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On the Design of Extended State-Dependent Differential Riccati Equation Controller for Nonlinear Reaction-Advection-Diffusion Partial Differential Equation with Multiple Delays

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Abstract

This paper proposes a sub-optimal Extended State-Dependent Differential Riccati Equation (ESDDRE) controller for nonlinear Reaction-Advection-Diffusion (R-A-D) Partial Differential Equation (PDE) systems with multiple delays. A State-Dependent Riccati Equation (SDRE) is a nonlinear version of Linear Quadratic Regulator (LQR) in optimal control and it is used to analyze nonlinear optimal control problems. Instead of the linearization or the Jacobin procedure, the ESDDRE technique applies a State-Dependent Coefficients (SDC) for parameterization to construct an Extended Pseudo-Linearization (EPL) representation. All of the multiple delays sections in this presentation can be located in the system matrices and input vectors. The control effort of ESDDRE method is derived based on the Hamiltonian equation and also cost function according to the PDE systems. In addition, the L_2 stability is guaranteed by Poincaré inequality and as well as Lyapunov function regarded on the ESDDRE control strategy for the closed-loop system. The simulation results for the nonlinear R-A-D partial differential equation with one and two constant delays indicate that the proposed ESDDRE controller technique is efficient.

Keywords: Extended state-dependent differential Riccati equation, Nonlinear reaction-advection-diffusion equation, Extended pseudo-linearization, Time-delay, Poincaré inequality.
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1. Introduction

Researchers in the field of control engineering are interested in nonlinear optimal controller and observer design, such as a State-Dependent Riccati Equation (SDRE) strategy. Instead of linearization or the Jacobin technique, the SDRE employs a State-Dependent Coefficient (SDC) parameterization as a Pseudo-Linearization (PL). Because the nonlinear characteristics of the system are consistently maintained by the PL, the SDRE holds robust features [1-4]. The PL is used for nonlinear systems without time-delay, whereas an Extended Pseudo-linearization (EPL) is utilized in nonlinear systems via time-delay. Time delays can be categorized as a distributed, variable, constant or multiple. The freedom degree of the SDRE controller and observer is limitless, because the EPL choice is also infinite and not unique [5]. In [6], an optimal extended version of state-dependent Riccati equation with an EPL and Nonlinear Impulsive-Observer (NIO) via state-dependent approaches are studied which has an adaptive control approach. The optimal SDRE controller/observer methods were designed to adjust the temperature and humidity using actuators of a motorized two-way valve and damper. Finally, the SDRE approach was compared with a Linear Quadratic Regulator (LQR) technique [7-9]. The SDRE and its differential version techniques are applied in [10, 11] to control nonlinear time-varying manipulators (flexible and rigid) using SDC parameterization. The SDRE approach for Ebola sickness is discussed to control the spread of virus cells in [12]. In [13], the DC micro-grid fault-tolerant regulation via the state-dependent Riccati equation techniques in a distributed network is examined.

In order to express in the solution of physical models and other issues involving functions of several variables such as fluid flow, elasticity, electrostatics, and electrodynamics, partial differential equations are introduced [14]. To solve the Partial Differential Equation (PDE) containing Dirichlet, Neumann, and Robin, initial/boundary conditions are investigated [15]. In order to regulate the boundary, a backstepping method is provided to the R-A-D equation [16]. In [17], stability analysis using the Lyapunov function is invoked for chemical reaction via partial differential equations. In [18], a T–S fuzzy model for the mobile actuator/sensor with nonlinear time-delay parabolic PDE system is offered. In [19], the existence, uniqueness, and asymptotic stability of traveling wave fronts in the nonlinear R-A-D equations with time-delay is established. In [20], delay-adaptive predictor feedback controller is presented for nonlinear R-A-D via a constant time-delay. This R-A-D system is examined as partial differential equations. For the global stability of the closed-loop system, the Lyapunov technique is investigated. In [21], the state feedback boundary controller of time fractional R-A-D equation with time-delay is considered via backstepping method. The design of the controller has been reviewed for the nonlinear R-A-D systems with normal time-delay based on recent researches. However, previous studies have not been examined the controller design involved with the SDRE technique for nonlinear R-A-D partial differential equations by a combination of normal or multiple time delays. The controller design for nonlinear R-A-D partial differential equations is very complicated because time-delay and nonlinear terms must be considered in controller strategy.

This article introduces a method called Extended State-Dependent Differential Riccati Equation (ESDDRE). The ESDDRE sub-optimal process is recommended for nonlinear R-A-D with time-delay. The mentioned R-A-D has partial differential equations. Currently, a design for pseudo-linearization of nonlinear R-A-D that has these conditions (partial differential equation systems and time-delay) was not proposed. So, an extended pseudo-linearization representation will be offered by using the ESDDRE technique. In the suggested method, the time-delay is assumed as retarded form, which can be seen in the state variables. The control effort obtained from the ESDDRE method is sub-optimal which uses the SDC matrices and optimality conditions. The Lyapunov function and also Poincaré inequality have been offered to prove the stability of L_2. In conclusion, simulations have been tested on the nonlinear R-A-D partial differential equation with single and multiple delays which confirming the ESDDRE's proper performance.

This article is divided into the following categories. A class of the nonlinear Reaction-Advection-Diffusion equations via multiple delays is considered, and the extended pseudo-linearization of R-A-D is defined in Section 2. The proposed sub-optimal controller design procedure with essential theorems for verifying the optimality and stability of the ESDDRE approach are explained in Section 3. Simulation results for two nonlinear R-A-D partial differential equations with one and two known and constant delays will be illustrated in Section 4. In Section 5, the conclusions are analysed.

2. Nonlinear Reaction-Advection-Diffusion Equation with Multiple Delays

A form of the nonlinear Reaction-Advection-Diffusion equations with multiple known and constant delays is defined as:

$$\begin{aligned} x_t(z,t) &= C_1 x_z(z,t) + C_2 x_{zz}(z,t) \\ &+ f\big(x(z,t), x(z,t-\tau_1), x(z,t-\tau_2), \dots, x(z,t-\tau_m), u(z,t)\big) \end{aligned} \quad (1)$$

In (1), $x(z,t) \in R^{n\times 1}$is the state vector. Also, $x(z,t) \in R^{n\times 1}$ is assumed as smooth according to $z$and t. $\tau_1, \tau_2, \dots, \tau_m$ are time delays which (2) assumed to be positive and constant. Moreover, $x_t(z,t)$ and $x_z(z,t)$ are the derivative of $x(z,t)$ with respect to t and z. Also, $x_{zz}(z,t)$ is the second derivative of $x(z,t)$ with respect to z. $f(.)$ is a nonlinear smooth function where $f(0, x(z,t-\tau_1), x(z,t-\tau_2), \dots, x(z,t-\tau_m), 0) = 0$. $u(z,t) \in R^{p\times 1}$ is the control input vector. $C_1 > 0$and $C_2 > 0$ are known matrices with appropriate dimensions. t_F is supposed fixed and defined as the ultimate (final) time. In (2), the initial condition is considered as follows:

$$x(z,t) = x_0(z) - max\{\tau_1, \dots, \tau_m\} \le t \le 0 \quad (2)$$

The homogeneous Dirichlet boundary condition is considered as:

$$\begin{aligned} &x(0,t) = 0, x_z(0,t) = 0 \\ &x(l,t) = 0, x_z(l,t) = 0 \end{aligned} \quad (3)$$

Moreover, $x(z,t_F) = x_F(z)$ called final state and it is considered as fixed.

A) Extended Pseudo-Linearization (EPL)

The nonlinear R-A-D equations with multiple delays (1) and affine input could be revised as follows:

$$\begin{aligned} x_t(z,t) &= C_1 x_z(z,t) + C_2 x_{zz}(z,t) \\ &+ f_A\big(x(z,t), x(z,t-\tau_1), x(z,t-\tau_2), \dots, x(z,t-\tau_m)\big) \\ &+ B\big(x(z,t), x(z,t-\tau_1), x(z,t-\tau_2), \dots, x(z,t-\tau_m)\big)u(z,t) \end{aligned} \quad (4)$$

The EPL display of (4) is written as:

$$\begin{aligned} x_t(z,t) &= C_1 x_z(z,t) + C_2 x_{zz}(z,t) \\ &+ A\big(x(z,t), x(z,t-\tau_1), x(z,t-\tau_2), \dots, x(z,t-\tau_m)\big)x(z,t) \\ &+ B\big(x(z,t), x(z,t-\tau_1), x(z,t-\tau_2), \dots, x(z,t-\tau_m)\big)u(z,t) \end{aligned}$$

where $A(.)$ is state-dependent matrix. $B(.)$ is input matrix of system. All time delays are considered in $A(.)$ and $B(.)$ matrices. To make it easier to write and follow equations, the notation is defined as

$$(\bar{x}) \triangleq \left(x(z,t), x(z,t-\tau_1), x(z,t-\tau_2), \dots, x(z,t-\tau_m)\right) \tag{5}$$

Proposition 1 [5]: Consider Ω is a limited and open sub-set of R^n Euclidean space. Also, it is including the origin such that $0 \in \Omega \subseteq R^n$. Remark $f_A\left(x(z,t), x(z,t-\tau_1), x(z,t-\tau_2), \dots, x(z,t-\tau_m)\right) \in C^m, m \geq 1$. At that point, there is at least one the EPL as follows:

$$\begin{aligned} &A\left(x(z,t), x(z,t-\tau_1), x(z,t-\tau_2), \dots, x(z,t-\tau_m)\right) \\ &= \int_0^1 \left.\frac{\partial f_A\left(x(z,t), x(z,t-\tau_1), x(z,t-\tau_2), \dots, x(z,t-\tau_m)\right)}{\partial x(t)}\right|_{x(t)=\eta x(t)} d\eta \end{aligned} \tag{6}$$

In (6), η is a dummy variable defined for the integration.

Proposition 2 [6]: Consider two EPL forms for $f_A(\bar{x})$ as $f_A(\bar{x}) = A_1(\bar{x})x(z,t)$ and $f_A(\bar{x}) = A_2(\bar{x})x(z,t)$. For any $\gamma \in R, A(\bar{x},\gamma)$, there is an EPL form as $A(\bar{x},\gamma) = \gamma A_1(\bar{x}) + (1-\gamma)A_2(\bar{x})$ displays enormous EPL forms of $f_A(\bar{x})$.

Proposition 3 [1, 13]: Consider $A(\bar{x})$ is continuous with regard to $x(z,t)$.

- The following definitions have the same meaning:
- The EPL (6) is a point-wise stabilizable parameterization of the nonlinear PDE system (1) in Ω with time-delay.
- The pair $\{A(\bar{x}), B(\bar{x})\}$ is stabilizable in the linear form for all $\bar{x} \in \Omega$.
- The state-dependent controllability matrix that is defined in (7) for all $\bar{x} \in \Omega$ is full rank.

$$\varphi_{cont}(\bar{x}) = [B(\bar{x}) | A(\bar{x})B(\bar{x}) | \dots | A^{n-1}(\bar{x})B(\bar{x})] \tag{7}$$

3. Extended State-Dependent Differential Riccati Equation Controller

The optimal control theory for systems with partial differential equations is more complicated than Ordinary Differential Equations (ODEs). For systems of PDEs, dynamic programming techniques, calculus of variations, and the principle of Pontryagin are presented. These methods depend on Euler-Lagrange, Hamilton-Jacobi-Bellman (HJB) equation, and Hamilton-Pontryagin [22, 23]. In the following sections, quadratic optimal control is introduced by proving the necessary optimality conditions for PDE systems. A quadratic cost function is presented as:

$$\begin{aligned} J &= \frac{1}{2}\int_0^{t_F}\int_0^{l}\left(x^T(z,t)Qx(z,t) + u^T(z,t)Ru(z,t)\right)dz\,dt \\ &+ \frac{1}{2}x^T(l,t_F)Fx(l,t_F) \end{aligned} \tag{8}$$

In (8) Q and $F$are gains that are considered as Positive Semi-Definite (PSD) matrices. R is defined as Positive Definite (PD) matrix. It should be noted, the mentioned gains have appropriate dimensions. Also, $u(z,t)$ regarded to the space of satisfactory controls showed by U. Similarly, J was assume as continuous and strictly convex function via quadratic. $u(z,t)$control law is the goal, which is the minimization of (8). Constraints are satisfied by considering the final state is fixed [22, 23].

Equation (9) demonstrates the function of Hamiltonian as:

$$\begin{aligned} &H(x, x_z, x_{zz}, u, p, z, t) = \\ &+\frac{1}{2}\left(x^T(z,t)Qx(z,t) + u^T(z,t)Ru(z,t)\right) \\ &+p^T(z,t)\begin{pmatrix} C_1 x_z(z,t) + C_2 x_{zz}(z,t) \\ +A(\bar{x})x(z,t) + B(\bar{x})u(z,t) \end{pmatrix} \end{aligned} \tag{9}$$

In (9), $p(z,t)$ is co-state vector. Thus, the optimal conditions are checked as follows:

$$\frac{\partial H(x, x_z, x_{zz}, u, p, z, t)}{\partial u} = 0 \tag{10}$$

which leads to

$$Ru(z,t) + B^T(\bar{x})p(z,t) = 0 \tag{11}$$

According to linearity and causality, the co-state$p(z,t)$introduced as:

$$p(z,t) = K(\bar{x})x(z,t) \tag{12}$$

In (12), $K(\bar{x})$ is considered as a PD matrix. Therefore, the control input is obtained by substituting (12) into (11) as:

$$u(z,t) = -R^{-1}B^T(\bar{x})K(\bar{x})x(z,t) \tag{13}$$

By calculating the derivative with regard to $t$from co-state, it is gained [22, 23]:

$$p_t(z,t) = -\left(\frac{\partial H}{\partial x} - \frac{\partial}{\partial z}\left(\frac{\partial H}{\partial x_z}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{\partial H}{\partial x_{zz}}\right)\right) \tag{14}$$

By replacing the derivatives, it is obtained:

$$\begin{aligned} &p_t(z,t) = \\ &-\left(Qx(z,t) + \left(\frac{\partial (A(\bar{x})x)}{\partial x}\right)^T p(z,t) + \left(\frac{\partial (B(\bar{x})u)}{\partial x}\right)^T p(z,t)\right) \\ &+C_1^T\left(\frac{\partial K(\bar{x})}{\partial z}x(z,t) + K(\bar{x})x_z(z,t)\right) \\ &-C_2^T\begin{pmatrix} \frac{\partial^2 K(\bar{x})}{\partial z^2}x(z,t) + 2\frac{\partial K(\bar{x})}{\partial z}x_z(z,t) \\ +K(\bar{x})x_{zz}(z,t) \end{pmatrix} \end{aligned} \tag{15}$$

which leads to:

$$\frac{\partial (A(\bar{x})x)}{\partial x} = A^T(\bar{x}) + \sum_{i=1}^{n} x_i \begin{bmatrix} \frac{\partial a_{1i}}{\partial x_1} & \cdots & \frac{\partial a_{1i}}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial a_{ni}}{\partial x_1} & \cdots & \frac{\partial a_{ni}}{\partial x_n} \end{bmatrix}^T \tag{16}$$

The result of (12) is calculated as follows:

$$\begin{aligned} p_t(z,t) &= K_t(\bar{x})x(z,t) + K(\bar{x})x_t(z,t) \\ &= K_t(\bar{x})x(z,t) + K(\bar{x})C_1x_z(z,t) \\ &+K(\bar{x})C_2x_{zz}(z,t) + K(\bar{x})A(\bar{x})x(z,t) \\ &-K(\bar{x})B(\bar{x})R^{-1}B^T(\bar{x})K(\bar{x})x(z,t) \end{aligned} \tag{17}$$

The following equation is attained by substituting (17) and (12) into (15):

$$\begin{aligned} &\left(\sum_{i=1}^{n} x_i \left(\frac{\partial A_i(\bar{x})}{\partial x}\right)^T K(\bar{x}) + \left(\frac{\partial (B(\bar{x})u)}{\partial x}\right)^T K(\bar{x})\right) x(z,t) \\ &+\left(C_2^T \frac{\partial^2 K(\bar{x})}{\partial z^2} - C_1^T \frac{\partial K(\bar{x})}{\partial z}\right) x(z,t) \\ &+\left(K(\bar{x})C_1 - C_1^T K(\bar{x}) + 2C_2^T \frac{\partial K(\bar{x})}{\partial z}\right) x_z(z,t) \\ &+\left(K(\bar{x})C_2 + C_2^T K(\bar{x})\right) x_{zz}(z,t) \\ &+\begin{pmatrix} K_t(\bar{x}) + A^T(\bar{x})K(\bar{x}) + K(\bar{x})A(\bar{x}) - \\ K(\bar{x})B(\bar{x})R^{-1}B^T(\bar{x})K(\bar{x}) + Q \end{pmatrix} x(z,t) \\ &= 0. \end{aligned} \tag{18}$$

The state vector will eventually reach to zero in a stable closed-loop system. Also, $x_z(z,t)$and $x_{zz}(z,t)$ with regard to z reach to zero. It is assumed that $B(\bar{x})u(z,t)$ and $f(\bar{x})$ are Lipshitz in (4). In the steady state, $\left(\frac{\partial (B(\bar{x})u)}{\partial x}\right)^T$ and $\sum_{i=1}^{n} x_i \left(\frac{\partial A_i(\bar{x})}{\partial x}\right)^T$ are bounded and reach to zero. Thus, matrix $K$and $K_t$will converge to zero with regard to z. Furthermore, $K(\bar{x})C_1 - C_1^T K(\bar{x}), K(\bar{x})C_2 + C_2^T K(\bar{x})$, $C_2^T \frac{\partial^2 K(\bar{x})}{\partial z^2} - C_1^T \frac{\partial K(\bar{x})}{\partial z}$, and $2C_2^T \frac{\partial K(\bar{x})}{\partial z}$similarly are bounded. Some of the elements indicated in (18) are omitted in this research in order to construct a sub-optimal controller [22, 23]. Finally, the ESDDRE is created as follows:

$$\begin{aligned} &K_t(\bar{x}) + A^T(\bar{x})K(\bar{x}) + K(\bar{x})A(\bar{x}) \\ &-K(\bar{x})B(\bar{x})R^{-1}B^T(\bar{x})K(\bar{x}) + Q = 0, \end{aligned} \tag{19}$$

that $K(\bar{x}_F) = F$.

A) Stability Analysis of Closed-Loop System

The stability of the closed-loop system is checked by the following theorem.

Theorem 1: The affine nonlinear R-A-D equation (4) with the point-wise controllable extended pseudo-linearization form (5), $C_2 > 0$, and the signal input (13) is concluded stable according to the ESDDRE (19).

Proof: By replacing the control effort (13) into the EPL display (5), the following closed-loop system is computed:

$$x_t(z,t) = C_1x_z(z,t) + C_2x_{zz}(z,t) + A_{CL}(\bar{x})x(z,t) \tag{20}$$

where $A_{CL}(\bar{x}) = A(\bar{x}) - B(\bar{x})R^{-1}B^T(\bar{x})K(\bar{x})$.

The Lyapunov function (21) assumed as:

$$V = \int_0^l x^T(z,t)K(\bar{x})x(z,t)dz \tag{21}$$

where $K(\bar{x})$ is the positive definite matrix solution of the ESDDRE (19), hence the Lyapunov function is positive definite. Time-derivative of the Lyapunov candidate produces as:

$$V_t = \int_0^l (x^T K(\bar{x})x_t + x_t^T K(\bar{x})x + x^T K_t(\bar{x})x)dz. \tag{22}$$

Then, by substituting (20) into (22) and employing chain rule, the following is reached:

$$\begin{aligned} V_t &= x^T K(\bar{x})C_1 x|_0^l + x^T K(\bar{x})C_2 x_z|_0^l \\ &- \int_0^l x_z^T K(\bar{x})C_2 x_z dz + \int_0^l x^T K_t(\bar{x})x dz. \\ &+x_z^T C_2^T K(\bar{x})x|_0^l - \int_0^l x_z^T C_2^T K(\bar{x})x_z dz \\ &+ \int_0^l x^T[K(\bar{x})A_{CL}(\bar{x}) + A_{CL}^T(\bar{x})K(\bar{x})]x dz \end{aligned} \tag{23}$$

Applying the definition of $A_{CL}(\bar{x})$ and (3), it is concluded:

$$\begin{aligned} V_t &= -\int_0^l x_z^T[K(\bar{x})C_2 + C_2^T K(\bar{x})]x_z dz \\ &+ \int_0^l x^T \begin{bmatrix} K_t(\bar{x}) + K(\bar{x})A(\bar{x}) \\ -2K(\bar{x})B(\bar{x})R^{-1}B^T(\bar{x})K(\bar{x}) \\ +A^T(\bar{x})K(\bar{x}) \end{bmatrix} x dz \end{aligned} \tag{24}$$

According to the ESDDRE (19), the $V_t$analyses to

$$\begin{aligned} V_t &\le -\int_0^l x_z^T[K(\bar{x})C_2 + C_2^T K(\bar{x})]x_z dz \\ &- \int_0^l x^T[Q + K(\bar{x})B(\bar{x})R^{-1}B^T(\bar{x})K(\bar{x})]x dz \end{aligned} \tag{25}$$

In (25), second term on right-hand side is negative. Now for L_2 stability, because $C_2 > 0$and $K(\bar{x}) \ge 0$. So, it is concluded:

$$\begin{aligned} &x_z^T[K(\bar{x})C_2 + C_2^T K(\bar{x})]x_z \\ &\qquad \ge 2\lambda_{min}\ (C_2)x_z^T K(\bar{x})x_z \end{aligned} \tag{26}$$

Therefore, (25) can be revised as:

$$V_t \le -2\lambda_{min}\ (C_2)\int_0^l x_z^T K(\bar{x})x_z dz. \tag{27}$$

The Poincaré inequality is introduced as (28) [23-25]:

$$\begin{aligned} &\int_0^l x^T x dz \le 2x^T(0)\,x(0) + 4\int_0^l x_z^T x_z dz \\ &\int_0^l x^T x dz \le 2x^T(l)\,x(l) + 4\int_0^l x_z^T x_z dz \end{aligned} \tag{28}$$

Applying zero boundary conditions (3), the Poincaré inequality in (27), and $K(\bar{x}) \ge 0$leads to the following conclusion:

$$V_t \le -\frac{2}{4}\lambda_{min}\ (C_2)\int_0^l x^T K(\bar{x})xdz = -\frac{1}{2}\lambda_{min}\ (C_2)V \tag{29}$$

and then

$$V(t) \le V(0)e^{-\frac{\lambda_{min}\ (C_2)}{2}t}, \tag{30}$$

and $\|x\| \le \|x(0)\|e^{-\frac{\lambda_{min}\ (C_2)}{2}t}$.

Consequently, the stability of the closed-loop system in L_2 is guaranteed when (19) is computed. The state vector of the system reaches to zero by considering the coefficient K (19) in the ESDDRE approach. Also, the cost function (8) will be minimized. As a result, stability and optimization are attained.

Remark 1: Various approaches that solve the differential Riccati equation are demonstrated in [26–29]. Furthermore, Riccati commands and several MATLAB toolboxes, including care (for steady-state mode), odericcati, and Matrix Equation Sparse Solver (MESS), have been implemented to solve it. The design of the suggested method is shown in multiple steps by Algorithm 1 in the following. The block diagram of the ESDDRE controller design is shown in Fig. 1.

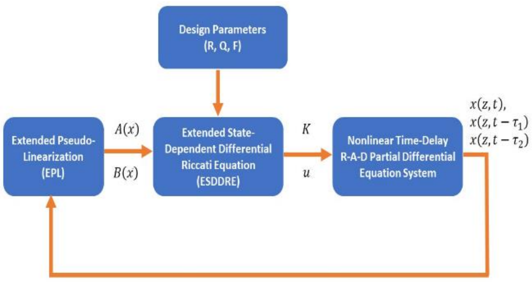

Fig. 1. Block diagram of the ESDDRE technique

Algorithm:

- Step 1: Select an EPL form (5) that satisfies controllability matrix for all $x, \bar{x}$ is full rank.
- Step 2: Choose matrices R, Q, and F.
- Step 3: Calculate the ESDDRE (19).
- Step 4: Compute the input signal (13).

4. Simulation Results

This part will be explained the simulation outcomes of the suggested method for two nonlinear R-A-D partial differential equations with known and constant delays. These examples can represent the real-world application in industrial systems with nonlinear time-delay R-A-D equations such as chemical reactors (packed-bed reactors, plug-flow reactors, continuous-stirred tank reactors, …), catalytic rod, prey-predator equation, Lotka-Volterra, three-beam laser welding, two-stage anaerobic digestion, population growth rate and etc [16, 17, 20, 21].

Example 1: Consider a nonlinear R-A-D equation with one known and constant delay as follows:

$$\begin{cases} \frac{\partial x_1(z,t)}{\partial t} = \frac{\partial^2 x_1(z,t)}{\partial z^2} - \frac{\partial x_1(z,t)}{\partial z} \\ \quad -0.001e^{10\frac{x_1(z,t-\tau)}{1+x_1(z,t-\tau)}}x_2(z,t) \\ \quad +10x_1(z,t-\tau)x_2(z,t) \\ \frac{\partial x_2(z,t)}{\partial t} = \frac{\partial^2 x_2(z,t)}{\partial z^2} - \frac{\partial x_2(z,t)}{\partial z} \\ \quad -0.001e^{10\frac{x_1(z,t-\tau)}{1+x_1(z,t-\tau)}}x_2(z,t) \\ \quad -x_1(z,t)+u(z,t), \end{cases} \tag{31}$$

with zero boundary condition and the following initial conditions:

$$\begin{aligned} x_1(z,t) &= 1 + sin\ \ (2\pi z), \quad -\tau \le t \le 0 \\ x_2(z,t) &= cos\ \ (1.5\pi z), \quad -\tau \le t \le 0 \end{aligned} \tag{32}$$

where $\tau = 0.2s$ is constant time-delay, $z \in [0,1]$, and $t \in [0,2]$. These are the choices for the EPL presentation:

$$A(\bar{x}) = \begin{bmatrix} 0 & -0.001e^{10\frac{x_1(z,t-\tau)}{1+x_1(z,t-\tau)}} + 10x_1(z,t-\tau) \\ -1 & -0.001e^{10\frac{x_1(z,t-\tau)}{1+x_1(z,t-\tau)}} \end{bmatrix} \\ B(\bar{x}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{33}$$

Form (33) is controllable for all $\bar{x}$. The controllability matrix is as follows, which has full rank.

$$\varphi_{cont}(\bar{x}) = [B(\bar{x})|A(\bar{x})B(\bar{x})] \\ = \begin{bmatrix} 0 & -0.001e^{10\frac{x_1(z,t-\tau)}{1+x_1(z,t-\tau)}} + 10x_1(z,t-\tau) \\ 1 & -0.001e^{10\frac{x_1(z,t-\tau)}{1+x_1(z,t-\tau)}} \end{bmatrix} \tag{34}$$

The controller parameters of the ESDDRE method are selected as $F = I$, $R = 1$, and $Q = 10I$. The stability condition of L_2 is satisfied because the value of $C_2 = I$ is positive. Furthermore, to solve the nonlinear R-A-D equation, the numerical and approximate method is applied. In this strategy, derivative approximation is used and then the nonlinear discrete state space model is edited and rewritten according to the $z$parameters [16, 17, 23]. The closed-loop answer of states and mesh form are shown for several z in Fig. 2 to 5. The control input is illustrated in Fig. 6. The proposed controller has a suitable response in terms of speed convergence and accuracy.

Example 2: Consider a nonlinear R-A-D equation with two known and constant delays as follows:

$$\begin{cases} \frac{\partial x_1(z,t)}{\partial t} = \frac{\partial^2 x_1(z,t)}{\partial z^2} - \frac{\partial x_1(z,t)}{\partial z} \\ \quad + x_1(z,t-\tau_1)x_2^2(z,t) \\ \quad + \sin\left(x_2(z,t-\tau_2)\right)x_1(z,t) + u_1(z,t) \\ \\ \frac{\partial x_2(z,t)}{\partial t} = \frac{\partial^2 x_2(z,t)}{\partial z^2} - \frac{\partial x_2(z,t)}{\partial z} \\ \quad - x_1(z,t)\left(1 + x_2^2(z,t-\tau_2)\right) + u_2(z,t), \end{cases} \quad (35)$$

with zero boundary condition and the following initial conditions:

$$x_1(z,t) = -cos(2.5\pi z), \quad -max\{\tau_1,\tau_2\} \leq t \leq 0 \quad (36)$$

where $\tau_1 = 0.1s$ and $\tau_2 = 0.2s$ is constant time-delay, $z \in [0,1]$ and $t \in [0,2]$. The EPL presentation is defined as follows:

$$A(\bar{x}) = \begin{bmatrix} sin(x_2(z,t-\tau_2)) & x_1(z,t-\tau_1)x_2(z,t) \\ -(1 + x_2^2(z,t-\tau_2)) & 0 \end{bmatrix} \quad (37)$$

$$B(\bar{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For all $\bar{x}$, form (37) is controllable. The controllability matrix is explained as follows that has full rank.

$$\varphi_{cont}(\bar{x}) = [B(\bar{x})|A(\bar{x})B(\bar{x})] \quad (38)$$

$$= \begin{bmatrix} 0 & -0.001e^{10\frac{x_1(z,t-\tau)}{1+x_1(z,t-\tau)}} + 10x_1(z,t-\tau) \\ 1 & -0.001e^{10\frac{x_1(z,t-\tau)}{1+x_1(z,t-\tau)}} \end{bmatrix}$$

The controller parameters of the ESDDRE approach are chosen as $F = I$, $R = I$, and $Q = 10I$. The L_2 stability condition is satisfied because the value of $C_2 = I$ is positive. In Fig. 7 to 10, the closed-loop response of states and mesh form are demonstrated for different z. In Fig. 11, the control inputs are displayed. Simulation results illustrate that the suggested controller answers appropriately in terms of speed convergence and accuracy with multiple delays. From the advantages and disadvantages of the ESDDRE controller design, the following items can be mentioned:

Advantages

- Nonlinear features of PDE system are preserved.
- Time-delay (single and multiple) is considered.
- The degrees of freedom of ESDDRE controller are infinite.
- The ESDDRE method is sub-optimal.
- The stability of ESDDRE controller is L_2 .

Disadvantages:

- The design process of the ESDDRE controller is very complex with a time-delay and PDE.
- The simulation time of ESDDRE controller design is high for the PDE systems with time-delay.
- Parametric uncertainties are not considered.

5. Conclusion

In this research, a nonlinear sub-optimal controller named extended state-dependent differential Riccati equation invokes for the Reaction-Advection-Diffusion partial differential equation with single and multiple delays. First, an extended pseudo-linearization presentation employing the method of state-dependent coefficients was presented for the nonlinear R-A-D. All multiple delay sections in this presentation are assumed as the retarded type, which are positioned in the matrices of system and input vectors. By using nonlinear R-A-D/PDE systems to describe a performance index and Hamiltonian equation, the control effort regarded on the ESDDRE is obtained. The Poincaré inequality and Lyapunov function, both considered in the ESDDRE control technique, ensure the closed-loop system's L_2 stability. The suggested ESDDRE controller methodology is effective according to simulation findings for the nonlinear R-A-D partial differential equation with one and two constant delays.

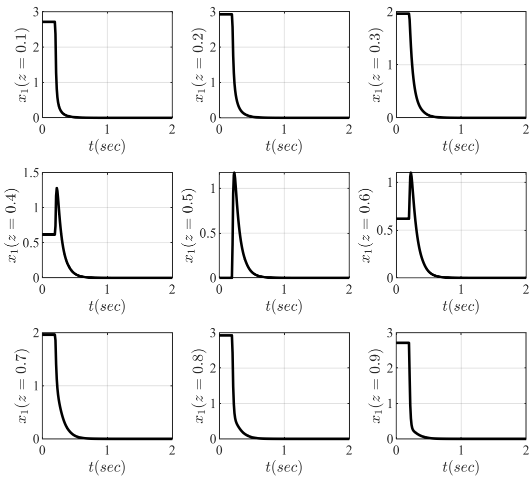

Fig. 2. State x_1 for different $z$in Example 1

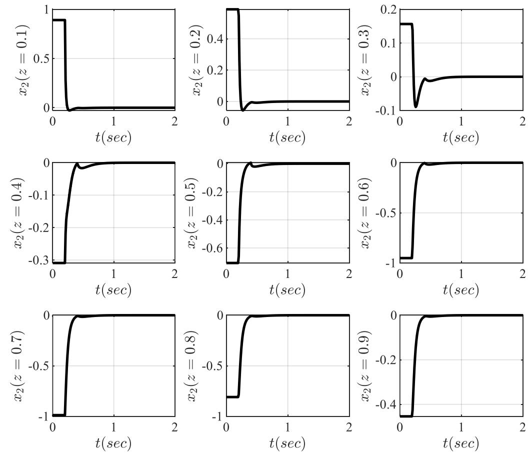

Fig. 3. State x_2 for different $z$in Example 1

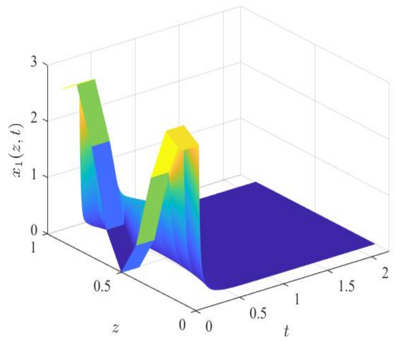

Fig. 4. State x_1 in Mesh form in Example 1

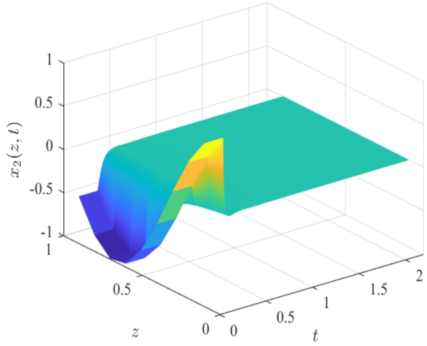

Fig. 5. State x_2 in Mesh form in Example 1

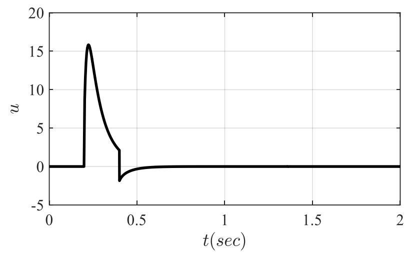

Fig. 6. Control input u in Example 1

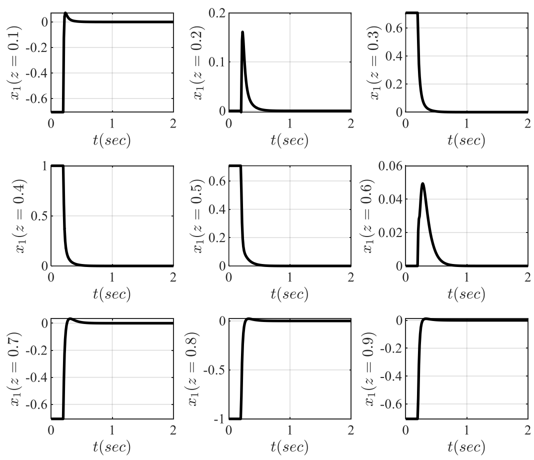

Fig. 7. State x_1 for different $z$in Example 2

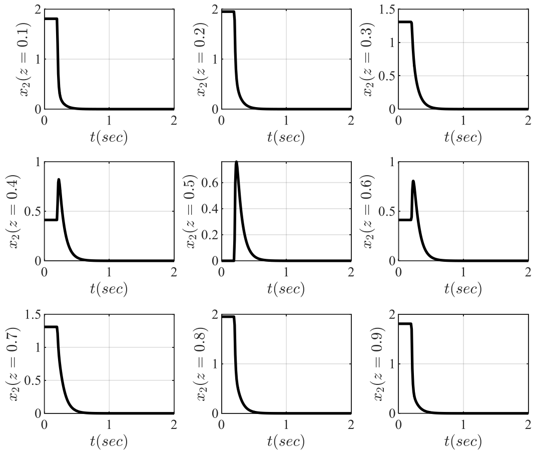

Fig. 8. State x_2 for different $z$in Example 2

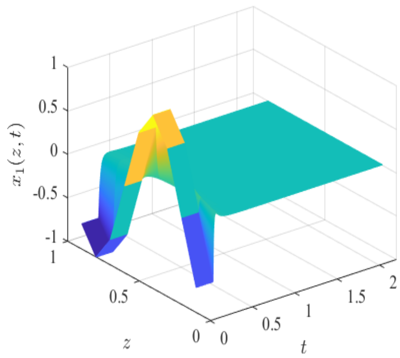

Fig. 9. State x_1 in Mesh form in Example 2

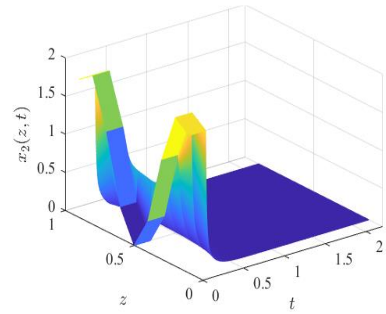

Fig. 10. State x_2 in Mesh form in Example 2

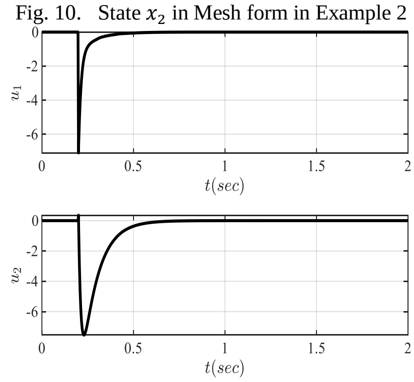

Fig. 11. Control inputs u_1 and $u_2$in Example 1

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