Original Research

Joint pricing and replenishment decisions in an inventory model for deteriorating items considering wastewater treatment: A case study

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Abstract

Wastewater treatment has been a challenge for beverage companies in recent years which has led them to establish specific units to purify the expired returning items before agricultural uses or disposal. This issue has never been studied from the perspective of inventory and production management. This paper proposes an inventory formulation in the case of Behnoush beverage company in Iran to determine the replenishment cycle and the optimal selling price of items when: (a) items deteriorate continuously and have their specific expiration dates, (b) a proportion of items that are expired are moved to the company and after the process of wastewater treatment, and (c) a trade credit period is offered to the buyer to make encouragement for ordering in larger quantities. Optimality conditions for the total cost function are elaborated. Thereafter, numerical experiments are adopted for validation. Eventually, managerial implications are outlined and findings of the study are summarized.

Keywords - Deteriorating Items; Inventory Control; Supply Chain; Trade Credit; Wastewater Treatment

INTRODUCTION

In today's market, perishable items such as dairy, vegetables, medicines, and fruits deteriorate due to damage, spoilage, dryness, and vaporization and this means their quality is decreased over time [1]. Controlling the progress of deterioration using inventory regulations prevents the loss of items. In a green inventory system, some expired items may be returned to the factory due to their demand forecast miscalculation. These items are fully deteriorated and should be treated before disposing them in wastewater or reusing them. In beverage companies, wastewater is produced due to cooling, washing, cleaning, bottle filling, utility operations, and recycling [2]. Waterborne bacteria and enteric viruses can infect and cause illnesses to people who drink untreated water. As wastewater depositions are the main reasons for human disease and environmental damages, treatment and recycling of the wastewater becomes important [3]. This can be intensified by environmental regulations which are becoming obligatory due to numerous environmental problems caused by industrialization. The chemical oxygen demand (COD) is a representative measure of the amount of oxygen that can be consumed by reactions in a measured solution. COD is an important factor to evaluate the quality of wastewater by providing a criterion to indicate the influence of sewage on the receiving body [4].

Behnoush is one of the first factories to produce beverage items in Iran. It continued its activity and produced fresh milk. The factory has the privilege of having the facilities needed to grow barley and malting. Figure 1 indicates some items produced in the Behnoush factory. Non-alcoholic drinks include the main items produced in Behnoush. According to environmental standards, beverage factories are required to purify it before pouring rotting liquids into municipal wastewater. So there is a section called a wastewater plant (See Figure 2) that purifies a large amount of fluid per day in the factory to control the COD of the returning items before disposing of them. Treated liquids under contract with the municipality can be used in the green space irrigation system and provide an external source of revenue for the company.

FIGURE 1
ITEMS PRODUCED IN BEHNOUSH FACTORY

FIGURE 2
THE WASTEWATER PLANT IN THE BEHNOUSH FACTORY

There is a lack of studies in considering wastewater treatment from an economic perspective. Keeping this fact in mind, three research questions arise: (1) How do the returning items affect the demand from the buyers and the cost of treatment? (2) What is the deterioration progress of beverage items? (3) What is the impact of offering a trade credit period from sellers to buyers? To address these questions, this paper introduces a novel price and returning items-dependent EOQ model for perishable items to minimize the total cost considering trade credit. Relict sections of this research are organized in the upcoming. The second section proposes a review of deteriorating inventory and sustainable inventory is proposed. Terminologies and assumptions are provided in Section 3 to formulate the model in different cases. Several models based on the value of trade credit, expiration date, and replenishment cycle are developed in Section 4. Theoretical results and optimal solutions using different theorems are described in Section 5. In Section 6, the case study is introduced and the models are solved with a real domain database. Moreover, a sensitivity analysis concerning the parameters is performed to indicate the impact of each parameter on the total cost function. Thereafter, managerial implications are provided based on the results obtained from the sensitivity analysis. In the last, Findings are recapitulated in Section 7.

LITERATURE REVIEW

Brief literature on deteriorating inventory and inventory models for returning items is reviewed in the following subsections. Eventually, research gaps are outlined and the effort of this study to contribute to the literature is addressed. The traditional EOQ model presumes the purchasing cost charged by the seller when items are received [5]. Although, in real domain markets, sellers offer a trade credit period for special incentives to the buyers to pay off [6]. This regulation attracts buyers as they can extend their sales time to a longer period and earn more interest [7]. This opportunity also allows buyers to order larger quantities that leading to decreasing the stock in hand [8]. This policy can be an efficacious idea for deteriorating items when they have specific expiration dates [9]. Moreover, selling price is another factor that can be significant in determining the demand. Tsao and Sheen [10] had been one of the first authors who developed a price-dependent demand for an inventory model. In this regard, a large number of papers have investigated inventory models for deteriorating items; however, the idea of considering expiration date using a time-varying deterioration rate was first elaborated by Sarkar [11] who took time-varying deterioration and delay in payment simultaneously. Wang et al. [12] extended the paper proposed by Sarkar [11] to expose an EOQ model considering the expiration date for items and trade credit policy. They proposed theorems that consist of conditions for the proof of the objective function's concavity. This paper extended their model by considering the trade credit period as a variable in the demand function and indicates that longer periods of credit encourages buyers to order more items. Considering variable deterioration and fixed lifetime for items, Sarkar et al. [13] developed a bi-level partial delay in payment. Later, Mahata and De [14] proposed a model for credit risk customers when an expiration date is specified for items. He et al. [15] optimized different decisions in a system with expiration dates coordinating two levels of the supply chain. Tiwari et al. [16] added partial backordering to a model considering maximum lifetime in the case of bi-level delay in payment. In another paper, Tiwari et al. [17] added carbon emissions and pricing to deteriorating inventory concurrently.

Shaikh et al. [18] provided a model when the deterioration progress follows a Weibull distribution function and trade credit policy is allowable. Moreover, the demand function is considered hybrid price and advertisement-dependent. Developing a price and advertisement-dependent demand function, Mashud et al. [19] proposed a model for two warehouses with non-instantaneous deterioration, partial backordering, and advance payments. Sepehri and Gholamian [20] took preservation technology and controllable carbon emissions into account concurrently. Mandal et al. [21] elaborated a model when the rate of deterioration is considered constant. In this paper, delay in payment is permissible and sustainability considerations are taken into account. Finally, Mahato and Mahata [22] developed a model utilizing credit and freshness-dependent demand when two levels of trade credit are considered for items with specific expiration dates.

Wastewater treatment in manufacturing companies has been attended in numerous papers. However, the main concentration in this field of study is on the biological aspect of the process and the challenge has never been attended from an economical perspective. This is the considerable gap in the literature in this field of research which is highlighted in this paper. The importance of establishing wastewater treatment plants for companies has been the subject of numerous papers. For instance, the life cycle assessment of items in construction or production companies when taking environmental considerations into account was developed by Morera et al. [23]. Wastewater treatment plants are not limited to beverage industries. In a paper proposed by Valta et al. [4], different food and beverage companies that have utilized wastewater treatment plants are investigated. Wastewater treatment that leads to recycling and reusing of beverage items have always been attended in research papers. Alkaya and Demirer [24] investigated a case study of a beverage company in Turkey that has implemented a system of preserving wastewaters.

Another paper proposed by (Huang et al. [25] concentrated on production disruptions that lead to producing waste items. In this work, an on-control state is proposed in the production systems when the demand is price-dependent and preservation technologies control the deterioration progress. One of the important works concentrated on the inventory of greenhouse gases emitted from wastewater treatment plants of Chinese industrial parks has been proposed by Hu et al. [26]. In this paper, aerobic biological treatment is employed for the treatment of wastewaters, and the total emission of this treatment method is compared with other methods. Another paper that outlined the economic aspect of waste management was developed by Tirkolaee et al. [27] who elaborated a model under uncertainties when location, allocation, and inventory decisions are objected to. Besides, environmental aspects are considered by developing a robust optimization approach. A selected literature review is brought in Table 1 to compare this study with near studies.

The papers reviewed above indicate that the number of papers investigated sustainable inventory models for deteriorating items has been increased in recent years. However, there are scarce papers that developed waste management in inventory models, especially under sustainability considerations. To contribute to the identified gap in the literature, this elaboration revisits the papers proposed by Wang et al. [12], Huang et al. [25], and Mandal et al. [21] which consist of three basic models for this elaboration and received attention in recent years. These papers help us to develop two different cases based on the values of the replenishment cycle and trade credit period. Besides, three subcases are derived based on the value of the expiration date. Studying the impact of the mentioned intervals on the total profit obtained by the seller leads us to develop a traditional EOQ model considering time-varying deterioration, price and returning items-dependent demand, and a single level of trade credit. The significant novelty of this paper is considering the revenue obtained from the wastewater treatment plants for irrigation uses.

TABLE 1
SELECTED REVIEWED PAPERS

Author(s)	Deterioration			Delay in	Waste	Carbon	Demand
	Constant	Time-	Stochastic				
Sarkar [11]		✓		✓			TD
Wang et al. [12]		✓		✓			CD
Sarkar et al. [13]		✓		✓			CO
Mahata and De [14]		✓		✓			CO
Huang et al. [25]	✓				✓	✓	PD
Tiwari et al. [17]	✓			✓		✓	PD
Shaikh et al. [18]			✓	✓			PD & AD
Mashud et al. [19]	✓			✓			PD & AD
Sepehri and Gholamian [20]	✓					✓	PD
Mahato and Mahata [22]		✓		✓			CD & FD
Mandal et al. [21]	✓			✓		✓	SD & AD
This paper		✓		✓	✓	✓	PD

CO: Constant; TD: Time-dependent; CD: Credit-dependent; PD: Price-dependent; AD: Advertisement-dependent; FD: Freshness (deterioration)-dependent; SD: Stock-dependent

PROBLEM DESCRIPTION AND MODEL DEVELOPMENT

From the previously reviewed papers, no research work has mentioned wastewater disposal or water purification concept as an environmental impact. This paper provides an integrated deteriorating inventory when the expiration date is considered regulating wastewater disposal policies. This study highlights the significance of the green water purification aspect of a beverage company. The purpose of this research is to provide a practical insight for beverage companies to decide on the replenishment cycles and the prices of their items. This paper presents optimality conditions. Finally, to validate the model, a case study is illustrated.

- *Problem description*

A sustainable EOQ model with a single seller and a single buyer is elaborated to cover the literature gap based on the model proposed by Wang et al. [12]. The seller makes effort to sell a single item to the buyer stimulating him/her by offering a period of delay in payment. Also, the orders are put initially when the buyer's demand follows a linearly price-dependent demand function. The item deteriorates with a time-varying demand function and approaches the full deterioration at the expiration date. The expired items are returned to the company to be disposed of. In this case, beverage items are taken to the wastewater treatment plant of the Behnoush Company, and the process of treatment starts. According to the contracts between Behnoush Company and the government, the purified water is utilized for irrigation uses and the company earns benefits from selling the treated wastewater to the government. Based on the paper proposed by Hu et al. [26], COD is considered a common indicator of the existence of substances in wastewater that can be controlled by different chemical methods. A standard value for the COD is proposed by the government which makes the wastewater appropriate for irrigation use. This standard is followed by the company. A brief indication of the stated problem is shown in Figure 3.

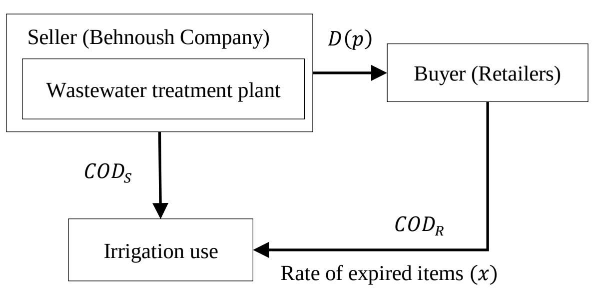

FIGURE 3
THE FLOW OF ITEMS IN THE INVENTORY SYSTEM

Notations: Table 2 summarizes terminologies that are necessary for developing a mathematical formulation.

TABLE 2
THE MODEL NOTATIONS

Variable	
T	Replenishment cycle (months)
p	Selling price per ($)
Parameters	
S	Period of delay in payment offered to the seller (unit time)
o	Ordering cost per order ($)
c	Purchasing cost per order ($)
h	Holding cost per unit per year ($)
m	Maximum lifetime (expiration date) of items (unit time)
C_e	Revenue obtained from selling treated wastewater per gallon ($)
C_p	Wastewater treatment cost for returning items per gallon ($)
COD_R	Density of Oxygen needed for the oxidation process (grams/liter)
COD_S	Healthy density of Oxygen needed for the oxidation process (grams/liter)
g	Demand's sensitivity coefficient to the returning items
x	Fraction of returning items
I_c	Rate of the interest charged
I_e	Rate of the interest earned
Expressions and Functions	
$D(p)$	Price-dependent demand function
$I(t)$	Inventory level in units at time t
$\theta(t)$	Time-varying deterioration rate at time t, where $0 \leq \theta(t) \leq 1$

- *Assumptions*

Considering a single item in the supply chain, a single seller (Behnoush Company) provides items for a single buyer (retailers).

- To stimulate the buyer for buying larger quantities of items, the seller permits the buyer to settle the payments within a specific trade credit period (S). In this regard, if the buyer pays off the purchasing cost within the period of credit, he/she can accumulate more revenue by the earned interest. Otherwise, the buyer has to pay a penalty cost as a charged interest to the seller. Based on the value of S and in comparison with the replenishment cycle T and the expiration date of items m, three subcases will be obtained [12].
- A specific maximum lifetime is considered for all deteriorating items. The rate of deterioration is considered as $\theta(t) = \frac{\lambda}{\lambda+m-t}$ where λ is a constant coefficient. The rate of deterioration approaches 1 when the maximum lifetime arrives. To make the problem tractable, we assume that the deterioration rate is the same as that in Sarkar [11] as follows.

$$\theta(t) = \frac{1}{1+m-t} \quad 0 \leq t \leq T \leq m \qquad (1)$$

- The demand rate is proposed as follows [10].

$$D(p) = \begin{cases} D_1(p) = kp^{-a} & t < m \\ D_2(p) = kp^{-a}e^{-gx} & t \geq m \end{cases} \qquad (2)$$

where k is the market scale, a is the demand's sensitivity coefficient to the selling price, and g is the demand's sensitivity coefficient to the proportion of returning items ($0 \leq a \leq 1$ and $0 \leq g \leq 1$.

- The replenishment is instantaneous.
- Shortages and lead-time are neglected.
- COD_R indicates the pollution rate of the wastewater that is intended to be treated. The higher the value of COD_R, more effort is required to treat the wastewater to make it appropriate for irrigation uses. COD_S is known as the standard value for the pollution of wastewaters that is healthy for irrigation uses. Therefore, Behnoush Company employs chemicals to approach COD_S and accumulate revenue from the government due to irrigation [26].

Based on the above-mentioned assumptions, two distinct cases are developed to formulate the mathematical model.

MATHEMATICAL MODEL

In this section, a sustainable EOQ model is elaborated to indicate how the delay in payment affects the total cost. Two cases are elaborated to cover all possible values of the replenishment cycle, trade credit period, and expiration date. The decreasing progress of the inventory level is studied to find the order quantity which is employed in formulating different costs associated with the inventory system. The level of inventory in hand is decreased by demand and deterioration in the interval of $[0, T]$. The following differential equations illustrate this mitigation of inventory level.

$$\frac{dI(t)}{dt} = -D(p) - \theta(t)I(t) \qquad 0 \leq t \leq T \qquad (3)$$

Considering boundary condition $I(T) = 0$, (3) can be solved and the following result is obtained.

$$I(t) = e^{-\delta} \int_t^T e^{\delta(t)} D(n) du \qquad (4)$$

where

$$\delta = \int_0^t \theta(u) du = \ln\left(\frac{1+m}{1+m-t}\right) \qquad (5)$$

From the values of T and S, we have two potential cases: (1) $T \leq S$, and (2) $T \geq S$. Let us discuss them separately.

Case 1 $(T \leq S)$

Different subcases of case 1 are illustrated as follows.

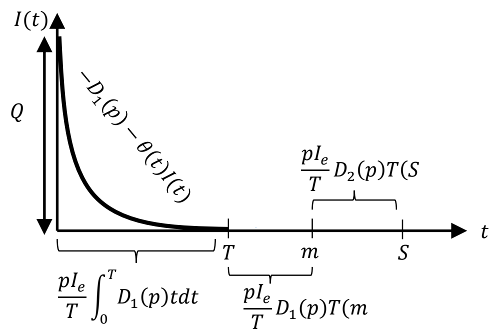

FIGURE 4
SUB-CASE 1-1 $(T \leq m \leq S)$

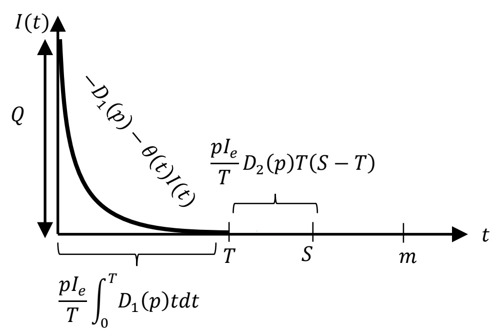

FIGURE 5
SUB-CASE 1-1 $(T \leq S \leq m)$

Sub-case 1.1 $(T \leq m \leq S)$

Figure 4 indicates that all items will be sold and the retailer earns interest by T. No interest is charged by the seller. Substituting (6) into (5), the inventory level at time t is obtained as

$$I(t) = \frac{1+m-t}{1+m} \int_t^T D_1(p) \frac{1+m}{1+m-u} du = \frac{1+m-t}{1+m} kp^{-a} \int_t^T \frac{1+m}{1+m-u} du = (1+m-t)kp^{-a} \ln\left(\frac{1+m-t}{1+m-T}\right) \qquad (6)$$

Therefore, the order quantity is

$$I(0) = Q = kp^{-a}(1+m) \ln\left(\frac{1+m}{1+m-T}\right) \qquad (7)$$

The abovementioned equation is employed to formulate holding costs as follows.

$$h\int_0^T I(t)dt = hkp^{-a}\left[\int_0^T (1+m-t)\ln\left(\frac{1+m-t}{1+m-T}\right)dt\right] = hkp^{-a}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right)+\frac{T^2}{4}-\frac{(1+m)T}{2}\right] \quad (8)$$

The factory can accumulate revenue due to selling the treated items for irrigation use and obtains revenue at the rate of C_e. Treatment cost which is posed to the system for decreasing the COD_R until COD_S which makes the treated wastewater healthy for irrigation uses is formulated as follows.

$$\frac{I(0)}{T}(C_p - C_e)x(COD_R - COD_S) = \frac{(C_p - C_e)}{T}x(COD_R - COD_S)kp^{-a}(1+m)\ln\left(\frac{1+m}{1+m-T}\right) \quad (9)$$

The earned interest by S is formulated as follows.

$$IE = \frac{pI_e}{T}\left[\int_0^T D_1(p)tdt + D_2(p)T(S-m) + D_1(p)T(m-T)\right] = \frac{pI_e}{T}kp^{-a}\left[(TS-Tm)e^{-gx}+Tm-\frac{T^2}{2}\right] \quad (10)$$

Eventually, ordering and purchasing costs are formulated as o and $cI(0)$ respectively. Hence, the total cost of the seller is formulated as follows.

$$TC_1(T,p) = \frac{o}{T}+\frac{h}{T}kp^{-a}\left[\frac{(1+m)^2}{2}ln\left(\frac{1+m}{1+m-T}\right)+\frac{T^2}{4}-\frac{(1+m)T}{2}\right]+\frac{c}{T}kp^{-a}(1+m)\,ln\left(\frac{1+m}{1+m-T}\right)+\frac{(C_p-C_e)}{T}x(COD_R - COD_S)kp^{-a}(1+m)\,ln\left(\frac{1+m}{1+m-T}\right)-\frac{pI_e}{T}kp^{-a}\left[(TS-Tm)e^{-gx}+Tm-\frac{T^2}{2}\right] \quad (11)$$

Sub-case 1.2 ($\boldsymbol{T \le S \le m}$)

From Figure 5, the inventory level at time t is calculated as

$$I(t) = \frac{1+m-t}{1+m}\int_t^T D_1(p)\frac{1+m}{1+m-u}du = \frac{1+m-t}{1+m}kp^{-a}\int_t^T\frac{1+m}{1+m-u}du = (1+m-t)kp^{-a}\ln\left(\frac{1+m-t}{1+m-T}\right) \quad (12)$$

Therefore, the order quantity is

$$I(0) = Q = kp^{-a}(1+m)\ln\left(\frac{1+m}{1+m-T}\right) \quad (13)$$

The holding cost per cycle is

$$h\int_0^T I(t)dt = hkp^{-a}\left[\int_0^T(1+m-t)\ln\left(\frac{1+m-t}{1+m-T}\right)dt\right] = ha\,p^{-b}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right)+\frac{T^2}{4}-\frac{(1+m)T}{2}\right] \quad (14)$$

The earned interest by S is formulated as follows.

$$IE = \frac{pI_e}{T}\left[\int_0^T D_1(p)tdt + D_1(p)T(S-T)\right] = \frac{pI_e}{T}kp^{-a}\left[TS-\frac{T^2}{2}\right] \quad (15)$$

Hence, the total cost of the seller is formulated as follows.

$$TC_2(T,p) = \frac{o}{T}+\frac{h}{T}kp^{-a}\left[\frac{(1+m)^2}{2}ln\left(\frac{1+m}{1+m-T}\right)+\frac{T^2}{4}-\frac{(1+m)T}{2}\right]+\frac{c}{T}kp^{-a}(1+m)\,ln\left(\frac{1+m}{1+m-T}\right)+\frac{(C_p-C_e)}{T}x(COD_R - COD_S)kp^{-a}(1+m)\,ln\left(\frac{1+m}{1+m-T}\right)-\frac{pI_e}{T}kp^{-a}\left[TS-\frac{T^2}{2}\right] \quad (16)$$

Case 2 ($S \le T$)

The only subcase of case 2 is illustrated in Figure 6.

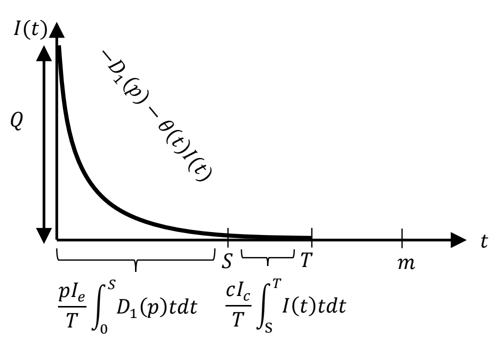

FIGURE 6
SUB-CASE 2-1 ($S \le T \le m$)

Sub-case 2.1 ($S \leq T \leq m$)

The inventory level (Figure 5) at time t is calculated as

$$I(t) = \frac{1+m-t}{1+m}\int_t^T D_1(p)\frac{1+m}{1+m-u}du = \frac{1+m-t}{1+m}kp^{-a}\int_t^T \frac{1+m}{1+m-u}du = (1+m-t)kp^{-a}\ln\left(\frac{1+m-t}{1+m-T}\right) \quad (17)$$

Therefore, the order quantity is

$$I(0) = Q = kp^{-a}(1+m)\ln\left(\frac{1+m}{1+m-T}\right) \quad (18)$$

The cost of maintaining inventory is calculated as follows.

$$h\int_0^T I(t)dt = hkp^{-a}\left[\int_0^T (1+m-t)\ln\left(\frac{1+m-t}{1+m-T}\right)dt\right] = hkp^{-a}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right] \quad (19)$$

The interest earned is calculated as follows.

$$IE = \frac{pI_e}{T}\int_0^S Dtdt = \frac{pI_e}{T}\int_0^S D_1(p)tdt = \frac{pI_e}{T}\left(\frac{kp^{-a}S^2}{2}\right) \quad (20)$$

In this sub-case, the buyer cannot pay off the purchase amount at time S, and must finance all items sold after time $T-S$ at an interest charged I_c per dollar per year.

$$IC = \frac{cI_c}{T}\int_S^T I(t)dt = \frac{1}{T}cI_ckp^{-a}\left[\frac{1}{4}(T-S)(T+S-2m-2) + \frac{1}{2}\left((m-S)+1\right)^2\ln\left(\frac{-1-m+S}{-1-m+T}\right)\right] \quad (21)$$

The annual total cost function is calculated as follows.

$$TC_3(T,p) = \frac{o}{T} + \frac{h}{T}kp^{-a}\left[\frac{(1+m)^2}{2}ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right] + \frac{c}{T}kp^{-a}(1+m)\,ln\left(\frac{1+m}{1+m-T}\right) + \frac{(C_p-C_e)}{T}x(COD_R - COD_S)kp^{-a}(1+m)\,ln\left(\frac{1+m}{1+m-T}\right) + \frac{1}{T}cI_ckp^{-a}\left[\frac{1}{4}(T-S)(T+S-2m-2) + \frac{1}{2}\left((m-S)+1\right)^2\ln\left(\frac{-1-m+S}{-1-m+T}\right)\right] - \frac{pI_e}{T}\left(\frac{kp^{-a}S^2}{2}\right) \quad (22)$$

The seller tends to optimize both p and T to find the minimum total cost. In the next section, we propose a solution procedure to characterize the selling price and replenishment cycle time simultaneously.

THEORETICAL RESULTS

The convexity of the total cost functions cannot be proved by derivations. The corollary proposed by Cambini and Martein [28] indicates that if both nominator $f(x)$ and denominator $g(x)$ of a fraction are convex and positive functions, then the whole fraction will be pseudo-convex.

$$q(x) = \frac{f(x)}{g(x)} \quad (23)$$

Therefore, we have

$$q_1(T,p) = \frac{TC_1(T,p)}{T} \quad (24)$$

and

$$q_2(T,p) = \frac{TC_2(T,p)}{T} \quad (25)$$

and

$$q_3(T,p) = \frac{TC_3(T,p)}{T} \quad (26)$$

Adopting the above-mentioned outcomes in (24)-(26), total cost functions for all subcases are rewritten and the convexity of these total cost functions are developed in the following theorems. Convexity proving procedure consists of necessary and sufficient conditions of convexity. In the following developed theorems, three bullets are indicated. The first two bullets illustrate the necessary conditions of convexity which are obtained when the second-order derivatives of the total cost function concerning variables are positive. The third bullet indicates the sufficient condition of convexity which is satisfied when the corresponding Hessian matrix of the total cost function concerning T and p is positive.

- *Theorem 1*

- Considering a desired positive value for p, $TC_1(T,p)$ is pseudo-convex concerning T.
- Considering a desired positive value for T, $TC_1(T,p)$ is pseudo-convex concerning p.
- $TC_1(T,p)$ is pseudo-convex with respect to T and p.

Refer to Appendix A for proof.

- *Theorem 2*

- Considering a desired positive value for p, $TC_2(T,p)$ is pseudo-convex concerning T.
Considering a desired positive value for T, $TC_2(T,p)$ is pseudo-convex concerning p.
- $TC_2(T,p)$ is pseudo-convex with respect to T and p.

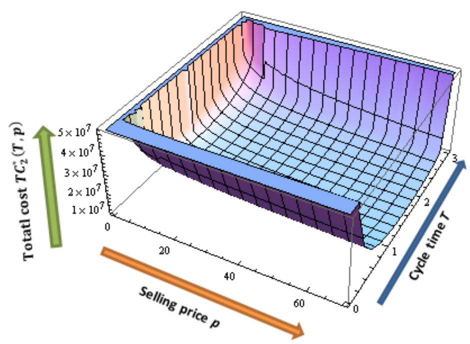

FIGURE 7
CONVEXITY OF $TC_1^*(T,p)$ WITH RESPECT TO p AND T

Refer to Appendix B for proof.

- *Theorem 3*

- Considering a desired positive value for p, $TC_3(T,p)$ is pseudo-convex concerning T.
- Considering a desired positive value for T, $TC_3(T,p)$ is pseudo-convex concerning p.
- $TC_3(T,p)$ is pseudo-convex with respect to T and p.

Refer to Appendix C for proof.

NUMERICAL EXPERIMENT

This section presents a scheme to solve three distinct inventory models that were propounded in previous sections to obtain the optimal solutions in three sub-cases. The case instances demonstrate the applicability of different sub-cases. The inventory model is utilized to regulate the trade credit period, find a better selling price, and optimize the total cost. The mathematical formulation should be simplified because of complex logarithm terms to be validated using numerical examples. To solve the model in all sub-cases, Mathematica version 11.2 is employed to find the solutions of the models as well as a parameters sensitivity analysis. The simplification process is illustrated using the McLaurin series.

$$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots \quad (27)$$

Hence, we have

$$\ln\left(\frac{1+m}{1+m-T}\right)=\left[m-\frac{m^2}{2}\right]-\left[(m-T)-\frac{1}{2}(m-T)^2\right] \quad (28)$$

Numerical examples are proposed using the abovementioned simplification.

- *Numerical example for Subcase 1.1* $(T\le m\le S)$

Consider $o=100$, $h=0.5$, $c=10$, $C_p=2$, $C_e=1$, $a=0.3$, $k=1000000$, $COD_R=400$, $COD_S=200$, $x=0.01$, $g=0.5$, $m=2$, $S=3$, and $I_e=0.1$ in appropriate units. Taking $\begin{cases}\frac{dq_1(T,p)}{dT}=0\\ \frac{dq_1(T,p)}{dp}=0\end{cases}$, we have $T^*=1.155$and $p^*=29.721$. Substituting T^* and p^* into the total cost, we have $TC_1^*(T,p)=3003377.47$. The sufficient conditions are held by $\frac{d^2TC_1(T,p)}{dT^2}=1225588.99>0$, $\frac{d^2TC_1(T,p)}{dp^2}=2956.09>0$, and $\frac{d^2TC_1(T,p)}{dpdT}=-3150.99<0$. Therefore the Hessian matrix determinant is

$|H_1| = 3702594954.75 > 0$. Furthermore, the convexity of $TC_1^*(T,p)$ with respect to selling price p and cycle time T is shown in Figure 7.

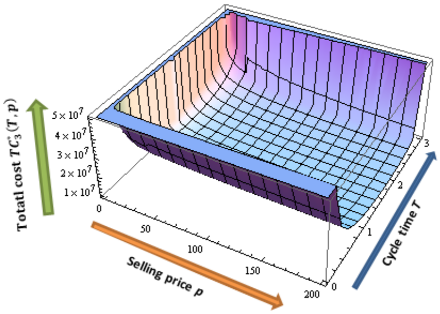

FIGURE 8
CONVEXITY OF $TC_2^*(T,p)$ WITH RESPECT TO p AND T

- *Numerical example for Subcase 1.2 ($T \le S \le m$)*

Consider $o = 10000000$, $h = 0.5$, $c = 10$, $C_p = 2$, $C_e = 1$, $a = 0.3$, $k = 1000000$, $COD_R = 400$, $COD_S = 200$, $x = 0.01$, $g = 0.5$, $m = 2$, $S = 1.5$, and $I_e = 0.1$ in appropriate units. Taking $\begin{cases} \frac{dq_2(T,p)}{dT} = 0 \\ \frac{dq_2(T,p)}{dp} = 0 \end{cases}$, we have $T^* = 1.097$ and $p^* = 80.315$.

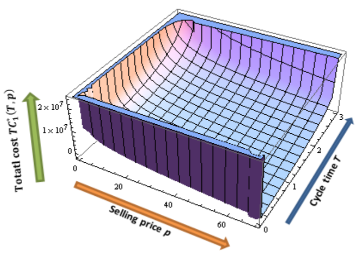

FIGURE 9
CONVEXITY OF $TC_3^*(T,p)$ WITH RESPECT TO p AND T

Substituting T^* and p^* into the total cost, we have $TC_2^*(T,p) = 2262068.89$. The sufficient conditions are held by $\frac{d^2TC_2(T,p)}{dT^2} = 15828116.23 > 0$, $\frac{d^2TC_2(T,p)}{dp^2} = 314.408 > 0$, and $\frac{d^2TC_2(T,p)}{dpdT} = 5400.07 > 0$. Therefore the Hessian matrix determinant is $|H_2| = 4947723611.63 > 0$. Furthermore, the convexity of $TC_2^*(T,p)$ with respect to selling price p and cycle time T is shown in Figure 8.

- *Numerical example for Subcase 2.1 ($S \le T \le m$)*

Consider $o = 10000000$, $h = 0.5$, $c = 10$, $C_p = 2$, $C_e = 1$, $a = 0.3$, $k = 1000000$, $COD_R = 400$, $COD_S = 200$, $x = 0.01$, $g = 0.5$, $m = 2$, $S = 0.5$, $I_c = 0.3$, and $I_e = 0.1$ in appropriate units. Taking $\begin{cases} \frac{dq_3(T,p)}{dT} = 0 \\ \frac{dq_3(T,p)}{dp} = 0 \end{cases}$, we have $T^* = 1.660$ and $p^* = 218.132$. Substituting T^* and p^* into the total cost, we have $TC_3^*(T,p) = 10016403.34$. The sufficient conditions are held by $\frac{d^2TC_3(T,p)}{dT^2} = 5781551.38 > 0$, $\frac{d^2TC_3(T,p)}{dp^2} = 33.871 > 0$, and $\frac{d^2TC_3(T,p)}{dpdT} = -1856.26 < 0$. Therefore the Hessian matrix determinant is $|H_3| = 192381225.60 > 0$. Furthermore, the convexity of $TC_3^*(T,p)$ with respect to selling price p and cycle time T is shown in Figure 9.

- *Sensitivity analysis*

This subsection provides the sensitivity analysis to study how relevant costs, demand scale, the fraction of returning items, and the interest earned rate affect the solution. One of the applicable sub-cases is discussed in **Error! Reference source not found.** in order to show the parameters attributions by changing them when other parameters are in their original values. All results are derived to achieve the minimum total cost. Some practical managerial insights are concluded from the sensitivity analysis.

TABLE 3
SENSITIVITY ANALYSIS

Sub-case 1.1 ($T \le m \le S$) $m = 2$ & $S = 3$			
Parameter	T^*	p^*	$TC_1^*(T,p)$
$o = 80$	1.155	29.721	3003360.16
$o = 100$	1.155	29.721	3003377.47
$o = 120$	1.155	29.721	3003394.79
$h = 0.2$	1.147	27.296	2927042.83
$h = 0.5$	1.155	29.721	3003377.47
$h = 0.8$	1.162	31.154	3083770.76
$c = 8$	1.157	25.166	2620759.02
$c = 10$	1.155	29.721	3003377.47
$c = 12$	1.153	33.638	3366473.26
$C_p = 1.5$	1.156	27.444	2814794.85
$C_p = 2$	1.155	29.721	3003377.47
$C_p = 2.5$	1.154	31.999	3187106.03
$k = 750000$	1.155	29.721	2252554.75
$k = 1000000$	1.155	29.721	3003377.47
$k = 1250000$	1.155	29.721	3754200.20
$COD_R = 300$	1.156	27.444	2814794.85
$COD_R = 400$	1.155	29.721	3003377.47
$COD_R = 500$	1.154	31.999	3187106.03
$x = 0.01$	1.155	29.721	3003377.47
$x = 0.02$	1.153	34.350	3363954.55
$x = 0.03$	1.151	38.998	3708170.54
$g = 0.4$	1.155	29.709	3003829.16
$g = 0.5$	1.155	29.734	3003377.47
$g = 0.6$	1.155	29.751	3002926.09
$I_e = 0.1$	1.155	29.721	3003377.47
$I_e = 0.2$	1.155	14.861	3697571.38
$I_e = 0.3$	1.155	9.907	5379575.32

THEORETICAL AND MANAGERIAL IMPLICATIONS

Based on the parameters attribution in **Error! Reference source not found.**, we can obtain the following theoretical and managerial insights.

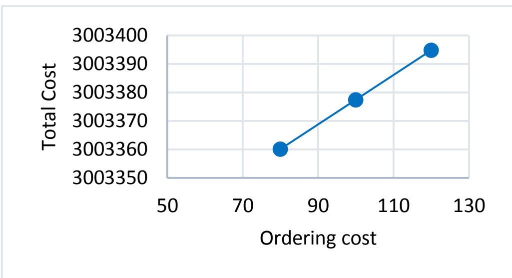

FIGURE 10
MPACT OF ORDERING COST ON THE TOTAL COST FUNCTION OF BEHNOUSH COMPANY

- Increasing ordering cost (o), purchasing cost (c), and holding cost (h) obviously increase the total cost posed to the Behnoush Company (See Figures 10, 11, and 12). In order to mitigate these relevant costs, Behnoush Company has

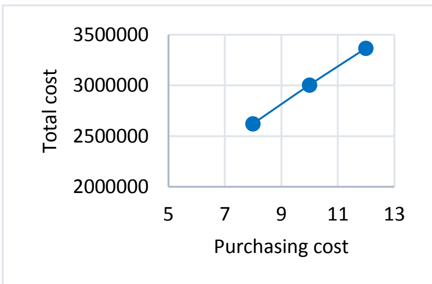

Figure 11
Impact of purchasing cost on the total cost function of Behnoush company

employed three different policies. Regarding the ordering cost, the company has decided to order items in larger quantities and with a combination of more than two specific items in an order, a significant decrease in the ordering cost is exposed. Purchasing cost which has the highest sensitivity in increasing the total cost can be decreased by developing a supplier selection model. In this regard, selecting suppliers that offer items with lower prices and acceptable qualities are preferable. Besides, Behnoush Company has investigated a policy to provide outsourcing contracts that allow other companies to hold the finished items of the company at lower prices. It is because there are a high number of pallets in the company and packaging, scheduling, and preserving of items requires a considerable investment.

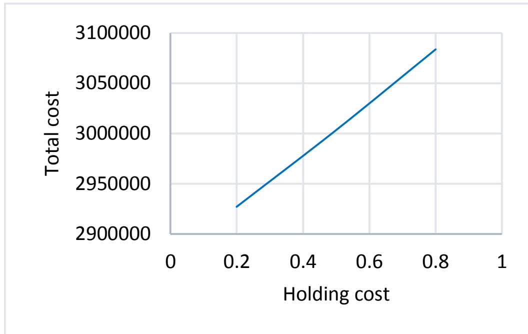

FIGURE 12
IMPACT OF HOLDING COST ON THE TOTAL COST FUNCTION OF BEHNOUSH COMPANY

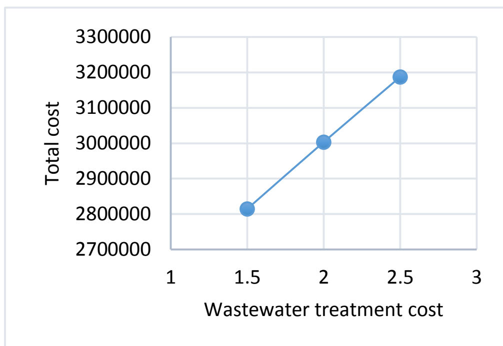

FIGURE 13
IMPACT OF WASTEWATER TREATMENT COST ON THE TOTAL COST FUNCTION OF BEHNOUSH COMPANY

- As unit wastewater treatment cost C_p increases, the total cost increases. It is quietly clear that when the company needs to invest more money in the disposal of wastewater, the total cost of its inventory system will be incremented. Hence, one way to reduce the disposal cost is to negotiate for contracts in which the disposed water can be used for some purposes such as irrigation. This will increase the amount of C_e and the factory can accumulate revenue due to this policy. In order to stimulate the revenue obtained from the wastewater treatment process, Behnoush Company has regulated beneficial contracts with the government to become involved in the irrigation of a larger area. The impact of wastewater treatment cost is illustrated in Figure 13.
- The increasing COD_R means that the deteriorated items are more polluted and more energy and purification stages are needed before the disposal. It is quietly clear that the factory needs to regulate its order quantities so fewer items will perish and the deteriorated returning items are not that polluted. These policies can decrease the value of x that is the fraction of returning items. Therefore, the total cost will be decreased. One of the solutions to this challenge is to investigate which minerals increase pollutes the water and increases COD_R. Besides, Behnoush Company has regulated a lot-sizing approach to decrease the proportion of returning items. These policies can decrease the value of x that is the fraction of returning items. Therefore, the total cost will be decreased. Moreover, they have invested in preservation technologies that lead to a decrease in the number of expired items. The impact of COD_R on the total cost of the Behnoush Company is illustrated in Figure 13.

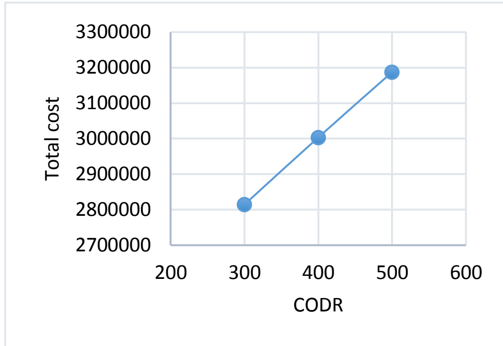

FIGURE 14
IMPACT OF COD_R ON THE TOTAL COST FUNCTION OF BEHNOUSH COMPANY

- Increasing the proportion of returning items (x) leads to a decrease in the total cost. It is because the company has sold more items and more returns are obtained. In this case, more wastewater treatment cost is posed to the company; however, the revenue from the treatment process is more than the treatment cost and this process has become beneficial to the company. The impact of the proportion of returning items on the total cost is illustrated in Figure 14.
- The rate of interest earned has a significant impact on the total cost of Behnoush Company. The reason for this circumstance is that the interest earned comprises a high proportion of the costs obtained by the company. Therefore, the company will accumulate more revenue from sales in the delayed period. However, there are specific restrictions for the earned interest. The impact of the rate of earned interest on the total cost is proposed in Figure 15

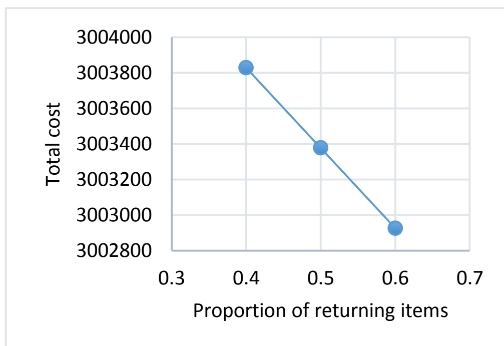

FIGURE 15
IMPACT OF THE PROPORTION OF RETURNING ITEMS ON THE TOTAL COST FUNCTION OF BEHNOUSH COMPANY

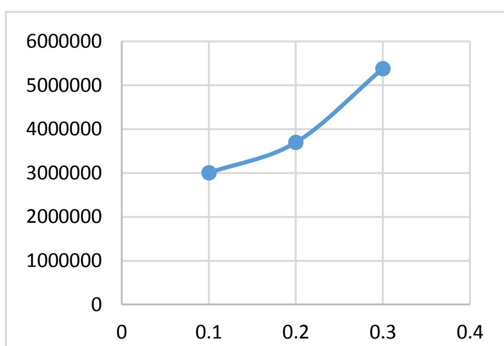

FIGURE 16
IMPACT OF THE RATE OF INTEREST EARNED ON THE TOTAL COST FUNCTION OF BEHNOUSH COMPANY

CONCLUSIONS

Increasing concerns toward developing sustainability and preservation methods, beverage companies are searching for ways to recycle their beverage returning items as items have specific expiration dates and the unsold items must be disposed of. Besides, due to environmental policies regulated by governments, disposing of expired items in a beverage company requires standard considerations. From the financial perspective, the companies offer periods of delay in payment to their buyers to stimulate them to purchase the items in larger quantities. Finding a framework that considers these challenges simultaneously was the subject of this paper. Developing a traditional inventory model for a case study leads this paper to address the mentioned challenges for deteriorating items with expiration dates considering the delay in payment, pricing, and wastewater treatment.

The outcome of this paper indicates that the role of trade credit policy in different cases varies according to the frequency of orders and the expiration date of each item. The challenge of wastewater treatment had been studied from a biological perspective and the economical aspect of the problem had never been outlined. Moreover, this paper elaborates on the possibility of selling the treated wastewater to the government for irrigation uses that results in accumulating revenue for Behnoush Company. For future investigations, one can develop the model of this research in numerous ways. For instance, one extension can be adding different permissible shortages, advance payments, or inflation rates. Partial trade credit is another possible extension. Good-credit and bad-credit customers are separated in this model. The worse-credit customers are the more prepayment of the total purchasing cost they should pay.

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APPENDIX A

Proof of Theorem 1

Using (24), two levels of partial derivatives of $q_1(T,p)$ concerning T are

$$\frac{dq_1(T,p)}{dT} = \frac{c(1+m)kp^{-a}}{1+m-T} + hkp^{-a}\left[\frac{1}{2}(-1-m) + \frac{(1+m)^2}{2(1+m-T)} + \frac{T}{2}\right] - kI_e p^{-1-a}[(S-m)e^{-gx} + m - T] + \frac{k(COD_R - COD_S)(C_p - C_e)(1+m)p^{-a}x}{1+m-T} \quad \text{(A1)}$$

and

$$\frac{d^2q_1(T,p)}{dT^2} = kI_e p^{-1-a} + \frac{kc(1+m)p^{-a}}{(1+m-T)^2} + khp^{-a}\left[\frac{1}{2} + \frac{(1+m)^2}{2(1+m-T)^2}\right] + \frac{k(COD_R - COD_S)(C_p - C_e)(1+m)p^{-a}x}{(1+m-T)^2} > 0 \quad \text{(A2)}$$

Therefore, the optimal solution of replenishment cycle time T^* is obtained from $\frac{dq_1(T,p)}{dT} = 0$

The derivatives of $q_1(T,p)$ concerning T^* and p^* is

$$\frac{d^2q_1(T,p)}{dTdp} = -\frac{kac(1+m)p^{-1-a}}{1+m-T} - kahp^{-1-a}\left[\frac{1}{2}(-1-m) + \frac{(1+m)^2}{2(1+m-T)} + \frac{T}{2}\right] - k(1-b)I_e p^{-a}[(S-m)e^{-gx} + m - T] - \frac{ka(COD_R - COD_S)(C_p - C_e)(1+m)p^{-1-a}x}{1+m-T} \quad \text{(A3)}$$

Equation (A2) represents the total cost function $q_1(T,p)$ is convex with respect to T.

The two levels of partial derivatives of $q_1(T,p)$ concerning p are

$$\frac{dq_1(T,p)}{dp} = -k(1-a)I_e p^{-a}\left((1-e^{-gx})mT + STe^{-gx} - \frac{T^2}{2}\right) - kac(1+m)p^{-1-a}\ln\left(\frac{1+m}{1+m-T}\right) - ka(COD_R - COD_S)(C_p - C_e)x(1+m)p^{-1-a}\ln\left(\frac{1+m}{1+m-T}\right) - kahp^{-1-a}\left[-\frac{1}{2}(1+m)T + \frac{T^2}{4} + \frac{1}{2}(1+m)^2\ln\left(\frac{1+m}{1+m-T}\right)\right] \quad \text{(A4)}$$

and

$$\frac{d^2q_1(T,p)}{dp^2} = ka(1-b)I_e p^{-1-a}\left((1-e^{-gx})mT + STe^{-gx} - \frac{T^2}{2}\right) + kac(a+1)(1+m)p^{-2-a}\ln\left(\frac{1+m}{1+m-T}\right) + ka(a+1)(COD_R - COD_S)(C_p - C_e)x(1+m)p^{-2-a}\ln\left(\frac{1+m}{1+m-T}\right) + ka(a+1)hp^{-2-a}\left[T(-2-2m+T) + 2(1+m)^2\ln\left(\frac{1+m}{1+m-T}\right)\right] > 0 \quad \text{(A5)}$$

If $\left[T(-2-2m+T) + 2(1+m)^2\ln\left(\frac{1+m}{1+m-T}\right)\right] \geq 0$, then the total cost function $TC_1(T,p)$ is convex with respect to p. Therefore, the optimal solution of p^* is obtained from $\frac{dq_1(T,p)}{dp} = 0$.

The optimality holds when the Hessian matrix of $q_1(T,p)$ is positive definite, where $\frac{d^2q_1(T,p)}{dT^2} > 0$ and $\frac{d^2q_1(T,p)}{dp^2} > 0$ are proved earlier. Also, the determinant of the Hessian matrix $|H_1| = \begin{vmatrix} \frac{d^2q_1(T,p)}{dT^2} & \frac{d^2q_1(T,p)}{dTdp} \\ \frac{d^2q_1(T,p)}{dTdp} & \frac{d^2q_1(T,p)}{dp^2} \end{vmatrix} = \Big\{\Big[kI_e p^{-1-a} + \frac{kc(1+m)p^{-a}}{(1+m-T)^2} + khp^{-a}\left[\frac{1}{2} + \frac{(1+m)^2}{2(1+m-T)^2}\right] + \frac{k(COD_R-COD_S)(C_p-C_e)(1+m)p^{-a}x}{(1+m-T)^2}\Big] \times \Big[ka(1-b)I_e p^{-1-a}\left((1-e^{-gx})mT + STe^{-gx} - \frac{T^2}{2}\right) + kac(a+1)(1+m)p^{-2-a}\ln\left(\frac{1+m}{1+m-T}\right) + ka(a+1)(COD_R - COD_S)(C_p - C_e)x(1+m)p^{-2-a}\ln\left(\frac{1+m}{1+m-T}\right) + ka(a+1)hp^{-2-a}\left[T(-2-2m+T) + 2(1+m)^2\ln\left(\frac{1+m}{1+m-T}\right)\right]\Big] - \Big[-\frac{kac(1+m)p^{-1-a}}{1+m-T} - kahp^{-1-a}\left[\frac{1}{2}(-1-m) + \frac{(1+m)^2}{2(1+m-T)} + \frac{T}{2}\right] - k(1-b)I_e p^{-a}[(S-m)e^{-gx} + m - T] - \frac{ka(COD_R-COD_S)(C_p-C_e)(1+m)p^{-1-a}x}{1+m-T}\Big]^2\Big\} > 0$ for any positive value of T and p.

Appendix B

Proof of Theorem 2

Using (25), two levels of partial derivatives of $q_2(T,p)$ concerning T are

$$\frac{dq_2(T,p)}{dT} = \frac{kcp^{-a}(1+m)}{1+m-T} + khp^{-a}\left[\frac{1}{2}(-1-m) + \frac{(1+m)^2}{2(1+m-T)} + \frac{T}{2}\right] - kI_e p^{1-a}(S-T) + \frac{k(COD_R-COD_S)(C_p-C_p)(1+m)p^{-a}x}{1+m-T} \quad \text{(B1)}$$

and

$$\frac{d^2q_2(T,p)}{dT^2} = kI_e p^{1-a} + \frac{kcp^{-a}(1+m)}{(1+m-T)^2} + khp^{-a}\left[\frac{1}{2} + \frac{(1+m)^2}{2(1+m-T)^2}\right] + \frac{k(COD_R-COD_S)(C_p-C_p)(1+m)p^{-a}x}{(1+m-T)^2} > 0 \quad \text{(B2)}$$

Therefore, the optimal solution of replenishment cycle time T^* is obtained from $\frac{dq_2(T,p)}{dT} = 0$

The derivatives of $q_2(T,p)$ concerning T^* and p^* is

$$\frac{d^2q_2(T,p)}{dTdp} = -\frac{kac(1+m)p^{-1-a}}{1+m-T} - kahp^{-1-a}\left[\frac{1}{2}(-1-m) + \frac{(1+m)^2}{2(1+m-T)} + \frac{T}{2}\right] - k(1-a)I_e p^{-a}(S-T) - \frac{ka(COD_R-COD_S)(C_p-C_e)(1+m)p^{-1-a}x}{1+m-T} \quad \text{(B3)}$$

Equation (B2) represents the total cost function $q_2(T,p)$ is convex with respect to T.

Using (25), the two levels of partial derivatives of $q_2(T,p)$ concerning p are

$$\frac{dq_2(T,p)}{dp} = -k(1-a)I_e p^{-a}\left(ST - \frac{T^2}{2}\right) - kac(1+m)p^{-1-a}\ln\left(\frac{1+m}{1+m-T}\right) - ka(COD_R - COD_S)(C_p - C_e)(1+m)p^{-1-a}x\ln\left(\frac{1+m}{1+m-T}\right) - kahp^{-1-a}\left[-\frac{1}{2}(1+m)T + \frac{T^2}{4} + \frac{1}{2}(1+m)^2\ln\left(\frac{1+m}{1+m-T}\right)\right] \quad \text{(B4)}$$

and

$$\frac{d^2q_2(T,p)}{dp^2} = ka(1-a)I_e p^{-1-a}\left(ST - \frac{T^2}{2}\right) + kac(1+a)(1+m)p^{-2-a}\ln\left(\frac{1+m}{1+m-T}\right) + ka(1+a)(COD_R - COD_S)(C_p - C_e)(1+m)p^{-2-a}x\ln\left(\frac{1+m}{1+m-T}\right) + ka(1+a)hp^{-2-a}\left[-\frac{1}{2}(1+m)T + \frac{T^2}{4} + \frac{1}{2}(1+m)^2\ln\left(\frac{1+m}{1+m-T}\right)\right] \quad \text{(B5)}$$

If $\left[-\frac{1}{2}(1+m)T + \frac{T^2}{4} + \frac{1}{2}(1+m)^2\ln\left(\frac{1+m}{1+m-T}\right)\right] > 0$, then the total cost function $TC_2(T,p)$ is convex with respect to p. Therefore, the optimal solution of p^* is obtained from $\frac{dq_2(T,p)}{dp} = 0$.

The optimality holds when the Hessian matrix of $q_2(T,p)$ is positive definite, where $\frac{d^2q_2(T,p)}{dT^2} > 0$ and $\frac{d^2q_2(T,p)}{dp^2} > 0$ are proved earlier. Also, the determinant of the Hessian matrix $|H_2| = \begin{vmatrix} \frac{d^2q_2(T,p)}{dT^2} & \frac{d^2q_2(T,p)}{dTdp} \\ \frac{d^2q_2(T,p)}{dTdp} & \frac{d^2q_2(T,p)}{dp^2} \end{vmatrix} = \Big\{\Big[kI_e p^{1-a} + \frac{kcp^{-a}(1+m)}{(1+m-T)^2} + khp^{-a}\left[\frac{1}{2} + \frac{(1+m)^2}{2(1+m-T)^2}\right] + \frac{k(COD_R-COD_S)(C_p-C_p)(1+m)p^{-a}x}{(1+m-T)^2}\Big] \times \Big[ka(1-a)I_e p^{-1-a}\left(ST - \frac{T^2}{2}\right) + kac(1+a)(1+$

$m)p^{-2-a}\ln\left(\frac{1+m}{1+m-T}\right)+ka(1+a)(COD_R-COD_S)(C_p-C_e)(1+m)p^{-2-a}x\ln\left(\frac{1+m}{1+m-T}\right)+ka(1+a)hp^{-2-a}\left[-\frac{1}{2}(1+m)T+\frac{T^2}{4}+\frac{1}{2}(1+m)^2\ln\left(\frac{1+m}{1+m-T}\right)\right]\Big]-\left[-\frac{kac(1+m)p^{-1-a}}{1+m-T}-kahp^{-1-a}\left[\frac{1}{2}(-1-m)+\frac{(1+m)^2}{2(1+m-T)}+\frac{T}{2}\right]-k(1-a)I_ep^{-a}(S-T)-\frac{ka(COD_R-COD_S)(C_p-C_e)(1+m)p^{-1-a}x}{1+m-T}\right]^2\Big\}>0$ for any positive value of T and p.

APPENDIX C

Proof of Theorem 3

Using (26), two levels of partial derivatives of $q_3(T,p)$ concerning T are

$$\frac{dq_3(T,p)}{dT}=\frac{kcp^{-a}(1+m)}{1+m-T}+khp^{-a}\left[-\frac{1}{2}(1+m)+\frac{(1+m)^2}{2(1+m-T)}+\frac{T}{2}\right]+kcI_cp^{-a}\left[\frac{(1+m-S)^2}{2(1+m-T)}+\frac{1}{4}(T-S)+\frac{1}{4}(-2-2m+S+T)\right]+\frac{k(COD_R-COD_S)(C_p-C_e)(1+m)xp^{-a}}{1+m-T} \quad \text{(C1)}$$

and

$$\frac{d^2q_3(T,p)}{dT^2}=\frac{kcp^{-a}(1+m)}{(1+m-T)^2}+khp^{-a}\left[\frac{1}{2}+\frac{(1+m)^2}{2(1+m-T)^2}\right]+kcI_cp^{-a}\left[\frac{1}{2}+\frac{(1+m-S)^2}{2(1+m-T)^2}\right]+\frac{k(COD_R-COD_S)(C_p-C_p)(1+m)p^{-a}x}{(1+m-T)^2}>0 \quad \text{(C2)}$$

Therefore, the optimal solution of replenishment cycle time T^* is obtained from $\frac{dq_3(T,p)}{dT}=0$

The derivatives of $q_3(T,p)$ concerning T^* and p^* is

$$\frac{d^2q_3(T,p)}{dTdp}=-\frac{kac(1+m)p^{-1-a}}{1+m-T}-kahp^{-1-a}\left[\frac{1}{2}(-1-m)+\frac{(1+m)^2}{2(1+m-T)}+\frac{T}{2}\right]-kacI_cp^{-1-a}\left[\frac{(1+m-S)^2}{2(1+m-T)}+\frac{1}{4}(T-S)+\frac{1}{4}(-2-2m+S+T)\right]-\frac{ka(COD_R-COD_S)(C_p-C_e)(1+m)p^{-1-a}x}{1+m-T} \quad \text{(C3)}$$

Equation (C2) represents the total cost function $q_2(T,p)$ is convex with respect to T.

Using (26), The two levels of partial derivatives of $q_3(T,p)$ concerning T are

$$\frac{dq_3(T,p)}{dp}=-\frac{1}{2}k(1-a)I_ep^{-a}S^2-kac(1+m)p^{-1-a}\ln\left(\frac{1+m}{1+m-T}\right)-kahp^{-1-a}\left[-\frac{1}{2}(1+m)T+\frac{T^2}{4}+\frac{1}{2}(1+m)^2\ln\left(\frac{1+m}{1+m-T}\right)\right]-ka(C_p-C_e)x(COD_R-COD_S)(1+m)p^{-1-a}\ln\left(\frac{1+m}{1+m-T}\right)-kacI_cp^{-1-a}\left[\frac{1}{4}(T-S)(-2-2m+S+T)+\frac{1}{2}(1+m-S)^2\ln\left(\frac{-1-m+S}{-1-m+T}\right)\right] \quad \text{(C4)}$$

and

$$\frac{d^2q_3(T,p)}{dp^2}=\frac{1}{2}ka(1-a)I_ep^{-1-a}S^2+kac(1+a)(1+m)p^{-2-a}\ln\left(\frac{1+m}{1+m-T}\right)+ka(a+1)hp^{-2-a}T\left[-\frac{1}{2}(1+m)T+\frac{T^2}{4}+\frac{1}{2}(1+m)^2\ln\left(\frac{1+m}{1+m-T}\right)\right]+ka(1+a)(C_p-C_e)x(COD_R-COD_S)(1+m)p^{-2-a}\ln\left(\frac{1+m}{1+m-T}\right)+kac(1+a)I_cp^{-2-a}\left[\frac{1}{4}(T-S)(-2-2m+S+T)+\frac{1}{2}(1+m-S)^2\ln\left(\frac{-1-m+S}{-1-m+T}\right)\right] \quad \text{(C5)}$$

If $\left[-\frac{1}{2}(1+m)T+\frac{T^2}{4}+\frac{1}{2}(1+m)^2 ln\left(\frac{1+m}{1+m-T}\right)\right]>0$ and $\left[\frac{1}{4}(T-S)(-2-2m+S+T)+\frac{1}{2}(1+m-S)^2\ln\left(\frac{-1-m+S}{-1-m+T}\right)\right]>0$, then the total cost function $TC_3(T,p)$ is convex with respect to p. Therefore, the optimal solution of p^* is obtained from $\frac{dq_3(T,p)}{dp}=0$.

The optimality holds when the Hessian matrix of $q_3(T,p)$ is positive definite, where $\frac{d^2q_3(T,p)}{dT^2}>0$ and $\frac{d^2q_3(T,p)}{dp^2}>0$ are proved earlier. Also, the determinant of the Hessian matrix $|H_3|=\begin{vmatrix}\frac{d^2q_3(T,p)}{dT^2} & \frac{d^2q_3(T,p)}{dTdp}\\ \frac{d^2q_3(T,p)}{dTdp} & \frac{d^2q_3(T,p)}{dp^2}\end{vmatrix}=\Big\{\Big[\frac{kcp^{-a}(1+m)}{(1+m-T)^2}+khp^{-a}\left[\frac{1}{2}+\frac{(1+m)^2}{2(1+m-T)^2}\right]+kcI_cp^{-a}\left[\frac{1}{2}+\frac{(1+m-S)^2}{2(1+m-T)^2}\right]+\frac{k(COD_R-COD_S)(C_p-C_p)(1+m)p^{-a}x}{(1+m-T)^2}\Big]\times\Big[\frac{1}{2}ka(1-a)I_ep^{-1-a}S^2+kac(1+a)(1+m)p^{-2-a}\ln\left(\frac{1+m}{1+m-T}\right)+ka(a+1)hp^{-2-a}T\left[-\frac{1}{2}(1+m)T+\frac{T^2}{4}+\frac{1}{2}(1+m)^2\ln\left(\frac{1+m}{1+m-T}\right)\right]+ka(1+a)(C_p-C_e)x(COD_R-COD_S)(1+m)p^{-2-a}\ln\left(\frac{1+m}{1+m-T}\right)+kac(1+a)I_cp^{-2-a}\left[\frac{1}{4}(T-S)(-2-2m+S+T)+\frac{1}{2}(1+m-S)^2\ln\left(\frac{-1-m+S}{-1-m+T}\right)\right]\Big]-$

$\left[-\frac{kac(1+m)p^{-1-a}}{1+m-T} - kahp^{-1-a}\left[\frac{1}{2}(-1-m) + \frac{(1+m)^2}{2(1+m-T)} + \frac{T}{2}\right] - kacI_c p^{-1-a}\left[\frac{(1+m-S)^2}{2(1+m-T)} + \frac{1}{4}(T-S) + \frac{1}{4}(-2-2m+S+T)\right] - \frac{ka(COD_R-COD_S)(C_p-C_e)(1+m)p^{-1-a}x}{1+m-T}\right]^2\Bigg\} > 0$ for any positive value of T and p.