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Non-Linear Analysis of Viscoelastic Rectangular Plates Subjected to In-Plane Compression

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Abstract: Geometrically nonlinear governing equations for a plate with linear viscoelastic material are derived. The material model is of Boltzmann superposition principle type. A third-order displacement field is used to model the shear deformation effects. For the solution of the nonlinear governing equations the Dynamic Relaxation (DR) iterative method together with the finite difference discretization technique is used. Finally, the numerical results for the critical buckling load for simply supported edge constraints are reported. In order to justify the accuracy of the results, the elastic plate critical buckling loads are obtained and compared with the existing results. The correlations are very satisfactory. The numerical results are presented for Classical Plate Theory (CPT), First order- Shear Defonmation Plate Theory (FSDT) and Higher Order- Shear Deformation Plate Theory (HSDT). In the case of thick plate the differences among the three theories are highlighted, however, for thin plate the variations are very small.

Keywords: Viscoelastic Plate, Mindlin Plate, Buckling, Rectangular Plate, Dynamic Relaxation, Higher-Order Shear Deformation

1. Introduction

With increasing application of composite structures made with polymeric matrix, the design and analysis of such structures is also growing. Fiber-reinforced laminated composite beams, plates and shells with epoxy matrix behave mostly elastically up to their point of fracture. On the other hand, such structures as mentioned above, with polymeric matrix, have inelastic behaviour even in room temperatures. Consequently, it is more appropriate to first analyze a rectangular plate made of viscoelastic material and then add the reinforcing elements.

The study of viscoelastic plates under dynamic loading has been carried out in various forms [1]. The stability of a viscoelastic plate with non-linear integrodifferential equations under dynamic loading is considered in [2]. The plate is assumed to be thin but with non-linear in-plane strain-displacement relations. In [3], the postbuckling behaviour of viscoelastic laminated plates taking into account the shear-deformations is presented. The higher-shear deformation theory, as originally developed by Reddy [4-6], is applied. The effects of viscoelastic behaviour are shown, and the results for lower order theories are compared with the results obtained using the higher order theory.

In [7], the dynamic stability of viscoelastic laminated plates, subjected to a harmonic in-plane excitation is considered. The viscoelastic behaviour is caused by the polymeric matrix of the fiber-reinforced laminated plate. The Boltzmann viscoelastic material model is used in the stress-strain relations of the laminated plate. This leads to an integro-differential equation of motion, obtained within the first-order shear deformation theory. The free vibration analysis of rectangular viscoelastic plates with simply supported boundary conditions is presented in [8]. In [8], a three dimensional deformation of linear viscoelastic material is considered but the normal stress in the plate thickness direction is neglected and a simplified form of the constitutive equations is used.

Mindlin theory of plates together with first-order shear deformation theory is applied. However, the same author has presented the free vibration analysis results of viscoelastic plates obtained by using a higher-order shear deformation theory [9]. However, the so called simple higher-order plate theory is employed where two terms are added to the displacement field equations in the plane of the plate and the deflections are taken as in the first-order shear deformation theory. An exact series form of solution is presented. The chaotic dynamic

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analysis of viscoelastic plates is presented in [10].

The dynamic buckling of viscoelastic plates with large deflection is investigated in [10] by using chaotic and fractal theory. The material behaviour is given in terms of Boltzmann superposition principal. In [11], the post buckling analysis of imperfect non-linear viscoelastic cylindrical panels is presented. The material in modeled according to the Schapery representation of non-linear viscoelasticity. They have developed solutions to calculate the growth of the initial imperfection in time by using the Donnell equilibrium equations for geometrically non-linear cylindrical panels. The buckling, vibration and damping characteristics of rectangular plates with composite stiff-layers and an isotropic viscoelastic core are studied under thermal loads using finite element method [12]. The inherent composite the damping and damping due to viscoelastic layer are compared and parametric study is conducted with ply lay-up, fiber angle and core thickness as parameters. In [13], the dynamic stability and nonlinear vibrations of viscoelastic orthotropic rectangular plates are investigated. The theory is based on the Kirchhoff-Love assumptions and Reissner-Mindlin generalized plate theory in a nonlinear geometry. The weakly singular Koltunov-Rzhanitsyn type kernel is utilized with three rheological parameters. A numerical method, based on quadrature formulae, is used.

A comparison of the results based on the above assumptions is presented and the implications of material viscoelasticity on vibration and dynamic stability are presented. Finally, the dynamic stability of a non-conservative viscoelastic rectangular plate is investigated in [14]. The thin-plate theory and the two-dimensional viscoelastic constitutive relation are used. The differential equation of motion of the viscoelastic rectangular plate subjected to uniformly distributed tangential follower force in Laplace domain is obtained. This equation is suitable for different viscoelastic models of differential nature. Based on the differential quadrature method, the generalized eigenequations of non-conservative viscoelastic rectangular plate with all edges simply supported, two opposite edges simply supported and other two edges clamped are established. The curves of real and imaginary parts of the first three-order dimensionless complex frequencies versus uniformly distributed tangential follower force are obtained. The factors influencing the dynamic stability of the viscoelastic rectangular plate are discussed.

In this paper the von Karman type of equilibriumequations taking into account the effect of shear deformations are used. The strain-displacement relations for the plate with large deflections, originally developed by Sander [15] are applied. The material behaviour is given in terms of Boltzmann superposition principle. The set of nonlinear plate governing equations are numerically solved using the DR iterative method in real and imaginary time domains. Finally, numerical results for the variations of dimensionless deflections, stress resultants and stress couples with time are illustrated.

2. Plate geometry

The geometry of the rectangular plate considered in this paper is shown in Fig. 1.

The plate is loaded in its plane and its buckling behaviour is studied. The Cartesian coordinate system (x,y,z) is used and the equations are derived in this system. The dimensions of the plate are 'a' parallel to the x-axis, 'b' parallel to the y-axis and the thickness is 'h'.

3. Non-Linear plate governing equations

Higher-order shear deformation effects are taken into account via third-order displacement fields. Nonlinear strain/displacement relations are of the same type as originally developed by Sanders [15] for thin shells. Obviously, in the present work, the out-of-planeshear strains are added to the in-plane strain/ displacement relations to account for the shear deformation effects. The curvature/ displacement relations are also given. The equilibrium equations which also represent the coupling between bending and stretching of the plate are stated. The standard solid model, i.e. Boltzmann superposition principle is used to represent the plate material behaviour. The time dependent stress resultant and stress couple/ strain relations are given. Finally, simply supported, clamped and free edge constraints are stated. The series of the equations mentioned above are given below in the order of their applications in the software developed for this purpose.

3.1. Equilibrium Equations

The equilibrium of a plate element in the x, y and z directions are considered. Taking moments about x and y axes provide two more equations. Consequently, there are five equilibrium equations for the plate element. They are stated as follows:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

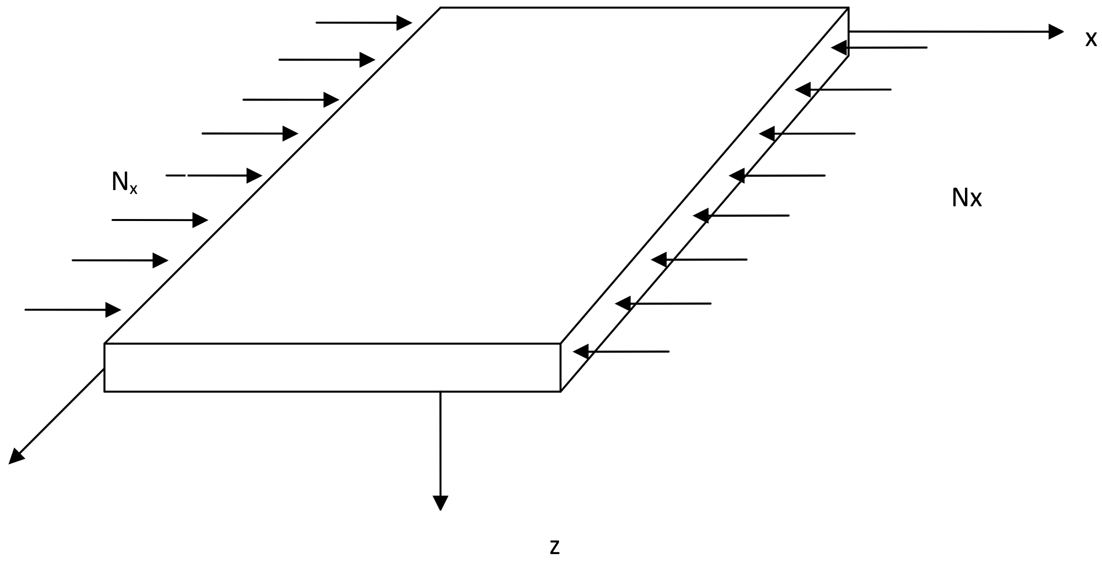


Fig. 1. Plate geometry and positive coordinate system

$$\frac{\partial N_y}{\partial y}+\frac{\partial N_{xy}}{\partial x}=0$$

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+N_x\frac{\partial^2 w}{\partial x^2}+ 2N_{xy}\frac{\partial^2 w}{\partial x\partial y}+N_y\frac{\partial^2 w}{\partial y^2}=0 \quad (1)$$

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x=0$$

$$\frac{\partial M_y}{\partial y}+\frac{\partial M_{xy}}{\partial x}-Q_y=0$$

In this equations N_s, M_s and Q_s are the components of stress resultants, stress couples and shear forces respectively. w is the deflection of the plate.

3.2. Higher-Order displacement field

In order to take an accurate measure of the shear deformations in the plate, the displacement field suggested by Reddy [5] is applied.

$$u(x,y,z,t)=u_0(x,y,t)+z\psi_{x0}(x,y,t)+ z^2\xi_{x0}(x,y,t)+z^3\zeta_{x0}(x,y,t)$$

$$v(x,y,z,t)=v_0(x,y,t)+z\psi_{y0}(x,y,t)+ z^2\xi_{y0}(x,y,t)+z^3\zeta_{y0}(x,y,t) \quad (2)$$

$$w(x,y,z,t)=w_0(x,y,t)+z\psi_{z0}(x,y,t)+ z^2\xi_{z0}(x,y,t)$$

A simplified form of the third expression of the Eqs. (2) is used:

$$w(x,y,t)=w_0(x,y,t)$$

where u, v and w are plate displacement components and u_o, v_o and w_o are plate mid-plane displacement components and ψ, ξ and ζ are plate mid-plane rotations.

3.3. Strain/ curvature relations

The nonlinear strain/curvature relations including three in-plane strains and two out-of-plane shear strains are given below [5]:

$$\varepsilon_x=\varepsilon_x^0+z(\kappa_x^0+z^2\kappa_x^2)$$

$$\varepsilon_y=\varepsilon_y^0+z(\kappa_y^0+z^2\kappa_y^2)$$

$$\varepsilon_{xy}=\varepsilon_{xy}^0+z(\kappa_{xy}^0+z^2\kappa_{xy}^2) \quad (3)$$

$$\varepsilon_{xz}=\varepsilon_{xz}^0+z^2\kappa_{xz}^2$$

$$\varepsilon_{yz}=\varepsilon_{yz}^0+z^2\kappa_{yz}^2$$

In which the plate mid-plane strain/ displacement relations are as follows [5]:

$$\varepsilon_x^0=\frac{\partial u}{\partial x}+\frac{1}{2}(\frac{\partial w}{\partial x})^2$$

$$\varepsilon_y^0=\frac{\partial v}{\partial y}+\frac{1}{2}(\frac{\partial w}{\partial y})^2$$

$$\varepsilon_{xy}^0=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \quad (4)$$
$$\varepsilon_{xz}^0=\psi_x+\frac{\partial w}{\partial x}$$
$$\varepsilon_{yz}^0=\psi_y+\frac{\partial w}{\partial y}$$

and the plate mid-plane curvature/ displacement relations are as stated below [5]:

$$\kappa_x^0=\frac{\partial \psi_x}{\partial x}$$
$$\kappa_y^0=\frac{\partial \psi_y}{\partial y} \quad (5)$$
$$\kappa_{xy}^0=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

and the following relations exist for:

$$\kappa_x^2=-\frac{4}{3h^2}\left(\frac{\partial \psi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)$$
$$\kappa_y^2=-\frac{4}{3h^2}\left(\frac{\partial \psi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)$$
$$\kappa_{xy}^2=-\frac{4}{3h^2}\left(\frac{\partial \psi_x}{\partial y}+\frac{\partial \psi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right)$$
$$\kappa_{xz}^2=-\frac{4}{h^2}\left(\psi_x+\frac{\partial w}{\partial x}\right)$$
$$\kappa_{yz}^2=-\frac{4}{h^2}\left(\psi_y+\frac{\partial w}{\partial y}\right)$$

3.4. Linear viscoelastic material constitutive law

The standard solid (Boltzmann superposition principle) linear viscoelastic stress-strain relations as given below are used.

$$\sigma_x=\frac{Y(0)}{1-\mu^2}(1-R^*)(\varepsilon_x+\mu\varepsilon_y)$$
$$\sigma_y=\frac{Y(0)}{1-\mu^2}(1-R^*)(\varepsilon_y+\mu\varepsilon_x)$$
$$\sigma_{xy}=\frac{Y(0)}{(1+\mu)}(1-R^*)\varepsilon_{xy} \quad (6)$$
$$\sigma_{xz}=\frac{Y(0)}{(1+\mu)}(1-R^*)\varepsilon_{xz}$$
$$\sigma_{yz}=\frac{Y(0)}{(1+\mu)}(1-R^*)\varepsilon_{yz}$$

In these equations, μ is Poisson's ratio, k^2 is the shear correction factor which in higher order shear deformation theory is taken to be 2/3 and $Y(t)=A+Be^{-\alpha t}$ is relaxation function for standard solid media and R^* is an integration operator in time domain:

$$R^*\left(f(t)\right)=\alpha\frac{Y(0)-Y(\infty)}{Y(0)}\int_0^t f(\tau(\tau^{-\alpha(t-\tau)}d\tau\tau \quad (7)$$

The stress resultants are obtained by the following equations:

$$N_x=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_x dz$$
$$N_y=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_y dz \quad (8)$$
$$N_{xy}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{xy} dz$$

The shear force expressions in terms of the shear stresses are given as the following:

$$Q_x=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{xz}dz$$
$$Q_x=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{yz}dz \quad (9)$$

The stress couples are stated as below:

$$M_x=\int_{-\frac{h}{2}}^{\frac{h}{2}}z\sigma_x dz$$
$$M_y=\int_{-\frac{h}{2}}^{\frac{h}{2}}z\sigma_y dz \quad (10)$$
$$M_{xy}=\int_{-\frac{h}{2}}^{\frac{h}{2}}z\sigma_{xy} dz$$

Finally, the constitutive equations in terms of the stress resultants, stress couples and shear forces and plate mid-plane strains are presented below:

$$N_x=h\frac{Y(0)}{1-\mu^2}(1-R^*)\left(\varepsilon_x^0+\mu\varepsilon_y^0\right) \quad (11a)$$
$$N_y=h\frac{Y(0)}{1-\mu^2}(1-R^*)\left(\varepsilon_y^0+\mu\varepsilon_x^0\right) \quad (11b)$$
$$N_{xy}=\frac{h}{2}\frac{Y(0)}{1+\mu}(1-R^*)\varepsilon_{xy}^0$$
$$Q_x=\frac{h}{3}\frac{Y(0)}{1+\mu}(1-R^*)\varepsilon_{xz}^0$$
$$Q_y=\frac{h}{3}\frac{Y(0)}{1+\mu}(1-R^*)\varepsilon_{yz}^0 \quad (12)$$
$$M_x=h^3\frac{Y(0)}{1-\mu^2}(1-R^*)\left(\frac{1}{12}\kappa_x^0+\frac{h^2}{80}\kappa_x^2+\frac{\mu}{12}\kappa_y^0+\frac{\mu h^2}{80}\kappa_y^2\right)$$

$$M_y = h^3 \frac{Y(0)}{1-\mu^2}(1-R^*)\left(\frac{1}{12}\kappa_y^0 + \frac{h^2}{80}\kappa_y^2 + \frac{\mu}{12}\kappa_x^0 + \frac{\mu h^2}{80}\kappa_x^2\right) \tag{13}$$

$$M_{xy} = \frac{h^3}{2}\frac{Y(0)}{1+\mu}(1-R^*)\left(\frac{1}{12}\kappa_{xy}^0 + \frac{h^2}{80}\kappa_{xy}^2\right)$$

3.5. Calculation of stress resultants and couples and shear forces

The stress resultants, stress couples and shear forces as stated in Eqs. (11), (12) and (13) in terms of the relaxation function and the integration operator have to be evaluated. As an example, the first expression of Eqs. (11), for simplicity, is restated below:

$$N_x = h\frac{Y(0)}{1-\mu^2}(1-R^*)\left(\varepsilon_x^0 + \mu\varepsilon_y^0\right) \tag{11a}$$

The integration operator in time domain, Eq. (7) is substituted in Eq. (11a):

$$N_x = \frac{1}{1-\mu^2}\left(\varepsilon_x^0 + \mu\varepsilon_y^0\right) - \alpha\frac{A}{A+B}e^{-\alpha t} \int_0^t \frac{1}{1-\pi^2}\ \varepsilon_x^0 + \mu\varepsilon_y^0)e^{\alpha\tau}d\tau \tag{14}$$

For the evaluation of the integration part of Eq. (14), the trapezoidal rule of integration is applied. The time t is defined in terms of time increment Δt and the real time iteration number n as the following relation:

$$t = n\Delta t \qquad (n = 1,2,...)$$

Consequently for:

$$(n=1) \Rightarrow N_x^1 = N_x^0 - \alpha\frac{A}{A+B}e^{-\alpha\Delta t}\left(\frac{\Delta t}{2}\right)\left[2N_x^0 + 2N_x^1 e^{\alpha\Delta t}\right]$$

$$(n=2) \Rightarrow N_x^2 = N_x^0 - \alpha\frac{A}{A+B}e^{-2\alpha\Delta t}\left(\frac{2\Delta t}{2}\right)\left[2N_x^0 + N_x^1 e^{\alpha\Delta t} + 2N_x^2 e^{2\alpha\Delta t}\right] \ \ldots\ldots$$

Where N_x^o is the constant term in every iteration. Finally, the stress resultant in the *x-direction* for the n^{th} real time iteration is obtained to be as follows:

$$N_x^n = N_x^0 - \alpha\frac{A}{A+B}e^{-n\alpha\Delta t}\left(\frac{n\Delta t}{2}\right)\left[2N_x^0 + N_x^1 e^{\alpha\Delta t} + N_x^2 e^{2\alpha\Delta t} + \ldots\ldots + 2N_x^n e^{n\alpha\Delta t}\right]$$

In general the equation is in the form given below:

$$N_x^n = N_x^0 - \alpha\frac{A}{A+B}e^{-n\alpha\Delta t}\left(\frac{n\Delta t}{2}\right)\left[2N_x^0 + \sum_{m=1}^{n-1} N_x^m e^{m\alpha\Delta t} + 2N_x^n e^{n\alpha\Delta t}\right]$$

After simplification, the equation for N_x^o becomes:

$$N_x^n = \left[\frac{A+B-n\alpha A\Delta t}{A+B+n\alpha A\Delta t}e^{-n\alpha\Delta t}\right]N_x^0 - \left[\frac{n\alpha A\Delta t}{2(A+B+n\alpha A\Delta t)}e^{-n\alpha\Delta t}\right]\sum_{m=1}^{n-1} N_x^m e^{m\alpha\Delta t} \tag{15a}$$

A similar procedure is utilized for determining the remaining equations for stress resultants as stated below:

$$N_y^n = \left[\frac{A+B-n\alpha A\Delta t}{A+B+n\alpha A\Delta t}e^{-n\alpha\Delta t}\right]N_y^0 - \left[\frac{n\alpha A\Delta t}{2(A+B+n\alpha A\Delta t)}e^{-n\alpha\Delta t}\right]\sum_{m=1}^{n-1} N_y^m e^{m\alpha\Delta t} \tag{15b}$$

$$N_{xy}^n = \left[\frac{A+B-n\alpha A\Delta t}{A+B+n\alpha A\Delta t}e^{-n\alpha\Delta t}\right]N_{xy}^0 - \left[\frac{n\alpha A\Delta t}{2(A+B+n\alpha A\Delta t)}e^{-n\alpha\Delta t}\right]\sum_{m=1}^{n-1} N_{xy}^m e^{m\alpha\Delta t}$$

The shear forces are:

$$Q_x^n = \left[\frac{A+B-n\alpha A\Delta t}{A+B+n\alpha A\Delta t}e^{-n\alpha\Delta t}\right]Q_x^0 - \left[\frac{n\alpha A\Delta t}{2(A+B+n\alpha A\Delta t)}e^{-n\alpha\Delta t}\right]\sum_{m=1}^{n-1} Q_x^m e^{m\alpha\Delta t}$$

$$Q_y^n = \left[\frac{A+B-n\alpha A\Delta t}{A+B+n\alpha A\Delta t}e^{-n\alpha\Delta t}\right]Q_y^0 - \left[\frac{n\alpha A\Delta t}{2(A+B+n\alpha A\Delta t)}e^{-n\alpha\Delta t}\right]\sum_{m=1}^{n-1} Q_y^m e^{m\alpha\Delta t} \tag{16}$$

The stress couple equations are:

$$M_x^n = \left[\frac{A+B-n\alpha A\Delta t}{A+B+n\alpha A\Delta t}e^{-n\alpha\Delta t}\right]M_x^0 - \left[\frac{n\alpha A\Delta t}{2(A+B+n\alpha A\Delta t)}e^{-n\alpha\Delta t}\right]\sum_{m=1}^{n-1} M_x^m e^{m\alpha\Delta t}$$

$$M_y^n = \left[\frac{A+B-n\alpha A\Delta t}{A+B+n\alpha A\Delta t}e^{-n\alpha\Delta t}\right]M_y^0 - \left[\frac{n\alpha A\Delta t}{2(A+B+n\alpha A\Delta t)}e^{-n\alpha\Delta t}\right]\sum_{m=1}^{n-1} M_y^m e^{m\alpha\Delta t} \tag{17}$$

$$M_{xy}^n = \left[\frac{A+B-n\alpha A\Delta t}{A+B+n\alpha A\Delta t}e^{-n\alpha\Delta t}\right]M_{xy}^0 - \left[\frac{n\alpha A\Delta t}{2(A+B+n\alpha A\Delta t)}e^{-n\alpha\Delta t}\right]\sum_{m=1}^{n-1} M_{xy}^m e^{m\alpha\Delta t}$$

3.6. Plate boundary conditions

Various plate edge conditions may how be defined. These can include simply supported and clamped or a combinations of these two and free edge. However, here only a simply supported edge condition is considered. This condition may be applied to all the four edges of the plate or at least one edge must have this constraint. Consequently, the constraints that the clamped condition imposes on the displacements and stress couples along the plate edges may be expressed as follows:

Along the x-axis:

$$\psi_y = w = 0$$
$$N_{xy} = M_x = 0\ ,\ N_x = T \tag{18}$$

Along the y-axis:

$$\psi_x = w = 0$$
$$N_y = N_{xy} = M_y = 0 \tag{19}$$

4. Numerical solution technique

The solution procedure for the plate equations is outlined here. The DR numerical method is used together with the finite-difference discretization technique to obtain numerical results for the problem.

4.1. Dynamic relaxation technique

The DR technique which initially was developed to study the tidal flow was soon realized that it can simply be applied to static problems [16]. This is done by adding damping and inertia terms to the right-hand-side of the equilibrium equations, Eqs. (1), in order to convert the continuous equations to discrete format, so that the Eqs. (1) become:

$$\begin{aligned}
&\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = \rho_u \frac{\partial^2 u}{\partial t^2} + K_u \frac{\partial u}{\partial t} \\
&\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = \rho_v \frac{\partial^2 v}{\partial t^2} + K_v \frac{\partial v}{\partial t} \\
&\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + \\
&N_y \frac{\partial^2 w}{\partial y^2} = \rho_w \frac{\partial^2 w}{\partial t^2} + K_w \frac{\partial w}{\partial t} \\
&\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = \rho_{\psi x} \frac{\partial^2 \psi_x}{\partial t^2} + K_{\psi x} \frac{\partial \psi_x}{\partial t} \\
&\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = \rho_{\psi y} \frac{\partial^2 \psi_y}{\partial t^2} + K_{\psi y} \frac{\partial \psi_y}{\partial t}
\end{aligned} \tag{20}$$

In order to complete the transformation process, the velocity and the acceleration terms must be replaced with the following approximate finite-difference expressions:

$$\frac{\partial \alpha}{\partial t} = 0.5[(\frac{\partial \alpha}{\partial t})^a + (\frac{\partial \alpha}{\partial t})^b]$$
$$\frac{\partial^2 \alpha}{\partial t^2} = [(\frac{\partial \alpha}{\partial t})^a + (\frac{\partial \alpha}{\partial t})^b] / \delta t \tag{21}$$

Where $\alpha = u, v, w, \psi_x, \psi_y$ and the superscripts "a" and "b" represent respectively the values of the variable **a**fter and **b**efore the time increment δt. Thus, substituting Eqs. (21) into the RHS of Eqs. (20), they become:

$$\begin{aligned}
&u^a = \frac{1}{1+k_u^*}\left((1-k_u^*)u^b + \frac{\delta t}{\rho_u}\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y}\right)\right) \\
&v^a = \frac{1}{1+k_v^*}\left((1-k_v^*)v^b + \frac{\delta t}{\rho_v}\left(\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x}\right)\right) \\
&w^a = \frac{1}{1+k_w^*}\left((1-k_w^*)w^b + \frac{\delta t}{\rho_w}\left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \right.\right. \\
&\qquad \left.\left. N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2}\right)\right) \\
&\psi_x^a = \frac{1}{1+k_{\psi x}^*}\left((1-k_{\psi x}^*)\psi_x^b + \frac{\delta t}{\rho_{\psi x}}\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x\right)\right) \\
&\psi_y^a = \frac{1}{1+k_{\psi y}^*}\left((1-k_{\psi y}^*)\psi_y^b + \frac{\delta t}{\rho_{\psi y}}\left(\frac{\partial M_x}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y\right)\right)
\end{aligned} \tag{22}$$

Where:

$$K_\alpha^* = \frac{k_\alpha \delta t}{2\rho_\alpha}\ (\alpha = u, v, w, \psi_x, \psi_y)$$

In order to be able to calculate the displacements and rotations from the five velocity components of Eqs. (22), simple integration expressions are used, as stated below:

$$\begin{aligned}
&u^a = u^b + \delta t u^a \\
&v^a = v^b + \delta t v^a \\
&w^a = w^b + \delta t w^a \\
&\psi_x^a = \psi_x^b + \delta t \psi_x^a
\end{aligned} \tag{23}$$

In Eqs. (23), the superscripts 'a' and 'b' are similar to those defined under Eqs. (21). Eqs. (22) and (23) constitute the complete set of transformations and modifications required for the application of the DR algorithm in the solution of the plate non-linear equations. The DR algorithm comprises the following simple sequence of operation on Eqs. (3)-(5), (11)-(15), (22) and (23).

Step 1: Set all variables except the transverse pressure, N_x to zero.

Step 2: Calculate velocities using Eqs. (22).

Step 3: Integrate velocities to obtain displacement util-izing Eqs. (23).

Step 4: Apply displacement boundary conditions along the x and y axes using the appropriate parts of Eqs. (18) and (19).

Step 5: Calculate the mid-plane strains and curvatures using Eqs. (3), (4) and (5).

Step 6: Calculate the stress resultants, shear forces and stress couples using Eqs. (11), (12) and (13).

Step 7: Apply stress couple boundary conditions using the appropriate parts of Eqs. (18) and (19).

Step 8: Check if velocities are negligibly small, i.e. $\leq 10^{-6}$ on all the interior nodes, and the static solution is obtained.

Step 9: If step 8 is satisfied store the results otherwise return to step 2 and repeat the sequence.

Step 10: Increment the real time and repeat steps (2)-(9) until the maximum specified time is reached, then print out all the results for the specified real time.

The computational steps described above are dependent on the proper choice of damping factors, densities and the fictitious time increment which all together add up to eleven parameters. In the present analysis the diversity of choice is reduced by using fictitious densities in conjunction with a unit time increment as adopted by Cassell and Hobbs [17]. Therefore, only the five damping factors, k_α where, $\alpha = u, v, w, \psi_x, \psi_y$ have to be selected using the following simple relation.

$$k_\alpha = (\rho_\alpha)_{\min} / F \qquad (24)$$

The factor F is in the range: $0.01 \leq F \leq 1$ that in the present analysis the value 0.04 suits best and it gives a minimum number of iterations.

5. Finite- Difference forms of plate equations

In the process of numerical solution, after transforming the equilibrium equations into their DR format, the velocity expressions were obtained. The differential terms on the RHS of the velocity expressions must be discretized so that numerical solution can be carried out. Consequently, the DR equations are first non-dimensionalized and then Eqs. (22), (23), (4), (5), (11)-(13) are discretized. The discretization technique may be finite-difference, finite-element, boundary element, etc. The technique adopted here is the finite-difference which suits the DR method and has also been previously used in [18, 19] and many others, some of which are given in [19]. In most DR applications an interlacing finite-difference is used, here however, due to minor differences in the final results, non-interlacing meshes are used.

6. Numerical results

Thin and moderately thick viscoelastic plate results for simply supported edges with in-plane fixed and free conditions are presented. Due to unavailability of viscoelastic buckling loads, elastic buckling loads are used for comparison purposes. CPT, FSDT and HSDT are used to generate three sets of results for plate centre deflections versus time, through-thickness in-plane displacements and normal stress and through-thickness in-plane and out-of-plane shear stresses for Polymethyl methacrylate (Plexiglass) or PMMA with A=2.24 MPa, *B*=2.24 GPa (material constants), μ=0.365, α=0.00931/*h*, (a) $h/L = 0.01$ and (b) $h/L = 0.1$.

6.1. Comparison results

Elastic buckling loads for CPT and HSDT are utilized for comparison purposes. Table 1 gives the in-plane buckling loads N_{cr} as determined using the HSDT and those given in [20] using the CPT. Four sets of results for different aspect ratios are presented. The correlations of the two sets of results are very satisfactory and that the CPT results are higher than the HSDT results. This illustrates a more accurate prediction by HSDT.

6.2. Deflection and stress results

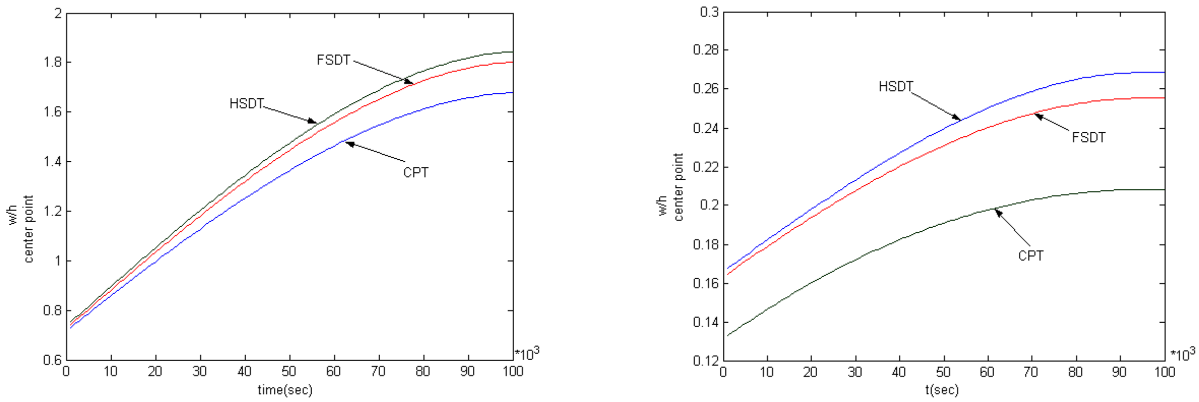
Figs. 2 (a,b) present dimensionless plate centre deflections versus time for simply supported plate with in-plane fixed movements for two slenderness ratios under uniformly distributed load. As predicted, the differences in deflections among the various theories are minor for thin plate, Fig. 2(a). However, there are significant differences between the CPT results and the present FSDT and HSDT, Fig. 2(b).

Dimensionless deflections versus time curves for simply supported plate with in-plane free edge conditions are presented in Figs. 3. The deflection values are larger, in this case, compared to those presented in Figs. 2. The deflection results for thin plates for the three plate theories are very close to one another. On the other hand, for the thick plate, the CPT results are much smaller than those predicted by the other two theories. From the physical considerations point of view, such behaviour is expected. In deed, this differences show the need for HSDT in thick plates.The through-thickness in-plane displacements in the *x*-direction as obtained from CPT and HSDT analyses are illustrated in Figs. 4. The CPT and FSDT results are very similar, consequently, only CPT results are included. These results are presented at time t=6000 minutes for thin plate in Fig. 4(a) and for thick viscoelastic plate in Fig. 4(b). Similar results to those illustrated in Fig. 4, are presented in Figs. 5 for elastic plates. The through-thickness in-plane normal stress in the *x*- direction is shown in Fig. 6 for CPT and HSDT.

The linear and non-linear natures of the stresses for the two plate theories are evident. Figs. 7 show the in-plane and out-of-plane shear stresses in the thickness direction of the plate, using HSDT. The non-linear behaviour of the stress variations is pronounced.

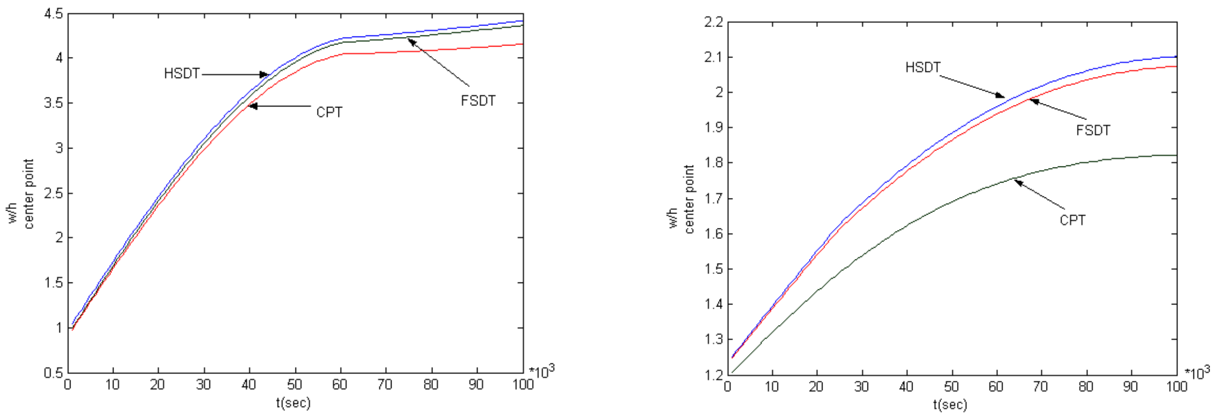
Table 1. Comparison of elastic buckling loads for CPT and HSDT (present results) E = 69GPa, μ = 0.25, simply supported

a(cm)	b(cm)	h(cm)	w/h (centre point)	N_{cr}(Kn/m) HSDT	N_{cr}(Kn/m) CPT
50	50	1	0.0246	942.425	968
100	50	1	0.0428	938.482	961
175	50	1	0.0612	1420.012	1438
240	50	1	0.1098	998.164	1004



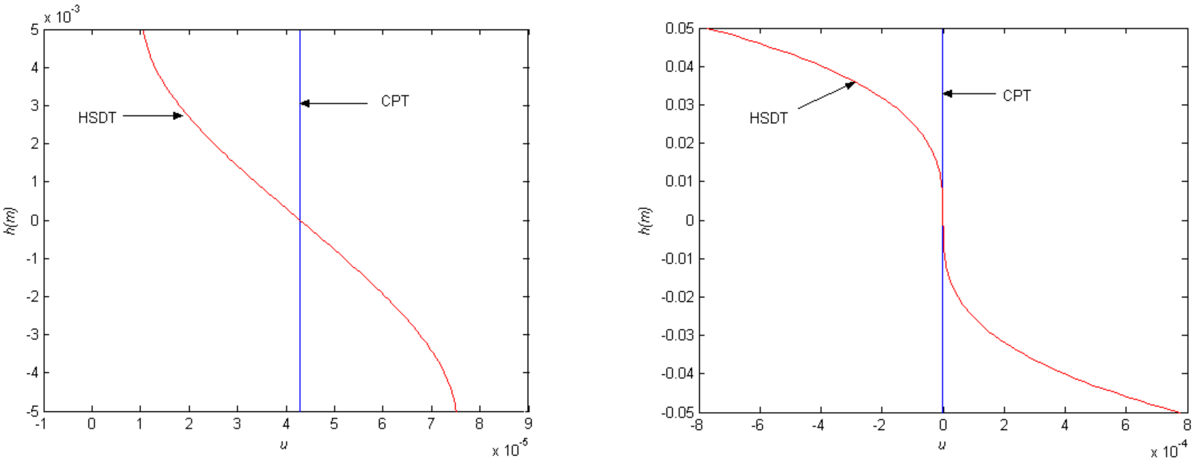
(a) (b)

Fig. 2. Comparison of dimensionless plate centre deflections versus time for a simply supported plate with out-of-plane fixed edges for various plate theories. A= 2.24MPa, B=2.24GPa, μ =0.365, α =0.0093/h, (a) h/L=0.01 and (b) h/L=0.1



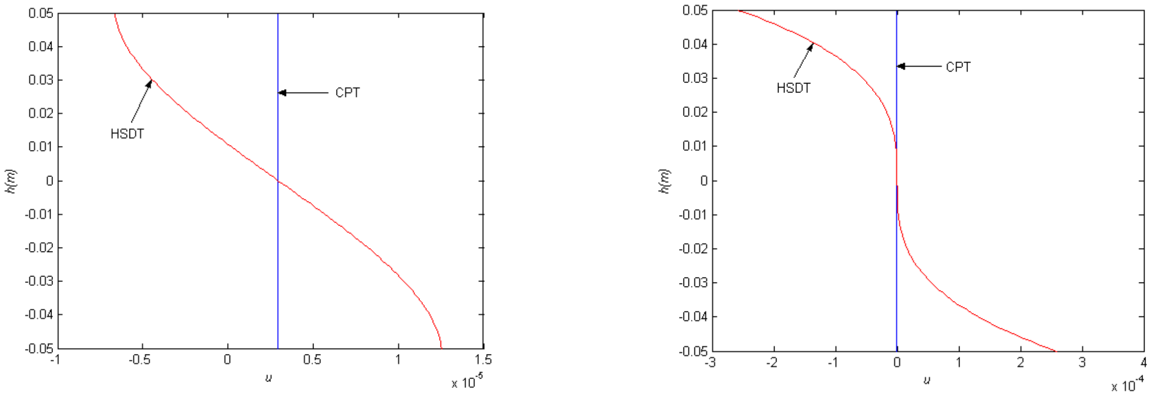
(a) (b)

Fig. 3. Comparison of dimensionless plate centre deflections versus time for a simply supported plate with out-of-plane free edges along the *x*-axis for various plate theories. A= 2.24MPa, B=2.24GPa, μ =0.365, α =0.0093/h, (a) h/L=0.01 and (b) h/L=0.1



(a) (b)

Fig. 4. Comparison of through thickness in-plane displacements for a simply supported plate with out-of-plane fixed edges for various plate theories. A=2.24MPa, B=2.24GPa, μ =0.365, α =0.0093/h, (a) h/l=0.01and (b) h/L=0.1



(a) (b)

Fig. 5. Comparison of through thickness in-plane displacements for a simply supported elastic plate with out-of-plane fixed edges for various plate theories. A=2.24MPa, μ =0.365, (a) h/L=0.01 and (b). h/L=0.1

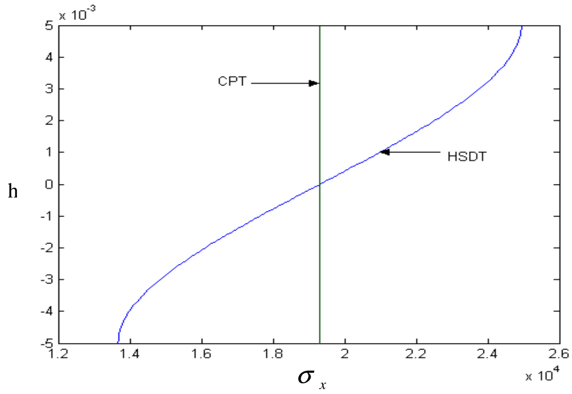


Fig. 6. Comparison of through thickness in-plane normal stresses for a simply supported plate with out-of-plane fixed edges for various plate theories. A=2.24MPa, B=2.24GPa, μ =0.365, α =0.0093/h and h/L=0.1

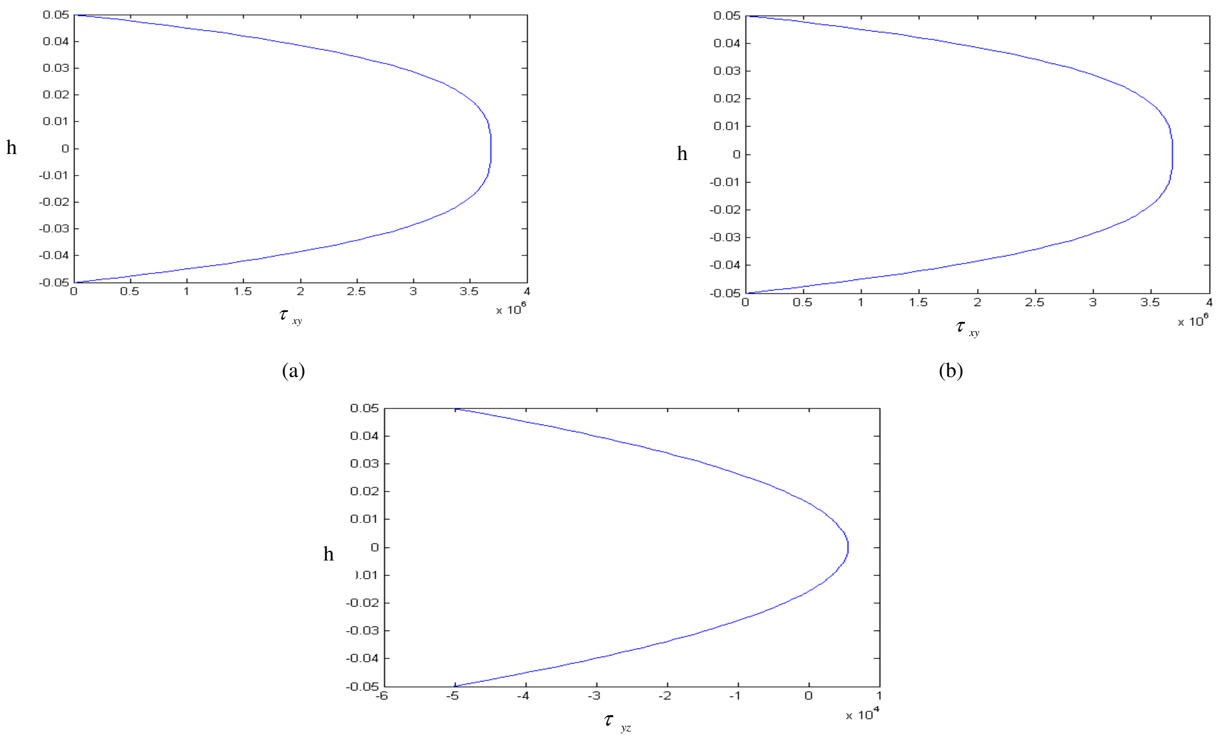


Fig. 7. Shear stresses for a simply supported plate with out-of-plane fixed edges for various plate theories. A=2.24MPa, B=2.24GPa, μ =0.365, α =0.0093/h and h/L=0.1, (a) in-plane σ_{xy}, (b) out-of-plane σ_{xz} and (c) out-of-plane σ_{yz}

7. Conclusions

The DR iterative numerical method has been applied for the first time to solve viscoelastic material plate non-linear governing equations. For this purpose, the plate equilibrium equations have been transformed into DR format. Then, the governing equations are discretized using the finite-difference discretization technique. The loading is a uniformly distributed pressure and the boundary condition is simply supported in-plane fixed and free. Using higher-order displacement field in non-linear strain/ displacement relations is the other originality of the present work. The standard Boltzmann superposition principle viscoelastic material model is adopted. In order to generate numerical results, the finite difference equations are then used to prepare computer software. Because of having a fictitious and a real time a much longer computer time is required to obtain numerical results, although, the number of iteration are very low for every static solution in fictitious time increment. The elastic results for buckling loads are compared with existing results and the correlations are very satisfactory. However, the viscoelastic plate results are compared for three different plate theories (i.e. CPT, FSDT and HSDT). The plate centre deflections versus time, through-thickness displacements, normal stresses, in-plane and out-of-plane shear stresses are presented graphically. The differences in the results for the three theories used are very significant for thick plates, but small variations observed for thin plates. The present results can be used as a benchmark for other researchers.

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Nomenclature

a,b	plate dimensions
$A(0)$	elastic extensional stiffness
$\overline{A}(0) = A(0)h_o^2 \;/ E(0)$	non-dimensional elastic extensi-onal stiffness
$D(0)$	elastic bending stiffness
$\overline{D}(0) = D(0)h_o^3 / E(0)$	non-dimensional elastic bending stiffness
$K_u, K_v, K_w, K_{\psi_x}, K_{\psi_y}$	damping factors
$K_u^*, K_v^*, K_w^*, K_{\psi_x}^*, K_{\psi_y}^*$	non- dimensional damping factors
M_x, M_y, M_{xy}	stress couples
N_x, N_y, N_{xy}	stress resultants
Q_x, Q_y	shear stress resultants

R^*	integral operator
$t_r = \frac{1}{\alpha}$	relaxation time
u, v, w	displacement components
u_o, v_o, w_o	plate mid-plane displacement components
$Y(t)$	relaxation function
$Y(0)$	relaxation function at time zero
$Y(\infty)$	relaxation function at time infinity
z	distance from plate mid-plane
α	inverse of relaxation time
δt	time increment
$\varepsilon_x, \varepsilon_y, \varepsilon_z, \varepsilon_{xy}, \varepsilon_{xz}, \varepsilon_{yz}$	normal and shear strain components
$\varepsilon_x^0, \varepsilon_y^0, \varepsilon_z^0, \varepsilon_{xy}^0, \varepsilon_{xz}^0, \varepsilon_{yz}^0$	plate mid-plane normal and shear strain components
k_x^o, k_y^o, k_{xy}^o	plate mid-plane curvature components
μ	Poissons ratio
$\rho_u, \rho_v, \rho_w, \rho_{\psi_x}, \rho_{\psi_y}$	fictitious densities
$\sigma_x, \sigma_y, \sigma_z, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}$	normal and shear stress components

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