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Duals and approximate duals of g-frames in Hilbert spaces

M. Mirzaee Azandaryani[a*], Amir Khosravi[b]

[a] *Department of Mathematics, Faculty of Science, University of Qom, Qom, Iran.*
[b] *Faculty of Mathematical Sciences and Computer, Kharazmi University, Tehran, Iran.*

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Abstract. In this paper we get some results and applications for duals and approximate duals of g-frames in Hilbert spaces. In particular, we consider the stability of duals and approximate duals under bounded operators and we study duals and approximate duals of g-frames in the direct sum of Hilbert spaces. We also obtain some results for perturbations of approximate duals.

Keywords: Frame, g-frame, duality, approximate duality.

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1. Introduction

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [3] in 1952 to study some problems in nonharmonic Fourier series, reintroduced in 1986 by Daubechies, Grossmann and Meyer [2].
Let H be a Hilbert space and let I be a finite or countable index set. A family $\mathcal{F} = \{f_i\}_{i\in I} \subseteq H$ is a *frame* for H, if there exist $0 < A \leqslant B < \infty$, such that

$$A\|f\|^2 \leqslant \sum_{i\in I} |\langle f, f_i\rangle|^2 \leqslant B\|f\|^2,$$

*Corresponding author.
E-mail address: morteza_ma62@yahoo.com(M. Mirzaee Azandaryani).

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for each $f \in H$. In this case we say that $\mathcal{F}$ is an (A, B) frame. If only the right-hand side inequality is required, it is called a *Bessel* sequence. If $\mathcal{F}$ is a Bessel sequence, then the *synthesis operator* $T_{\mathcal{F}} : \ell^2(I) \longrightarrow H$ which is defined by $T_{\mathcal{F}}(\{c_i\}_{i\in I}) = \sum_{i\in I} c_i f_i$ is bounded. We say that a Bessel sequence $\{g_i\}_{i\in I}$ is an *alternate dual* or a *dual* for the Bessel sequence $\{f_i\}_{i\in I}$, if for each $f \in H$, we have $f = \sum_{i\in I}\langle f, f_i\rangle g_i$ or equivalently $f = \sum_{i\in I}\langle f, g_i\rangle f_i$.

Many generalizations of frames have been introduced. The general one is g-frame (see [9]). Let H_i be a Hilbert space, for each $i \in I$ and let $L(H, H_i)$ be the set of all bounded operators from H into H_i. We call $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in I\}$ a *g-frame* for H with respect to $\{H_i : i \in I\}$ if there exist two positive constants A and B such that

$$A\|f\|^2 \leqslant \sum_{i\in I} \|\Lambda_i f\|^2 \leqslant B\|f\|^2,$$

for each $f \in H$. If only the second inequality is required, we call it a *g-Bessel sequence.* A g-Bessel sequence $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in I\}$ is called an *alternate g-dual* or a *g-dual* of Λ if $f = \sum_{i\in I}\Gamma_i^*\Lambda_i f = \sum_{i\in I}\Lambda_i^*\Gamma_i f$, for each $f \in H$. The *synthesis operator* for a g-Bessel sequence $\Lambda = \{\Lambda_i\}_{i\in I}$ is $T_\Lambda : \oplus_{i\in I} H_i \longrightarrow H$, $T_\Lambda(\{f_i\}_{i\in I}) = \sum_{i\in I}\Lambda_i^* f_i$.

Approximate duality of frames was recently investigated by Christensen and Laugesen in [1]. Next, we introduced and characterized approximate duality for g-frames (see [4]). Approximate duals help us to get useful results for perturbations and reconstruction of signals (especially when it is difficult to find an alternate dual).

2. Duality and approximate duality of g-frames

In this section we obtain some results and applications for duals and approximate duals. First we recall the definitions of approximate duals and approximate g-duals from [1] and [4], respectively:

Definition 2.1

(i) Let $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{G} = \{g_i\}_{i\in I}$ be two Bessel sequences for H. Suppose that $R_{\mathcal{GF}} = T_{\mathcal{G}}T_{\mathcal{F}}^*$. We say that $\mathcal{F}$ and $\mathcal{G}$ are *approximately dual frames* if $\|Id_H - R_{\mathcal{GF}}\| < 1$ or $\|Id_H - R_{\mathcal{FG}}\| < 1$. In this case we call $\mathcal{G}$ an *approximate dual* of $\mathcal{F}$.

(ii) Two g-Bessel sequences Λ and Γ are *approximately dual* g-frames if $\|Id_H - S_{\Gamma\Lambda}\| < 1$ or $\|Id_H - S_{\Lambda\Gamma}\| < 1$, where $S_{\Gamma\Lambda} = T_\Gamma T_\Lambda{}^*$. In this case, we say that Γ is an *approximate dual* g-frame or *approximate g-dual* of Λ.

Since $\|Id_H - S_{\Gamma\Lambda}\| = \|(Id_H - S_{\Gamma\Lambda})^*\| = \|Id_H - S_{\Lambda\Gamma}\|$, the conditions in the above definition are equivalent. Since $\|Id_H - S_{\Lambda\Gamma}\| < 1$, $S_{\Lambda\Gamma}$ is invertible with $S_{\Lambda\Gamma}{}^{-1} = \sum_{n=0}^{\infty}(Id_H - S_{\Lambda\Gamma})^n$. Therefore every $f \in H$ can be reconstructed as

$$\sum_{n=0}^{\infty} S_{\Lambda\Gamma}(Id_H - S_{\Lambda\Gamma})^n f = f = \sum_{n=0}^{\infty}(Id_H - S_{\Lambda\Gamma})^n S_{\Lambda\Gamma} f.$$

Note that if $\{f_i\}_{i\in I}$ is a Bessel sequence for H and Λ_{f_i} is a functional on H defined by $\Lambda_{f_i}(f) = \langle f, f_i\rangle$, then $\{\Lambda_{f_i}\}_{i\in I}$ is a g-Bessel sequence for H. Now it is easy to see that if $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ are approximately dual frames, then $\{\Lambda_{f_i}\}_{i\in I}$ is an approximate dual g-frame of $\{\Lambda_{g_i}\}_{i\in I}$.

We proved in [4, Proposition 2.3] that if Λ and Γ are approximately dual g-frames, then

Λ and Γ are g-frames. In the following theorem, we obtain this result using the fact that a g-Bessel sequence is a g-frame if and only if it has a g-dual.

Theorem 2.2 Let Λ and Γ be approximately dual g-frames. Then

(i) Λ and Γ have at least one alternate g-dual and they are g-frames.
(ii) If ${\psi_i}^N = \Gamma_i + \sum_{n=1}^{N} \Gamma_i(Id_H - S_{\Lambda\Gamma})^n$, then $\Psi_N = \{{\psi_i}^N\}_{i\in I}$ is a g-frame with $\lim_{N\to\infty} S_{\Lambda\Psi_N} = Id_H$ and for each signal $f \in H$, we have $\lim_{N\to\infty} S_{\Lambda\Psi_N} f = f$.

Proof. (i) Since Λ and Γ are approximately g-duals, $\|S_{\Lambda\Gamma} - Id_H\| < 1$ and $\|S_{\Gamma\Lambda} - Id_H\| < 1$. Hence by Newmann algorithm, $S_{\Lambda\Gamma}$ and $S_{\Gamma\Lambda}$ are invertible. Now for each $f \in H$, we have $\sum_{i\in I} \Lambda_i^* \Gamma_i S_{\Lambda\Gamma}^{-1} f = f$ and $\sum_{i\in I} \Gamma_i^* \Lambda_i S_{\Gamma\Lambda}^{-1} f = f$. Thus $\{\Gamma_i S_{\Lambda\Gamma}^{-1}\}_{i\in I}$ and $\{\Lambda_i S_{\Gamma\Lambda}^{-1}\}$ are g-duals of Λ and Γ, respectively. Now if $\Phi_i = \Gamma_i S_{\Lambda\Gamma}^{-1}$ and $\Phi = \{\Phi_i\}_{i\in I}$, then

$$\|f\| = \|S_{\Phi\Lambda} f\| = \sup_{\|g\|=1} \Big| \sum_{i\in I} \langle \Lambda_i f, \Phi_i g\rangle \Big| \leqslant \Big(\sum_{i\in I} \|\Lambda_i f\|^2 \Big)^{\frac{1}{2}} \sup_{\|g\|=1} \Big(\sum_{i\in I} \|\Phi_i g\|^2 \Big)^{\frac{1}{2}}.$$

Thus if D is an upper bound for Φ, then $\frac{1}{D}\|f\|^2 \leqslant \sum_{i\in I} \|\Lambda_i f\|^2$. This shows that $\frac{1}{D}$ is a lower bound for Λ. Similarly we can see that if B is an upper bound for $\{\Lambda_i S_{\Gamma\Lambda}^{-1}\}_{i\in I}$, then $\frac{1}{B}$ is a lower bound for Γ.
(ii) Let D be an upper bound for Γ and $f \in H$. Then

$$\begin{aligned} \sum_{i\in I} \|\psi_i^N f\|^2 = \sum_{i\in I} \Big\| \sum_{n=0}^{N} \Gamma_i (Id_H - S_{\Lambda\Gamma})^n f \Big\|^2 &\leqslant \sum_{n=0}^{N} \sum_{i\in I} \|\Gamma_i (Id_H - S_{\Lambda\Gamma})^n f\|^2 \\ &\leqslant \Big(D \sum_{n=0}^{N} \|Id_H - S_{\Lambda\Gamma}\|^{2n} \Big) \|f\|^2. \end{aligned}$$

Hence $\Psi_N = \{\psi_i^N\}_{i\in I}$ is a g-Bessel sequence. Now similar to the proof of Proposition 2.3 in [4], we can see that $\|Id_H - S_{\Lambda\Psi_N}\| \leqslant \|Id_H - S_{\Lambda\Gamma}\|^{N+1}$. Therefore for each $f \in H$, $\|f - S_{\Lambda\Psi_N} f\| \leqslant \|Id_H - S_{\Lambda\Gamma}\|^{N+1} \|f\|$. Since $\|Id_H - S_{\Lambda\Gamma}\| < 1$, we have

$$\lim_{N\to\infty} \|f - S_{\Lambda\Psi_N} f\| \leqslant \lim_{N\to\infty} \|Id_H - S_{\Lambda\Gamma}\|^{N+1} \|f\| = 0,$$

so $\lim_{N\to\infty} S_{\Lambda\Psi_N} f = f$. Also it follows from $\|Id_H - S_{\Lambda\Psi_N}\| < 1$ that Λ and Ψ_N are approximately dual g-frames and by part (i), they are g-frames. ■

Let J be a finite or countable index set and let $\Phi_j = \{\Lambda_{ij} \in L(H_j, H_{ij}) : i \in I\}$ be a g-Bessel sequence for H_j, with upper bound B_j such that $B = sup\{B_j : j \in J\} < \infty$. Then $\{\Phi_j\}_{j\in J}$ is called a *B-bounded family of g-Bessel sequences* or shortly *B-BFGBS*. Let Φ_j be an (A_j, B_j) g-frame such that $A = \inf\{A_j : j \in J\} > 0$ and $B = sup\{B_j : j \in J\} < \infty$. Then $\{\Phi_j\}_{j\in J}$ is called an *(A,B)-bounded family of g-frames* or shortly *(A,B)-BFGF* (see [5, 6]).

Proposition 2.3 Suppose that $\Phi_j = \{\Lambda_{ij} \in L(H_j, H_{ij}) : i \in I\}$ and $\Psi_j = \{\Gamma_{ij} \in L(H_j, H_{ij}) : i \in I\}$ are two g-Bessel sequences such that $\{\Phi_j\}_{j\in J}$ and $\{\Psi_j\}_{j\in J}$ are BFGBS. Let $\alpha = \sup_{j\in J}\{\|S_{\Phi_j\Psi_j} - Id_{H_j}\|\}$. Then

(i) $\{\oplus_{j\in J}\Lambda_{ij}\}_{i\in I}$ is an approximate dual g-frame of $\{\oplus_{j\in J}\Gamma_{ij}\}_{i\in I}$ if and only if $\alpha < 1$.
(ii) $\{\oplus_{j\in J}\Lambda_{ij}\}_{i\in I}$ is a g-dual of $\{\oplus_{j\in J}\Gamma_{ij}\}_{i\in I}$ if and only if $\alpha = 0$.

Proof. (i) Let $\oplus_{j\in J}\Phi_j = \{\oplus_{j\in J}\Lambda_{ij}\}_{i\in I}$ and $\oplus_{j\in J}\Psi_j = \{\oplus_{j\in J}\Gamma_{ij}\}_{i\in I}$. Then by Theorem 2.1 in [5], $\oplus_{j\in J}\Phi_j$ and $\oplus_{j\in J}\Psi_j$ are g-Bessel sequences. Now it is easy to see that

$$\|S_{(\oplus_{j\in J}\Phi_j)(\oplus_{j\in J}\Psi_j)} - Id_{\oplus_{j\in J}H_j}\| = \sup_{j\in J}\{\|S_{\Phi_j\Psi_j} - Id_{H_j}\|\} = \alpha,$$

so $\oplus_{j\in J}\Phi_j$ and $\oplus_{j\in J}\Psi_j$ are approximately dual g-frames if and only if $\alpha < 1$.
(ii) The equality obtained in part (i) implies that $S_{(\oplus_{j\in J}\Phi_j)(\oplus_{j\in J}\Psi_j)} = Id_{\oplus_{j\in J}H_j}$ if and only if $\alpha = 0$ and this means that $\oplus_{j\in J}\Phi_j$ is a g-dual of $\oplus_{j\in J}\Psi_j$ if and only if $\alpha = 0$. ■

Remark 1 *Note that in the above proposition $\alpha = 0$ is equivalent to say that Ψ_j is a g-dual of Φ_j, for each $j \in J$, so it follows from part* (ii) *of the above proposition that Ψ_j is a g-dual of Φ_j, for each $j \in J$ if and only if $\{\oplus_{j\in J}\Gamma_{ij}\}_{i\in I}$ and $\{\oplus_{j\in J}\Lambda_{ij}\}_{i\in I}$ are g-duals (see also [5]). But $\alpha < 1$ is not equivalent to the approximate duality of Φ_j and Ψ_j, for each $j \in J$. As we see in [4, Example 2.9] that if $\Phi_j = \{\Lambda_{ij} \in L(H_j, H_{ij}) : i \in I\}$ is an approximate dual g-frame of itself, then it is not necessarily true that $\{\oplus_{j\in J}\Lambda_{ij}\}_{i\in I}$ is an approximate dual g-frame of itself.*

Proposition 2.4

(i) Let Γ and Λ be two g-Bessel sequences such that Λ_i is a partial isometric operator, for each $i \in I$. Then Λ and Γ are approximately g-duals (resp. g-duals) if and only if $\{\Lambda_i^*\Gamma_i\}_{i\in I}$ and $\{\Lambda_i^*\Lambda_i\}_{i\in I}$ are approximately g-duals (resp. g-duals).

(ii) Let T be an isometric operator on H. If Λ is an approximate g-dual (resp. a g-dual) of Γ, then $\{\Lambda_i T\}_{i\in I}$ is an approximate g-dual (resp. a g-dual) of $\{\Gamma_i T\}_{i\in I}$.

(iii) Let T be a co-isometric operator on H. If Λ is an approximate g-dual (resp. a g-dual) of Γ, then $\{\Lambda_i T^*\}_{i\in I}$ is an approximate g-dual (resp. a g-dual) of $\{\Gamma_i T^*\}_{i\in I}$.

Proof. (i) It is easy to see that $\Phi = \{\Lambda_i^*\Gamma_i\}_{i\in I}$ and $\Psi = \{\Lambda_i^*\Lambda_i\}_{i\in I}$ are g-Bessel sequences. Since Λ_i's are partial isometry, by Theorem 2.3.3 in [8], we have $\Lambda_i\Lambda_i^*\Lambda_i f = \Lambda_i f$, for each $f \in H$, so

$$S_{\Phi\Psi}f = \sum_{i\in I}\Gamma_i^*\Lambda_i\Lambda_i^*\Lambda_i f = \sum_{i\in I}\Gamma_i^*\Lambda_i f = S_{\Gamma\Lambda}f.$$

Hence Λ and Γ are approximately g-duals (resp. g-duals) if and only if Φ and Ψ are approximately g-duals (resp. g-duals).
(ii) Let B and D be upper bounds of Λ and Γ, respectively. Then it is clear that $\Phi = \{\Lambda_i T\}_{i\in I}$ and $\Psi = \{\Gamma_i T\}_{i\in I}$ are g-Bessel sequences with upper bounds $B\|T\|^2$ and $D\|T\|^2$, respectively. Then for each $f \in H$, we have

$$\|S_{\Psi\Phi}f - f\| = \left\|T^*\Big[(\sum_{i\in I}\Gamma_i^*\Lambda_i T)f - Tf\Big]\right\| = \|T^*(S_{\Gamma\Lambda} - Id_H)Tf\| \leqslant \|S_{\Gamma\Lambda} - Id_H\|\|f\|.$$

Since $\|S_{\Gamma\Lambda} - Id_H\| < 1$ (resp. $S_{\Gamma\Lambda} = Id_H$), we get $\|S_{\Psi\Phi} - Id_H\| < 1$ (resp. $S_{\Psi\Phi} = Id_H$) and the result follows.
(iii) The result follows from part (ii) by considering T^* instead of T. ■

Corollary 2.5 Suppose that $\Phi_j = \{\Lambda_{ij} \in L(H_j, H_{ij}) : i \in I\}$ and $\Psi_j = \{\Gamma_{ij} \in L(H_j, H_{ij}) : i \in I\}$ are two g-Bessel sequences such that $\{\Phi_j\}_{j\in J}$ and $\{\Psi_j\}_{j\in J}$ are BFGBS.

(i) Let Λ_{ij} be a partial isometric operator, for each $i \in I$. Then $\oplus_{j\in J}\Phi_j$ and $\oplus_{j\in J}\Psi_j$

are approximately g-duals (resp. g-duals) if and only if $\{(\oplus_{j\in J}\Lambda_{ij}^*)(\oplus_{j\in J}\Gamma_{ij})\}_{i\in I}$ and $\{(\oplus_{j\in J}\Lambda_{ij}^*)(\oplus_{j\in J}\Lambda_{ij})\}_{i\in I}$ are approximately g-duals (resp. g-duals).

(ii) Let T_j be an isometric operator on H_j. If Φ_j is a g-dual of Ψ_j (resp. an approximate g-dual of Ψ_j with $\sup_{j\in J}\{\|S_{\Phi_j\Psi_j} - Id_{H_j}\|\} < 1$), for each $j \in J$, then $\{(\oplus_{j\in J}\Lambda_{ij})(\oplus_{j\in J}T_j)\}_{i\in I}$ is a g-dual (resp. an approximate g-dual) of $\{(\oplus_{j\in J}\Gamma_{ij})(\oplus_{j\in J}T_j)\}_{i\in I}$.

(iii) Let T_j be a co-isometric operator on H_j. If Φ_j is a g-dual of Ψ_j (resp. an approximate g-dual of Ψ_j with $\sup_{j\in J}\{\|S_{\Phi_j\Psi_j} - Id_{H_j}\|\} < 1$), for each $j \in J$, then $\{(\oplus_{j\in J}\Lambda_{ij})(\oplus_{j\in J}T_j^*)\}_{i\in I}$ is an approximate g-dual (resp. a g-dual) of $\{(\oplus_{j\in J}\Gamma_{ij})(\oplus_{j\in J}T_j^*)\}_{i\in I}$.

Proof. Note that if T_j is an isometric operator on H_j, then $\oplus_{j\in J}T_j$ is an isometric operator on $\oplus_{j\in J}H_j$. Also if Λ_{ij} is a partial isometric operator, for each $i \in I$, then $\oplus_{j\in J}\Lambda_{ij}$ is also a partial isometric operator. Now we can get the result from Propositions 2.3 and 2.4. ■

Proposition 2.6

(i) Let T be an isometric operator on H. If $\{f_i\}_{i\in I}$ is an approximate dual (resp. a dual) of $\{g_i\}_{i\in I}$, then $\{T^*f_i\}_{i\in I}$ is an approximate dual (resp. a dual) of $\{T^*g_i\}_{i\in I}$.

(ii) Let T be a co-isometric operator on H. If $\{f_i\}_{i\in I}$ is an approximate dual (resp. a dual) of $\{g_i\}_{i\in I}$, then $\{Tf_i\}_{i\in I}$ is an approximate dual (resp. a dual) of $\{Tg_i\}_{i\in I}$.

Proof. (i) Let $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{G} = \{g_i\}_{i\in I}$ be Bessel sequences. Then it is clear that $T^*\mathcal{F} = \{T^*f_i\}_{i\in I}$ and $T^*\mathcal{G} = \{T^*g_i\}_{i\in I}$ are Bessel sequences. Now for each $f \in H$ we have

$$\|R_{(T^*\mathcal{G})(T^*\mathcal{F})}f - f\| = \|\sum_{i\in I}\langle f, T^*f_i\rangle T^*g_i - f\| = \|T^*(S_{\mathcal{G}\mathcal{F}} - Id_H)Tf\| \leqslant \|S_{\mathcal{G}\mathcal{F}} - Id_H\|\|f\|.$$

This yields that $\|R_{(T^*\mathcal{G})(T^*\mathcal{F})} - Id_H\| \leqslant \|S_{\mathcal{G}\mathcal{F}} - Id_H\| < 1$, so $T^*\mathcal{F}$ and $T^*\mathcal{G}$ are approximately dual frames.

(ii) We can get the result by considering T^* instead of T in part (i). ■

The following example shows that the converse of Proposition 2.6 (also Proposition 2.4) is not necessarily true.

***Example* 2.7** Let $T : \ell^2(\mathbb{N}) \longrightarrow \ell^2(\mathbb{N})$ be the unilateral shift operator on $\ell^2(\mathbb{N})$, i.e., $T(\{\alpha_i\}_{i=1}^\infty) = (0, \alpha_1, \alpha_2, \ldots)$. T is isometric and its adjoint operator is the bilateral shift operator on $\ell^2(\mathbb{N})$, i.e., $T^* : \ell^2(\mathbb{N}) \longrightarrow \ell^2(\mathbb{N})$, $T^*(\{\alpha_i\}_{i=1}^\infty) = (\alpha_2, \alpha_3, \ldots)$. Now let $f_i = \{\delta_{ij}\}_{j=1}^\infty$, for $i \geqslant 2$. Then $\{T^*f_i\}_{i=2}^\infty$ is an orthonormal basis, so it is a dual (also an approximate dual) of itself but $\{f_i\}_{i=2}^\infty$ is not an approximate dual of itself because it is not a frame.

Now we recall the following definition from [7]:

Definition 2.8 Let Λ be a sequence, $0 \leqslant \lambda_1, \lambda_2 < 1$ and $\{c_i\}_{i\in I}$ be a sequence of positive numbers in $\ell^2(I)$. We say that Γ is a $(\lambda_1, \lambda_2, \{c_i\}_{i\in I})$-*perturbation* of Λ if for each $i \in I$, we have

$$\|\Lambda_i f - \Gamma_i f\| \leqslant \lambda_1\|\Lambda_i f\| + \lambda_2\|\Gamma_i f\| + c_i\|f\| \qquad \forall f \in H.$$

Recall that for an (A,B) g-frame $\Lambda = \{\Lambda_i\}_{i\in I}$, the operator $S_\Lambda = T_\Lambda T_\Lambda^*$ is called the *g-frame operator* of Λ and if $\widetilde{\Lambda} = \{\widetilde{\Lambda_i}\}_{i\in I}$, where $\widetilde{\Lambda_i} = \Lambda_i S_\Lambda^{-1}$, then $\widetilde{\Lambda}$ is called the *canonical g-dual* of Λ which is an $(\frac{1}{B}, \frac{1}{A})$ g-frame.
In part (i) of the following proposition we give a direct proof for the result obtained in [4, Corollary 3.8].

Proposition 2.9

(i) Let Λ be an (A,B) g-frame and Γ be a g-Bessel sequence with upper bound D. If Γ is a $(\lambda_1, \lambda_2, \{c_i\}_{i\in I})$-perturbation of Λ with $\Big(\lambda_1\sqrt{B} + \lambda_2\sqrt{D} + (\sum_{i\in I} c_i^2)^{\frac{1}{2}}\Big) < \sqrt{A}$, then Γ is an approximate dual g-frame of $\widetilde{\Lambda}$.

(ii) Let $\{\Phi_j\}_{j\in J}$ be an (A,B)-BFGF and $\{\Psi_j\}_{j\in J}$ be a D-BFGBS. If $\{c_i\}_{i\in I}$ is a sequence of positive numbers in $\ell^2(I)$ such that Ψ_j is a $(\lambda_1, \lambda_2, \{c_i\}_{i\in I})$-perturbation of Φ_j, for each $j \in J$ and $\Big(\lambda_1\sqrt{B} + \lambda_2\sqrt{D} + (\sum_{i\in I} c_i^2)^{\frac{1}{2}}\Big) < \sqrt{A}$, then $\{\oplus_{j\in J}\Gamma_{ij}\}_{i\in I}$ is an approximate dual g-frame of $\{\oplus_{j\in J}\widetilde{\Lambda_{ij}}\}_{i\in I}$.

Proof. (i) Since $\frac{1}{A}$ is an upper bound for $\widetilde{\Lambda}$, for each $f \in H$, we have

$$\|T_{\widetilde{\Lambda}} T_\Gamma^* f - T_{\widetilde{\Lambda}} T_\Lambda^* f\|^2 \leqslant \frac{1}{A}\sum_{i\in I} \|(\Gamma_i - \Lambda_i)f\|^2$$

$$\leqslant \frac{1}{A}\sum_{i\in I}(\lambda_1\|\Lambda_i f\| + \lambda_2\|\Gamma_i f\| + c_i\|f\|)^2 \leqslant \frac{1}{A}\Big(\sum_{i\in I}\lambda_1^2\|\Lambda_i f\|^2 + \sum_{i\in I}\lambda_2^2\|\Gamma_i f\|^2$$

$$+ \sum_{i\in I} c_i^2\|f\|^2 + 2\lambda_1\lambda_2(\sum_{i\in I}\|\Lambda_i f\|^2)^{\frac{1}{2}}(\sum_{i\in I}\|\Gamma_i f\|^2)^{\frac{1}{2}} + 2\lambda_1(\sum_{i\in I}\|\Lambda_i f\|^2)^{\frac{1}{2}}(\sum_{i\in I} c_i^2)^{\frac{1}{2}}\|f\|$$

$$+ 2\lambda_2(\sum_{i\in I}\|\Gamma_i f\|^2)^{\frac{1}{2}}(\sum_{i\in I} c_i^2)^{\frac{1}{2}}\|f\|\Big) \leqslant \frac{1}{A}\Big(\lambda_1^2 B\|f\|^2 + \lambda_2^2 D\|f\|^2 + \sum_{i\in I} c_i^2\|f\|^2$$

$$+ 2\lambda_1\lambda_2\sqrt{BD}\|f\|^2 + 2\lambda_1\sqrt{B}\|f\|^2(\sum_{i\in I} c_i^2)^{\frac{1}{2}} + 2\lambda_2\sqrt{D}(\sum_{i\in I} c_i^2)^{\frac{1}{2}}\|f\|^2\Big) = R\|f\|^2,$$

where $R = \frac{1}{A}\Big(\lambda_1\sqrt{B} + \lambda_2\sqrt{D} + (\sum_{i\in I} c_i^2)^{\frac{1}{2}}\Big)^2$. Thus we have $\|S_{\widetilde{\Lambda}\Gamma} - Id_H\| \leqslant \sqrt{R} < 1$, so Γ is an approximate dual g-frame of $\widetilde{\Lambda}$.
(ii) By Theorem 2.1 in [5], $\oplus_{j\in J}\Phi_j = \{\oplus_{j\in J}\Lambda_{ij}\}_{i\in I}$ and $\oplus_{j\in J}\Psi_j = \{\oplus_{j\in J}\Gamma_{ij}\}_{i\in I}$ are (A,B) g-frame and g-Bessel sequence with upper bound D, respectively. Also Corollary 3.1 in [5] yields that $\oplus_{j\in J}\Psi_j$ is a $(\lambda_1, \lambda_2, \{c_i\}_{i\in I})$-perturbation of $\oplus_{j\in J}\Phi_j$. Now since (by Proposition 3.3 in [5]) $\{\oplus_{j\in J}\widetilde{\Lambda_{ij}}\}_{i\in I}$ is the canonical g-dual of $\oplus_{j\in J}\Phi_j$, part (i) implies that $\{\oplus_{j\in J}\Gamma_{ij}\}_{i\in I}$ is an approximate dual g-frame of $\{\oplus_{j\in J}\widetilde{\Lambda_{ij}}\}_{i\in I}$. ■

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