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On the averaging of differential inclusions with Fuzzy right hand side with the average of the right hand side is absent

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Abstract

In this article we consider the averaging method for differential inclusions with fuzzy right-hand side for the case when the limit of a method of an average does not exist.

Keywords: differential inclusion, averaging method, fuzzy set, R-solution.

1. Introduction

One possibility of modeling uncertainty in a dynamical system is to replace functions in the problem

$$\dot{x}(t) = f(t, x(t)), \quad x(0) = x_0 \in R^n \tag{1.1}$$

by set-valued functions [12,13,19]. This leads to the following (generalized) initial value problem

$$\dot{x}(t) \in F(t, x(t)), \quad x(0) = x_0 \in R^n, \tag{1.2}$$

where F is a set-valued function [3,7,18].

A reasonable generalization of "set-valued" modeling, which takes aspects of gradedness into account, is the replacement of sets by fuzzy sets, i.e. (1.2) becomes the fuzzy initial value problem

$$\dot{x}(t) \in \widetilde{F}(t, x(t)), \; x(0) = x_0 \in E^n, \tag{1.3}$$

with a fuzzy function $\widetilde{F}$ [1,2,5,6,8,11].

As it is known, for usual differential inclusions the average method is well justified [15]. Therefore we in the given paper will justify a possibility of application of a method of an average for differential inclusions with a fuzzy right-hand side.

2. Preliminaries

Let $conv(R^n)$ be the family of all nonempty compact convex subsets of R^n with the Hausdorff metric

$$h(A,B) = \max\left\{\max_{a\in A}\min_{b\in B}\|a-b\|, \max_{b\in B}\min_{a\in A}\|a-b\|\right\},$$

where $\|\cdot\|$ denotes the usual Euclidean norm in R^n.

Let E^n be the family of mappings $x: R^n \to [0,1]$ satisfying the following conditions:

1) x is normal, i.e. there exists an $\xi_0 \in R^n$ such that $x(\xi_0) = 1$;

2) x is fuzzy convex, i.e. $x(\lambda\xi + (1-\lambda)\varsigma) \geq \min\{x(\xi), x(\varsigma)\}$ whenever $\xi, \varsigma \in R^n$ and $\lambda \in [0,1]$;

3) x is upper semicontinuous, i.e. for any $\xi_0 \in R^n$ and $\varepsilon > 0$ exists $\delta(\xi_0, \varepsilon) > 0$ such that

$$x(\xi) < x(\xi_0) + \varepsilon \text{ whenever } \|\xi - \xi_0\| < \delta, \; \xi \in R^n;$$

4) the closure of the set $cl\{\xi \in R^n : x(\xi) > 0\}$ is compact.

Let $\hat{0}$ be the fuzzy mapping defined by $\hat{0}(\xi) = 0$ if $\xi \neq 0$ and $\hat{0}(0) = 1$.

Definition 2.1. The set $\{y \in R^n : x(y) \geq \alpha\}$ is called the α -level $[x]^\alpha$ of a mapping $x \in E^n$ for $0 < \alpha \leq 1$. The closure of the set $\{y \in R^n : x(y) > 0\}$ is called the 0 -level $[x]^0$ of a mapping $x \in E^n$.

Theorem 2.1. [14] If $x \in E^n$ then

1) $[x]^\alpha \in conv(R^n)$ for all $0 \le \alpha \le 1$;

2) $[x]^{\alpha_2} \subset [x]^{\alpha_1}$ for all $0 \le \alpha_1 \le \alpha_2 \le 1$;

3) if $\{\alpha_k\} \subset [0,1]$ is a nondecreasing sequence converging to $\alpha > 0$, then $[x]^\alpha = \bigcap_{k \ge 1} [x]^{\alpha_k}$.

Conversely, if $\{A^\alpha : 0 \le \alpha \le 1\}$ is the family of subsets of R^n satisfying the conditions 1) - 3) then there exists $x \in E^n$ such that $[x]^\alpha = A^\alpha$ for $0 < \alpha \le 1$ and $[x]^0 = \overline{\bigcup_{0<\alpha\le 1} A^\alpha} \subset A^0$.

Define the metric $D : E^n \times E^n \to R_+$ by the equation $D(x, y) = \sup_{\alpha \in [0,1]} h([x]^\alpha, [y]^\alpha)$.

Using the results of [17], we know that

(1) (E^n, D) is a complete metric space,

(2) $D(x+z, y+z) = D(x, y)$ for all $x, y, z \in E^n$,

(3) $D(kx, ky) = |k| D(x, y)$ for all $x, y \in E^n$, $k \in R$.

Let I be an interval in R.

Definition 2.2. A mapping $f : I \to E^n$ is called continuous at point $t_0 \in I$ provided for any $\varepsilon > 0$ there exists $\delta > 0$ such that $D(f(t), f(t_0)) < \varepsilon$ whenever $|t - t_0| < \delta,\ t \in I$. A mapping $f : I \to E^n$ is called continuous on I if it is continuous at every point $t_0 \in I$.

Definition 2.3. [14] A mapping $f : I \to E^n$ is called measurable on I if for any $\alpha \in [0,1]$ the multivalued mapping $f_\alpha(t) = [f(t)]^\alpha$ is Lebesgue measurable.

Definition 2.4. [14] A mapping $f : I \to E^n$ is called integrably bounded on I if there exists a Lebesgue integrable function $k(t)$ such that $\|x\| \le k(t)$ for all $x \in f_0(t),\ t \in I$.

Definition 2.5. [14] An element $g \in E^n$ is called an integral of $f : I \to E^n$ over I if $[g]^\alpha = (A)\int_I f_\alpha(t)dt$ for any $\alpha \in (0,1]$, where $(A)\int_I f_\alpha(t)dt$ $(A) is the Aumann integral [4].

Theorem 2.2. [14] If a mapping $f : I \to E^n$ is measurable and integrably bounded then f is integrable over I.

Now, consider the Cauchy problem with small parameter

$$\dot{x} \in \varepsilon \Phi(t, x), \quad x(0) = x_0, \tag{2.1}$$

where $\varepsilon > 0$ is a small parameter, $\Phi : R_+ \times R^n \to E^n$ is a fuzzy mapping, $x_0 \in E^n$.

We interpret [2,5,6] the equation (2.1) as a family of differential inclusions

$$\dot{x}_\alpha \in \varepsilon \Phi_\alpha(t, x_\alpha(t)) \equiv \varepsilon[\Phi(t, x_\alpha(t))]^\alpha, \; x_0^\alpha \in [x_0]^\alpha \tag{2.2}$$

where the subscript α indicates that the α-level set of a fuzzy set is involved (the system (2.2) can only have any significance as a replacement for (2.1) if the solutions generate fuzzy sets (fuzzy R-solution) [8]).

In the articles [9] associate with the inclusion (2.1) the following averaged differential inclusion

$$\dot{y} \in \varepsilon \overline{\Phi}(y), \qquad y(0) = x_0, \tag{2.3}$$

where $$\lim_{T\to\infty} D\left(\overline{\Phi}(x), \frac{1}{T}\int_0^T \Phi(t, x)dt\right) = 0. \tag{2.4}$$

Here the integral of the fuzzy mapping is understood in sense [14].

3. Main Result

In this article we consider the case when the limit (2.4) does not exist but there exist fuzzy mappings $\Phi^-, \Phi^+ : R^n \to E^n$ such that

$$\lim_{T\to\infty} \beta\left(\Phi^-(x), \frac{1}{T}\int_0^T \Phi(t, x)dt\right) = 0, \tag{3.1}$$

$$\lim_{T\to\infty} \beta\left(\frac{1}{T}\int_0^T \Phi(t,x)dt, \Phi^+(x)\right) = 0 \ , \tag{3.2}$$

where $\beta(\cdot,\cdot)$ is the semideviation of the elements in the sense of fuzzy metric:

$$\beta(A,B) = \sup_{\alpha\in[0,1]} \sup_{a\in[A]^\alpha} \inf_{b\in[B]^\alpha} \|a-b\| .$$

Along with the differential inclusion (2.1) we will consider the following differential inclusions:

$$\dot{x}^- \in \varepsilon\Phi^-(x^-), \quad x^-(0) = x_0 , \tag{3.3}$$

$$\dot{x}^+ \in \varepsilon\Phi^+(x^+), \quad x^+(0) = x_0 . \tag{3.4}$$

Theorem 3.1. Let in the domain $Q = \{(t,x) | \ t\in R_+ , \ x\in G \subset E^n \}$ the following hold:

1) the fuzzy mapping $\Phi(t,x)$ is uniformly bounded with constant M, measurable in t, satisfies the Lipschitz condition in x with constant λ;

2) the fuzzy mapping $\Phi^-(x)$ is uniformly bounded with constant M, satisfies the Lipschitz condition in x with constant λ;

3) uniformly with respect to x in the domain G the limit (3.1) exists;

4) for any $x_0 \in G' \subset G$ and $t \geq 0$ the R-solution of the inclusion (3.3) $R^-(t)$ together with a σ-neighborhood belong to the domain G.

Then for any $\eta \in (0,\sigma]$ and $L > 0$ there exists $\varepsilon^0(\eta, L) > 0$ such that for all $\varepsilon \in (0,\varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$

$$R^-(t) \subset R(t) + \hat{S}_\eta(\hat{0}), \tag{3.5}$$

where $[\hat{S}_\eta(\hat{0})]^\alpha \equiv S_\eta(0)$ for all $\alpha \in [0,1]$.

Proof. Let $\alpha \in [0,1]$ is arbitrary. Divide the interval $[0, L\varepsilon^{-1}]$ on the partial intervals with the points $t_i = \dfrac{iL}{\varepsilon m}, \ i = \overline{0,m}, \ m \in N$. Let $x^-(t)$ be a solution of the inclusion

$$\dot{x}^- \in \varepsilon[\Phi^-(x^-)]^\alpha, \quad x^-(0) = x_0 . \tag{3.6}$$

Then there exists a measurable selector $\bar{u}(t)\in\left[\Phi^{-}\left(x^{-}(t)\right)\right]^{\alpha}$ such that

$$x^{-}(t)=x^{-}(t_i)+\varepsilon\int_{t_i}^{t}\bar{u}(\tau)d\tau,\ t\in[t_i,\,t_{i+1}],\ x^{-}(0)=x_0. \tag{3.7}$$

Consider the following function

$$y^1(t)=y^1(t_i)+\varepsilon u_i(t-t_i),\quad t\in[t_i,\,t_{i+1}],\ y^1(0)=x_0, \tag{3.8}$$

where

$$\left\|\frac{L}{\varepsilon m}u_i-\int_{t_i}^{t_{i+1}}\bar{u}(t)dt\right\|=\min_{u\in\left[\Phi^{-}\left(y^1(t_i)\right)\right]^{\alpha}}\left\|\frac{L}{\varepsilon m}u-\int_{t_i}^{t_{i+1}}\bar{u}(t)dt\right\|. \tag{3.9}$$

As in (3.9) the function being minimized is strongly convex and the set $\left[\Phi^{-}\left(x^{-}(t_i)\right)\right]^{\alpha}$ is compact and convex then there exists the point u_i.

Let $\delta_i=\left\|x^{-}(t_i)-y^1(t_i)\right\|$, then for $t\in[t_i,t_{i+1}]$ we have

$$\left\|x^{-}(t)-y^1(t_i)\right\|\le\left\|x^{-}(t)-x^{-}(t_i)\right\|+\left\|x^{-}(t_i)-y^1(t_i)\right\|\le\delta_i+M(t-t_i); \tag{3.10}$$

$$h\left(\left[\Phi^{-}\left(x^{-}(t)\right)\right]^{\alpha},\left[\Phi^{-}\left(y^1(t_i)\right)\right]^{\alpha}\right)\le\lambda\left[\delta_i+\varepsilon M(t-t_i)\right]. \tag{3.11}$$

From (3.9), (3.11) follow that

$$\left\|\int_{t_i}^{t_{i+1}}\bar{u}(t)dt-\int_{t_i}^{t_{i+1}}u_i dt\right\|\le h\left(\int_{t_i}^{t_{i+1}}\left[\Phi^{-}\left(x^{-}(t)\right)\right]^{\alpha}dt,\int_{t_i}^{t_{i+1}}\left[\Phi^{-}\left(y^1(t_i)\right)\right]^{\alpha}dt\right)\le$$

$$\le\int_{t_i}^{t_{i+1}}h\left(\left[\Phi^{-}\left(x^{-}(t)\right)\right]^{\alpha},\left[\Phi^{-}\left(y^1(t_i)\right)\right]^{\alpha}\right)dt\le\lambda\left[\delta_i(t_{i+1}-t_i)+\frac{\varepsilon M(t_{i+1}-t_i)^2}{2}\right]=$$

$$=\lambda\left[\delta_i\frac{L}{\varepsilon m}+\frac{L^2M}{2\varepsilon m^2}\right]. \tag{3.12}$$

Taking into account (3.7), (3.8) and (3.12) we get the following estimate:

$$\delta_{i+1}\le\delta_i+\varepsilon\lambda\left[\delta_i\frac{L}{\varepsilon m}+\frac{ML^2}{2\varepsilon m^2}\right]=\frac{\lambda ML^2}{2m^2}+\left(1+\frac{\lambda L}{m}\right)\delta_i\le$$

$$\leq \frac{ML}{2m}\left[\left(1+\frac{\lambda L}{m}\right)^{i+1}-1\right] \leq \frac{ML}{2m}\left(e^{\lambda L}-1\right). \tag{3.13}$$

As

$$\left\|x^{-}(t)-x^{-}(t_i)\right\| = \varepsilon \left\|\int_{t_i}^{t} \bar{u}(\tau)d\tau\right\| \leq \frac{ML}{m}, \qquad \left\|y^1(t)-y^1(t_i)\right\| \leq \frac{ML}{m},$$

so then using (3.13) we obtain

$$\left\|x^{-}(t)-y^1(t)\right\| \leq \frac{ML}{m}+\frac{ML}{m}+\frac{ML}{2m}\left(e^{\lambda L}-1\right)=\frac{ML}{2m}\left(e^{\lambda L}+3\right). \tag{3.14}$$

From the condition 2) of the theorem follows that for any $\eta_1 > 0$ there exists $\varepsilon^0(L, \eta_1) > 0$ such that for all $\varepsilon \leq \varepsilon^0$ the inclusion holds

$$\left[\Phi^{-}\left(y^1(t_i)\right)\right]^{\alpha} \subset \frac{\varepsilon m}{L}\int_{t_i}^{t_{i+1}}\left[\Phi\left(\tau, y^1(t_i)\right)\right]^{\alpha} d\tau + S_{\eta_1}(0). \tag{3.15}$$

So there exists a measurable function $u^1(t) \in \left[\Phi\left(t, y^1(t_i)\right)\right]^{\alpha}$, $t \in [t_i, t_{i+1}]$ such that

$$\left\|\frac{\varepsilon m}{L}\int_{t_i}^{t_{i+1}} u^1(t)dt - u_i\right\| \leq \eta_1 .$$

Consider the function

$$x^1(t) = x^1(t_i) + \varepsilon\int_{t_i}^{t} u^1(\tau)d\tau, \quad t \in [t_i, t_{i+1}], \; x^1(0) = x_0 . \tag{3.16}$$

Then from (3.15), (3.16) follows that

$$\left\|x^1(t_i)-y^1(t_i)\right\| \leq L\eta_1 .$$

As

$$\left\|x^1(t)-x^1(t_i)\right\| \leq \frac{M L}{m},$$

we obtain the following inequalities:

$$\left\|x^1(t)-y^1(t)\right\| \leq \frac{\lambda ML}{m} + L\eta_1 , \tag{3.17}$$

$$h\left(\left[\Phi\left(t,x^{1}(t)\right)\right]^{\alpha},\left[\Phi\left(t,y^{1}(t_{i})\right)\right]^{\alpha}\right)\leq\frac{\lambda ML}{m}+\lambda L\eta_{1}=\lambda L\left(\frac{M}{m}+\eta_{1}\right). \qquad (3.18)$$

From the inequality (3.18) and the way of choosing the function $u^{1}(t)$ we get

$$\rho\left(\dot{x}^{1}(t),\varepsilon\left[\Phi\left(t,x^{1}(t)\right)\right]^{\alpha}\right)\leq\varepsilon\lambda L\left(\frac{M}{m}+\eta_{1}\right),$$

where $\rho(a,B)=\min_{b\in B}\|a-b\|,\quad a\in R^{n},\ B\in conv\left(R^{n}\right).$

According to [7] there exists such a solution $x(t)$ of the inclusion (2.1) that

$$\left\|x(t)-x^{1}(t)\right\|\leq\varepsilon\lambda L\left(\frac{M}{m}+\eta_{1}\right)\int_{0}^{t}e^{\varepsilon\lambda(t-\tau)}d\tau\leq L\left(\frac{M}{m}+\eta_{1}\right)\left(e^{\lambda L}-1\right). \qquad (3.19)$$

From (3.14), (3.17), (3.19) follows that

$$\left\|x^{-}(t)-x(t)\right\|\leq\left(3e^{\lambda L}+5\right)\frac{ML}{2m}+L\eta_{1}e^{\lambda L}.$$

Choosing $m\geq\left(3e^{\lambda L}+5\right)\dfrac{ML}{\eta}$ and $\eta_{1}\leq\dfrac{\eta}{2Le^{\lambda L}}$, we get

$$\left\|x^{-}(t)-x(t)\right\|\leq\eta$$

and

$$\left[R^{-}(t)\right]^{\alpha}\subset\left[R(t)\right]^{\alpha}+S_{\eta}(0).$$

Since $\alpha\in[0,1]$ is arbitrary, we obtain $R^{-}(t)\subset R(t)+\hat{S}\left(\hat{0}\right)$. The theorem is proved.

Theorem 3.2. Let in the domain Q the following hold:

1) the mapping $\Phi(t,x)$ is uniformly bounded, measurable in t, satisfies the Lipschitz condition in x;

2) the mapping $\Phi^{+}(x)$ is uniformly bounded, satisfies the Lipschitz condition in x;

3) uniformly with respect to x in the domain G the limit (3.2) exists;

4) for any $x_0 \in G' \subset G$ and $t \geq 0$ the R-solution of the inclusion (3.4) $R^+(t)$ together with a σ-neighborhood belong to the domain G.

Then for any $\eta \in (0,\sigma]$ and $L > 0$ there exists $\varepsilon^0(\eta, L) > 0$ such that for all $\varepsilon \in (0,\varepsilon^0]$ and $t \in \left[0, L\varepsilon^{-1}\right]$

$$R(t) \subset R^+(t) + \hat{S}_\eta(\hat{0}). \tag{3.20}$$

The proof of the theorem is carried on similarly to the proof of the theorem 3.1.

Remark 3.1. In the capacity of the mappings $\Phi^-(x)$ and $\Phi^+(x)$ one can use the superior and inferior limit of the sequence of sets [10]:

$$\underline{\lim}_{T\to\infty} D\left(\overline{\Phi}^{\,-}(x), \frac{1}{T}\int_0^T \Phi(t,x)dt\right) = 0, \qquad \overline{\lim}_{T\to\infty} D\left(\overline{\Phi}^{\,+}(x), \frac{1}{T}\int_0^T \Phi(t,x)dt\right) = 0.$$

The sets $\overline{\Phi}^{\,-}(x)$ and $\overline{\Phi}^{\,+}(x)$ are the maximum and the minimum with respect to the inclusion among the sets $\Phi^-(x)$ and $\Phi^+(x)$, that is for any $\Phi^-(x)$ and $\Phi^+(x)$ the inclusions hold

$$\Phi^-(x) \subset \overline{\Phi}^{\,-}(x), \qquad \overline{\Phi}^{\,+}(x) \subset \Phi^+(x).$$

Remark 3.2. If the limit (2.4) exists then $\overline{\Phi}^{\,-}(x) = \overline{\Phi}^{\,+}(x) = \overline{\Phi}(x)$ and from theorems 3.1, 3.2 the theorem [9] follows.

4. Conclusion

It is also possible to use the partial averaging of the differential inclusions with fuzzy right-hand side, i.e. to average only some summands or factors. Such variant of the averaging method also leads to the simplification of the initial inclusion and happens to be useful when the average of some functions does not exist or their presence in the system does not complicate its research.

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