.4}\end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists V \in eO(Y, f(x)))(\exists W \in eO(Y, f(y)))(e\text{-}cl(V) \cap e\text{-}cl(W) = \emptyset) \ldots (1)$

$$\left.\begin{array}{r}(V \in eO(Y, f(x)))(W \in eO(Y, f(y))) \\ f \text{ is } w.e.i.\end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists G \in eO(X,x))(\exists H \in eO(X,y))(f[G] \subseteq e\text{-}cl(V))(f[H] \subseteq e\text{-}cl(W)) \ldots (2)$

$(1),(2) \Rightarrow (\exists G \in eO(X,x))(\exists H \in eO(X,y))(f[G] \cap f[H] = \emptyset)$

$\Rightarrow (\exists G \in eO(X,x))(\exists H \in eO(X,y))(f[G \cap H] = \emptyset)$

$\Rightarrow (\exists G \in eO(X,x))(\exists H \in eO(X,y))(G \cap H = \emptyset)$

Then X is e-T_2. ∎

We recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) | x \in X\}$ of the product space $X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 4.6 The graph $G(f)$ of a function $f : X \to Y$ is said to be completely e-closed (briefly *c.e.c.*) if for each $(x,y) \in (X \times Y) \setminus G(f)$, there exist $U \in eO(X,x)$ and $V \in eO(Y,y)$ such that $(e\text{-}cl(U) \times e\text{-}cl(V)) \cap G(f) = \emptyset$.

Lemma 4.7 The graph of a function $f : X \to Y$ is completely e-closed if and only if for each $(x,y) \in (X \times Y) \setminus G(f)$, there exist $U \in eO(X,x)$ and $V \in eO(Y,y)$ such that $f[e\text{-}cl(U)] \cap e\text{-}cl(V) = \emptyset$.

Proof. *Necessity.* Let $(x,y) \in (X \times Y) \setminus G(f)$.

$$\left.\begin{array}{r}(x,y) \in (X \times Y) \setminus G(f) \\ G(f) \text{ is } c.e.c.\end{array}\right\} \Rightarrow$$
$\Rightarrow (\exists U \in eO(X,x))(\exists V \in eO(Y,y))([e\text{-}cl(U) \times e\text{-}cl(V)] \cap G(f) = \emptyset)$
$\Rightarrow (\exists U \in eO(X,x))(\exists V \in eO(Y,y))(f[e\text{-}cl(U)] \cap e\text{-}cl(V) = \emptyset)$.

Sufficiency. Let $(x,y) \in (X \times Y) \setminus G(f)$.

$$\left.\begin{array}{r}(x,y) \in (X \times Y) \setminus G(f) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow (\exists U \in eO(X,x))(\exists V \in eO(Y,y))(f[e\text{-}cl(U)] \cap e\text{-}cl(V) = \emptyset)$$
$\Rightarrow (\exists U \in eO(X,x))(\exists V \in eO(Y,y))([e\text{-}cl(U) \times e\text{-}cl(V)] \cap G(f) = \emptyset)$. ■

Theorem 4.8 If Y is e-T_2 and $f : X \to Y$ is weakly e-irresolute, then $G(f)$ is completely e-closed.

Proof. Let $(x,y) \in (X \times Y) \setminus G(f)$.

$$\left.\begin{array}{r}(x,y) \in (X \times Y) \setminus G(f) \Rightarrow (x,y) \notin G(f) \Rightarrow y \neq f(x) \\ Y \text{ is } e\text{-}T_2\end{array}\right\} \overset{\text{Lemma 4.4}}{\Rightarrow}$$
$\Rightarrow (\exists V \in eO(Y, f(x)))(\exists W \in eO(Y,y))(e\text{-}cl(V) \cap e\text{-}cl(W) = \emptyset) \ldots (1)$

$$\left.\begin{array}{r}V \in eO(Y, f(x)) \\ f \text{ is } w.e.i.\end{array}\right\} \overset{\text{Theorem 3.7}(b)}{\Rightarrow} (\exists U \in eO(X,x))(f[e\text{-}cl(U)] \subseteq e\text{-}cl(V)) \ldots (2)$$
$(1),(2) \Rightarrow (\exists U \in eO(X,x))(\exists W \in eO(Y,y))(f[e\text{-}cl(U)] \cap e\text{-}cl(W) = \emptyset)$
$\Rightarrow (\exists U \in eO(X,x))(\exists W \in eO(Y,y))(e\text{-}cl(U) \times e\text{-}cl(W)) \cap G(f) = \emptyset)$

Then $G(f)$ is completely e-closed. ■

Theorem 4.9 If a function $f : X \to Y$ is weakly e-irresolute injection and $G(f)$ is completely e-closed, then X is e-T_2.

Proof. Let $x, y \in X$ and $x \neq y$.

$$\left.\begin{array}{r}(x,y \in X)(x \neq y) \\ f \text{ is injective}\end{array}\right\} \Rightarrow \left.\begin{array}{r}f(x) \neq f(y) \Rightarrow (x, f(y)) \notin G(f) \\ G(f) \text{ is } c.e.c.\end{array}\right\} \overset{\text{Lemma 4.7}}{\Rightarrow}$$
$\Rightarrow (\exists U \in eO(X,x))(\exists V \in eO(Y, f(y)))(f[e\text{-}cl(U)] \cap e\text{-}cl(V) = \emptyset) \ldots (1)$

$$\left.\begin{array}{r}V \in eO(Y, f(y)) \\ f \text{ is } w.e.i.\end{array}\right\} \Rightarrow (\exists H \in eO(X,y))(f[H] \subseteq e\text{-}cl(V)) \ldots (2)$$
$(1),(2) \Rightarrow (\exists U \in eO(X,x))(\exists H \in eO(X,y))(f[e\text{-}cl(U)] \cap f[H] = \emptyset)$
$\Rightarrow (\exists U \in eO(X,x))(\exists H \in eO(X,y))(f[e\text{-}cl(U) \cap H] = \emptyset)$
$\Rightarrow (\exists U \in eO(X,x))(\exists H \in eO(X,y))(e\text{-}cl(U) \cap H = \emptyset)$
$\Rightarrow (\exists U \in eO(X,x))(\exists H \in eO(X,y))(U \cap H = \emptyset)$

This means that X is e-T_2. ■

Definition 4.10 A topological space X is said to be e-connected [5] if it cannot be written as the union of two nonempty disjoint e-open sets.

Theorem 4.11 If a function $f : X \to Y$ is weakly e-irresolute surjection and X is e-connected, then Y is e-connected.

Proof. Suppose that Y is not e-connected. Then

$$\left.\begin{array}{r}\Rightarrow (\exists U,V \in eO(Y) \setminus \{\emptyset\})(U \cap V = \emptyset)(U \cup V = Y) \Rightarrow U,V \in eR(Y) \setminus \{\emptyset\} \\ \text{Hypothesis}\end{array}\right\}$$
$\Rightarrow \left(f^{-1}[U], f^{-1}[V] \in eR(X) \setminus \{\emptyset\}\right)\left(f^{-1}[U \cap V] = f^{-1}[\emptyset]\right)\left(f^{-1}[U \cup V] = f^{-1}[Y]\right)$
$\Rightarrow \left(f^{-1}[U], f^{-1}[V] \in eR(X) \setminus \{\emptyset\}\right)\left(f^{-1}[U] \cap f^{-1}[V] = \emptyset\right)\left(f^{-1}[U] \cup f^{-1}[V] = X\right)$.

This means that X is not e-connected. ■

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