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Heat Conduction in Spherical Composite Vessels

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Abstract: This paper presents an analytical solution for two-dimensional conductive heat transfer in spherical composite pressure vessels. The vessels are in a spherical shape and fibers are winded in circumferential direction. The vessel is made of one-layer reinforced composite material. The analytical solution is obtained under the general boundary conditions which consist of convection, conduction and radiation inside/outside of vessel. The heat transfer equation for orthotropic conduction in spherical coordinates is derived and solved using separation of variables method based on the Legendre and Euler functions. Here, the effect of fiber's angle on heat conduction in orthotropic spherical pressure vessels is investigated in detail. These results can be used extensively for analyzing the thermal stress in this kind of vessels.

Keywords: Exact Analytical Solution, Legendre Function, Spherical Composite Vessel, Steady Heat Conduction

1. Introduction

Heat conduction in composite materials is particularly important for preventing thermal fracture, analyzing fiber placement in production processes, and controlling directional heat transfer through laminates by varying the angles and materials of the fibers. The problem of heat conduction in composite structures can be subdivided into: heat conduction in cartesian coordinate [1-5], cylindrical coordinate in $r-z$ [6-9], and $r-\varphi$ [10,11] directions and heat transfer in spherical shapes. Only few studies [12,13] have considered heat conduction of spherical layered materials ($r-\theta$). In these studies, the layer materials are isotropic in each layer. In this study, exact analytical solution for heat conduction in spherical composite vessel has been presented for the first time. Vessel is in spherical shape and fibers are winded in a circumferential direction. It is supposed that the vessel is made of one composite layer. The boundary conditions are the general linear boundary conditions which can be simplified to all mechanisms of heat transfer at the inside/outside of vessels. Analytical solution has been derived based on the separation method of variables. The effect of fibers' angle on temperature distribution in composite vessel has been investigated in details.

2. Heat Conduction in Spherical Composite

In this paper steady heat conduction in spherical composite laminate is investigated. ψ is the angle between fiber and horizontal

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direction and $r_o < r < r_{nl}, 0 < \theta < \pi, 0 < \varphi < 2\pi$. The Fourier low in spherical direction for orthotropic material will be:

$$\begin{pmatrix} q_r \\ q_\theta \\ q_\phi \end{pmatrix} = -\begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix} \begin{pmatrix} \dfrac{\partial T}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial T}{\partial \theta} \\ \dfrac{1}{r\sin\theta}\dfrac{\partial T}{\partial \phi} \end{pmatrix} \quad (1)$$

where k is the conductive heat transfer coefficient, T is the temperature and q is the heat flux. In order to study the heat conduction in composite laminates, two different coordinate systems, on-axis and off-axis, should be defined [7]. Off-axis coordinate system is defined to study the thermal properties in unique directions. According to thermodynamic reciprocity, the tensor of conductive heat coefficient should be symmetric:

$$k_{ij} = k_{ji} \quad (2)$$

On the other hand, the second law of thermodynamics caused that the diametric elements of this tensor be positive so the following relation must be satisfavtory:

$$k_{ii} k_{jj} > k^2_{ij} \qquad for \qquad i \neq j \quad (3)$$

Using the Clausius-Duhem inequality, the following inequalities for the conductive coefficients of orthotropic materials are achieved:

$$k_{(ii)} \geq 0 \quad (4a)$$

$$\frac{1}{2}(k_{(ii)}k_{(jj)} - k_{(ji)}k_{(ij)}) \geq 0 \quad (4b)$$

$$\varepsilon_{ijk} k_{(1j)} k_{(2j)} k_{(3j) \geq 0} \quad (4c)$$

where k_{ij} represents the symmetric part of tensor:

$$k_{(ij)} = k_{(ji)} = \frac{k_{ij} + k_{ji}}{2} \quad (5)$$

The heat conduction coefficients can be directly obtained from experimental measurements or be calculated based on the theoretical models. Applying the balance of energy in the elements of a sphere which has been shown in Fig. 1, the following equation will be achieved:

$$\frac{\partial q_\phi dA_\phi}{\partial \phi} d\phi + \frac{\partial q_\theta dA_\theta}{\partial \theta} d\theta + \frac{\partial q_r dA_r}{\partial r} dr = \rho c_p \frac{\partial T}{\partial t} dv \quad (6)$$

Surface areas and volume of sphere element are as follows:

$$\begin{aligned} dA_r &= r^2 \sin\theta d\phi d\theta \\ dA_\theta &= r\sin\theta d\phi d\theta \\ dA_\phi &= rd\theta dr \\ dv &= r^2 \sin\theta dr d\phi d\theta \end{aligned} \quad (7)$$

Quantities ρ and c_p in Eq. (6) are the density and specific heat capacity at constant pressure, respectively. Substituting Eq. (1) and Eq. (7) into Eq. (6) will result in:

$$\begin{aligned} &\bar{k}_{11} \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial T}{\partial r}) + \bar{k}_{22} \frac{1}{r^2 \sin\theta} \frac{\partial^2 T}{\partial \Phi^2} + \\ &\bar{k}_{33} \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \frac{\partial T}{\partial \theta}) + \frac{(\bar{k}_{12} + \bar{k}_{21})}{r\sin\theta}\frac{\partial^2 T}{\partial r \partial \phi} \\ &+ \bar{k}_{12} \frac{1}{r^2 \sin\theta}\frac{\partial T}{\partial \Phi} + \frac{(\bar{k}_{13} + \bar{k}_{31})}{r}\frac{\partial^2 T}{\partial r \partial \theta} + \\ &\bar{k}_{13}\frac{1}{r^2}\frac{\partial T}{\partial \theta} + (\bar{k}_{32} + \bar{k}_{23})\frac{1}{r^2 \sin\theta}\frac{\partial^2 T}{\partial \theta \partial \Phi} \\ &+ \bar{k}_{31}\frac{\cos\theta}{r\sin\theta}\frac{\partial T}{\partial r} = \rho C_p \frac{\partial T}{\partial t} \end{aligned} \quad (8)$$

Here, steady-state conductive heat transfer in the r and θ directions are considered as well. Thus, Eq. (8) can be simplified to:

$$\begin{aligned} &\bar{k}_{11}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial T}{\partial r}) + \\ &\bar{k}_{22}\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \frac{\partial T}{\partial \theta}) = 0 \end{aligned} \quad (9)$$

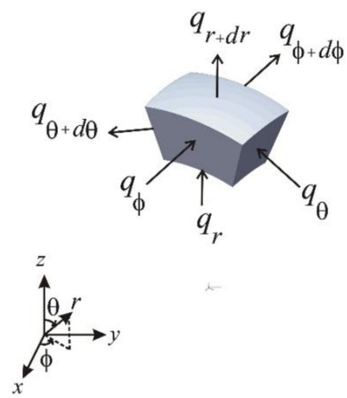

Fig. 1. Spherical Element.

In order to simplify formulations, the following terms are also considered:

$$\begin{cases} \overline{k_{11}} = k_{22} \\ \overline{k_{22}} = (C\,\text{os}\Psi)^2 k_{11} + (Sin\Psi)^2 k_{22}, \end{cases} \tag{10}$$

where ψ is the angle between the tangent line to fibers and the tangent line to Sphere in θ direction. Substituting the determined off-axis coefficients (Eq. 10) into energy equation (Eq. 9) results in:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial T}{\partial r})+\frac{1}{\mu^2}\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial T}{\partial\theta})=0 \tag{11}$$

where parameter μ is given by:

$$\mu_i = \sqrt{\frac{k_{22}}{m_l^2 k_{11} + n_l^2 k_{22}}} \tag{12}$$

The general linear boundary conditions inside and outside the sphere are in the following forms which can cover a wide range of applicable thermal conditions:

$$a_1 T(r_0,\theta) + b_1 \frac{\partial T}{\partial r}(r_0,\theta) = f_1(\theta) \tag{13a}$$

$$a_1 T(r_0,\theta) + b_1 \frac{\partial T}{\partial r}(r_0,\theta) = f_1(\theta) \tag{13b}$$

Note that $f_1(\theta)$, $f_2(\theta)$ are the arbitrary functions, the constants a_1, a_2 have the same dimension as convection coefficient and b_1, b_2 have the same dimension as conduction coefficient.

3. Analytical Solution

In this section, the analytical solution of steady temperature distribution under generalized linear boundary conditions is presented based on the separation of variable methods. By applying the separation of variable method on Eq. (11), the temperature distribution could be separated as two independent functions $R(r)$ and $\Theta(\theta)$:

$$T(r,\theta) = R(r)\Theta(\theta) \tag{14}$$

Substituting Eq. (14) into the Eq. (11), heat conduction equation has been separated as:

$$(r^2\frac{R''}{R}+2r\frac{R'}{R}) = -\frac{1}{\mu^2}(\frac{\cos\theta}{\sin\theta}\frac{\dot{\Theta}}{\Theta}+\frac{\ddot{\Theta}}{\Theta}) = \lambda \tag{15}$$

here λ is a constant. By supposing $x=\sin\theta$, the separated equation in θ direction can be solved as a Legendre equation:

$$\frac{\partial}{\partial x}(1-x^2\frac{\partial\Theta}{\partial x}) + n(n+1)\Theta = 0 \tag{16}$$

The solution of Eq. (16) is as follows:

$$\Theta(\theta) = \sum A_n P_n \left(Cos\,\theta\right) \tag{17}$$

where P_n indicates the Legendre function of degree n and order one, and A_n is the coefficient of Legendre series. Comparing Eq. (15) and Eq. (16), λ will be achieved as follows:

$$\lambda = \frac{n(n+1)}{\mu^2} \tag{18}$$

According to Eq. (15), the separated equation in r direction is an Euler equation with the following solution:

$$R_n(r) = \begin{cases} B_n r^{\frac{n}{\mu^2}} + C_n r^{\frac{-(n+1)}{\mu^2}} & for\ n \geq 1 \\ B_0 \ln r + C_0 & for\ n = 0 \end{cases} \tag{19}$$

The temperature distribution will be:

$$T(r,\theta) = (a_0 \ln(\frac{r}{r_{nl}}) + b_0) P_0(\cos\theta) + \sum_{n=1}^{\infty} \begin{pmatrix} (a_n (\frac{r}{r_{nl}})^{\frac{n}{\mu^2}} \\ +b_n (\frac{r}{r_{nl}})^{\frac{-(n+1)}{\mu^2}}) \end{pmatrix} P_n(\cos\theta) \tag{20}$$

Where:

$$a_\chi = A_\chi B_\chi \qquad \chi = 0, n \qquad b_\chi = A_\chi C_\chi \tag{21}$$

Applying the inside and outside boundary conditions in the direction of r the coefficients a_0, b_0, a_n, b_n are obtained as follows:

$$\begin{pmatrix} \left(a_1 \ln(\frac{r_0}{r_{nl}}) + b_1(\frac{1}{r_0})\right) \\ a_0^{(1)} + a_1 b_0^{(1)} \end{pmatrix} P_0(\cos\theta)$$

$$\begin{pmatrix} \left(a_1 r_0^{\frac{n}{\mu_m}} + b_1 \frac{n}{\mu_0^2} r_0^{\frac{n}{\mu_0^2}-1}\right) a_n^{(1)} + \\ \left(a_2 r_0^{\frac{-(n+1)}{\mu_0^2}} + b_2 \frac{-(n+1)}{\mu_0^2} r_0^{\frac{-(n+1)}{\mu_0^2}-1}\right) \end{pmatrix} P_n(\cos\theta) = f_1(\theta) \tag{22a}$$

$$\begin{pmatrix} \frac{b_2}{r_{nl}} a_0^{(n_l)} + \\ a_2 b_0^{(n_l)} \end{pmatrix} P_0(\cos\theta) + \sum_{n=1}^{\infty} \begin{pmatrix} \begin{pmatrix} a_2 r_{nl}^{\frac{n}{\mu_{n_l}^2}} \\ +b_2 \frac{n}{\mu_m^2} r_{nl}^{\frac{n}{\mu_{n_l}^2}-1} \end{pmatrix} a_n^{(n_l)} + \\ \begin{pmatrix} a_2 r_{nl}^{\frac{-(n+1)}{\mu_{n_l}^2}} + \\ b_2 \frac{-(n+1)}{\mu_{n_l}^2} r_{nl}^{\frac{-(n+1)}{\mu_{n_l}^2}-1} \end{pmatrix} b_n^{(n_l)} \end{pmatrix} P_n(\cos\theta) = f_2(\theta) \tag{22b}$$

Using the existing relations for orthogonal Legendre functions and rearranging the Eqs. (22a) and (22b), the unknown coefficients will be achieved.

- Resorting Eq. (22a) results in:

$$\left(a_1 \ln(\frac{r_0}{r_{nl}}) + b_1(\frac{1}{r_0})\right) a_0^{(1)} + a_1 b_0^{(1)} = F_0^0,$$

$$\left(a_1 r_0^{\frac{n}{\mu_m^2}} + b_1 \frac{n}{\mu_0^2} r_0^{\frac{n}{\mu_0^2}-1}\right) a_n^{(1)} + \begin{pmatrix} a_2 r_0^{\frac{-(n+1)}{\mu_0^2}} \\ +b_2 \frac{-(n+1)}{\mu_0^2} r_0^{\frac{-(n+1)}{\mu_0^2}-1} \end{pmatrix} b_n^{(1)} = F_n^0 \tag{23a}$$

- Similarly, Eq. (22b) results in:

$$\frac{b_2}{r_{nl}} a_0^{(n_l)} + a_2 b_0^{(n_l)} = F_0^{n_l}$$

$$\left(a_2 r_{nl}^{\frac{n}{\mu_m^2}} + b_2 \frac{n}{\mu_m^2} r_{nl}^{\frac{n}{\mu_m^2}-1} \right) a_n^{(n_l)} + \left(a_2 r_{nl}^{\frac{-(n+1)}{\mu_m^2}} + b_2 \frac{-(n+1)}{\mu_m^2} r_{nl}^{\frac{-(n+1)}{\mu_m^2}-1} \right) b_n^{(n_l)} = F_n^{n_l} \quad (23b)$$

Where:

$$F_\chi^0 = \frac{2n+1}{2} \times \int_0^\pi f_1(\theta) P_\chi(\cos(\theta)) \sin(\theta) d\theta \qquad \chi = 0, n$$

$$F_\chi^{n_l} = \frac{2n+1}{2} \times \int_0^\pi f_2(\theta) P_\chi(\cos(\theta)) \sin(\theta) d\theta \quad (24)$$

Eqs. (23a), and (23e) should be solved to determine the coefficients a_n and b_n.

4. Results and discussion

In this section, the researchers examined the ability of the current analytical solution to find the temperature distribution in composite vessels. The result of Arpaci [15] solution for the heat conduction in isotropic sphere is employed to validate the result of this paper. The results are completely coincident as shown in Fig. 2.

The geometry and environment para-meters of vessels are as follows:

$r_{nl} = 0.5mm,\ r_0 = 0.1mm,$

$T_\infty = 200^\circ C,\ h = 1W/m^2,$

$f_2(\theta) = 100 \sin(\theta),$

$k_{11} = 910W/m^{20}C,$

$$k_{22} = 514W/m^{20}C \quad (25)$$

Fig. 3. shows the variation of vessel temperature in radial direction for different values of θ angle under a specified angle $\psi = 90^\circ$; as it is seen from this figure, the temperature of laminate is higher in the case of $\theta = 90^\circ$ and decreases when the θ angle is near to $\theta = 0^\circ$.

The variation of laminate temperature respect to radius of the vessel is presented in Fig. 4. This figure is depicted for various ψ angle under the specific case of $\theta = 45^\circ$. Regarding this figure, the temperature of the vessel decreases with the growth of cone angle.

5. Conclusions

This paper presents an analytical solution for heat transfer in spherical composite vessels. This exact solution is derived based on an separation of variables method. Regarding to the exact analytical method which is used here, the results can be employed for verifying the numerical solution in this field. The temperature distribution in spherical laminates has an important role in analyzing of thermal stresses in spherical vessels. Using the exact results of this paper, the effects of fibers' placement angle of composite material on temperature distribution of reinforced composite vessel can be investigated as well as possible.

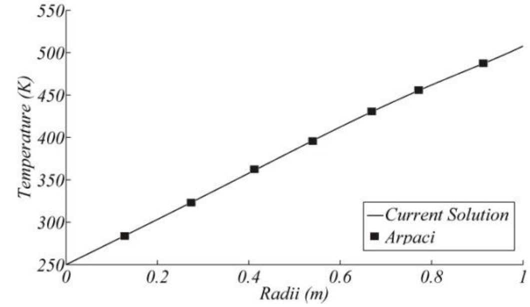

Fig. 2. Temperature distribution in an isotropic sphere.

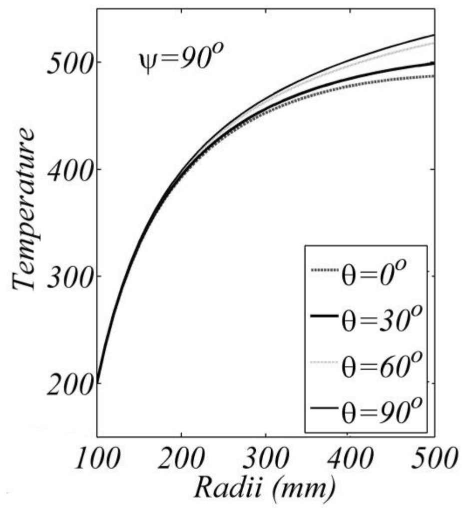

Fig. 3. Temperature distribution of the composite for different set of θ .

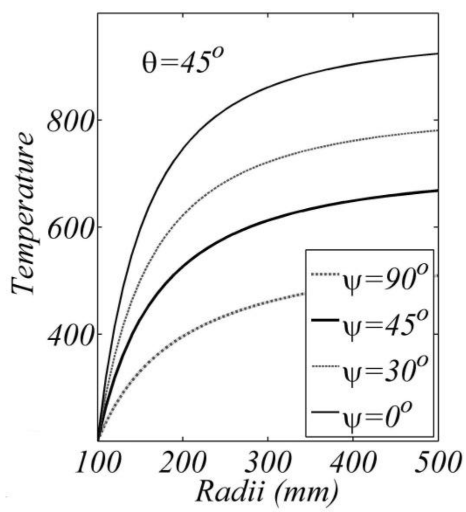

Fig. 4. Temperature distribution of the composite for different set of ψ .

References

[1] Salt, H., "Transient conduction in a two-dimensional composite slab-I. Theoreticaldevelopment of temperature modes", Int. J. Heat and Mass Transfer, 1983, 26, 1611–1616.

[2] Salt, H., "Transient conduction in a two-dimensional composite slab- II. Physicalinterpretation of temperature modes", Int. J. Heat and Mass Transfer, 1983, 26, 1617–1623.

[3] de Monte, F., "transient heat conduction in one-dimensional composite slab. a "natural" analytic approach.", int. j. heat and mass transfer, 2000, 43, 3607–3619.

[4] de Monte, F., "an analytic approach to the unsteady heat conduction processes in onedimensionalcomposite media", int. j.

heat and mass transfer, 2002, 45, 1333–1343.

[5] de Monte, F.,"unsteady heat conduction in two-dimensional two slab-shaped regions.exact closed-form solution and results", int. j. heat and mass transfer 2003, 46, 1455–1469.

[6] Kayhani, M.H.;Norouzi, M.; AmiriDelouei A.,"A general analytical solution for heatconduction in cylindrical multilayer composite laminates", Int. J. Therm. Sci.,2012, 52, 73–82.

[7] Kayhani, M.H.; Norouzi M.;AmiriDelouei A., "An exact solution of axi-symmetricconductive heat transfer in cylindrical composite laminate under the general boundarycondition", Word Acad. Sci. Eng. Technol., 2010, 69, 55–62.

[8] Kayhani, M. H.; Norouzi M.; Amiri Delouei A.,"on heat conduction problem in multi–layer composite pin fins", proceedings of third international conference of advancedcomposite materials engineering (comat), brasov, romani, 2010, 89 -94.

[9] AmiriDelouei, A.;Kayhani, M.H.;Norouzi M.,"exact analytical solution of unsteady axi-symmetric conductive heat transfer in orthotropic cylindrical composite laminates",int. j. heat and mass transfer, 2012, 55, 4427–4436.

[10] Norouzi, M.;RezaeiNiya, S.M.;Kayhani, M. H.; KarimiDemneh, M.; Naghavi, M. S., "Exactsolution of unsteady conductive heat transfer in cylindrical composite laminates", J. HeatTransfer (ASME), 2012, 134, 101301.

[11] Kayhani, M.H.;Shariati, M.;Norouzi, M.;KarimiDemneh, M.,"Exact solution of conductiveheat transfer in cylindrical composite laminate", Heat Mass Transfer, 2009, 46, 83–94.

[12] Singh, S.; Jain, P.K.;Rizwan-uddin, "Analytical solution to transient heat conduction inpolar coordinates with multiple layers in radial direction", Int. J. Thermal Sciences, 2008,47, 261–273.

[13] Jain, P.K.; Singh, S.;Rizwan-uddin, "An exact analytical solution for two dimensional, unsteady, multilayer heat conduction in spherical coordinates", Int. J. Heat and MassTransfer, 2010, 53, 2133–2142.

[14] Herakovich, C.T., “Mechanics of Fibrous Composites”, New York, Wiley, 1998.

[15] Arpaci, V.S.,Conducyion heat transfer, Addison-Wesley, Massachusetts, 1996.