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The interval Malmquist productivity index in DEA

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Abstract

One of the most popular approaches to measuring productivity changes is based on using Malmquist productivity indexes. In this paper we propose a method for obtaining interval Malmquist productivity index (IMPI). The classical DEA models have been before used for measuring the Malmquist productivity index. The current article extends DEA models for measuring the interval Malmquist productivity index by utilize the bounded DEA models instead of classical DEA models.

Keywords: DEA; interval efficiency; Malmquist productivity index

1. Introduction

Data envelopment analysis (DEA) is a non-parametric technique for measures the relative efficiencies of DMUs with multiple inputs and multiple outputs. DEA measures the relative efficiency of peer units when multiple outputs and multiple inputs are present fare et al. (1992, 1994a) develop a DEA-based Malmquist productivity index which measures the technical and productivity changes over time. The Malmquist index was first suggested by Malmquist (1953) as a quantity index for use in the analysis of consumption of inputs; Fare et al. (1992)

combined ideas on the measurement of efficiency from Farrell (1957) and the measurement of productivity from Caves et al. (1982) to construct a Malmquist productivity index directly from input and output data using DEA.

This paper extends the Malmquist index to interval Malmquist index by using the interval efficiency. The rest of the article is organized as follows. In section 2, we introduce the basic DEA models for measuring the best and worst relative efficiencies of DMUs, the bounded DEA models and also the Malmquist productivity index. Section 3 introduces the interval Malmquist productivity index. An illustrative application is presented in section 4. The paper is concluded in section 5.

2. Background

2.1. The best and the worst relative efficiency of DMUs

Suppose we have n DMUs, each DMU using m inputs to produce s outputs. The best efficiency of DMU_o relative to the other DMUs is evaluated by the following basic DEA model:

$$\begin{aligned} max \quad & \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}} \\ s.t. \quad & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1, \quad j=1,\dots,n, \\ & u_r, v_i \ge 0, \quad r=1,\dots,s, \quad i=1,\dots,m. \end{aligned} \tag{1}$$

Using the Charnes–Cooper transformation, the fractional programming (1) can be converted into the following linear model (2) which is called CCR multiplier model:

$$\begin{aligned} max \quad & \sum_{r=1}^{s} u_r y_{ro} \\ s.t. \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j=1,\dots,n, \\ & \sum_{i=1}^{m} v_i x_{io} = 1, \\ & u_r, v_i \ge 0, \quad r=1,\dots,s, \quad i=1,\dots,m. \end{aligned} \tag{2}$$

In model (2) DMU_o is evaluated efficient, when the optimal value of objective function equals 1, otherwise it is evaluated as inefficient.

The worst relative efficiency of DMU_o can be measured by following fractional programming model:

$$\begin{aligned} min \quad & \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}} \\ s.t. \quad & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \geq 1, \quad j=1,\dots,n, \\ & u_r, v_i \geq 0, \quad r=1,\dots,s, \quad i=1,\dots,m. \end{aligned} \tag{3}$$

Model (3) can be transformed into the following equivalent LP model:

$$\begin{aligned} min \quad & \sum_{r=1}^{s} u_r y_{ro} \\ s.t. \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \geq 0, \quad j=1,\dots,n, \\ & \sum_{i=1}^{m} v_i x_{io} = 1, \\ & u_r, v_i \geq 0, \quad r=1,\dots,s, \quad i=1,\dots,m. \end{aligned} \tag{4}$$

Determined efficiency by the LP model (4) equals 1, and then DMU_o is rated as efficient, otherwise is not rated as inefficient. The interval efficiency is given by two models (2) and (4) of the performance of each DMU. It is obvious that the best and the worst relative efficiencies are obtained of different constraints set (different feasible spaces). Therefore, they can not be used to form an efficiency interval for each DMU. In continue we present another approach for this aim (Wang, et al. 2007).

2.2. Bounded DEA model

In order to reasonably determine of the interval efficiency for each decision making unit, we use the bounded DEA models as follow:

Definition: An anti-ideal DMU (ADMU) is a virtual DMU which consumes the most inputs only to produce the least outputs.

According to the above definition, the inputs and outputs of the anti-ideal DMU determined by the following equations:

$$x_i^{\max} = \max_j \{x_{ij}\}, \quad i=1,\dots,m,$$

$$y_r^{\min} = \min_j \{y_{rj}\}, \quad r=1,\dots,s.$$

The best relative efficiency of the anti-ideal DMU is determined by the following fractional programming model (Wang, et al. 2007).

$$\begin{aligned} max \quad & \theta_{ADMU} = \frac{\sum_{r=1}^{s} u_r y_r^{\min}}{\sum_{i=1}^{m} v_i x_i^{\max}} \\ s.t. \quad & \theta_j = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1, \quad j = 1,\dots,n, \\ & u_r, v_i \ge 0, \quad r = 1,\dots,s, \quad i = 1,\dots,m. \end{aligned} \tag{5}$$

The model (5) can be replaced with the following LP model:

$$\begin{aligned} max \quad & \sum_{r=1}^{s} u_r y_r^{\min} \\ s.t. \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j = 1,\dots,n, \\ & \sum_{i=1}^{m} v_i x_i^{\max} = 1, \\ & u_r, v_i \ge 0, \quad r = 1,\dots,s, \quad i = 1,\dots,m. \end{aligned} \tag{6}$$

The efficiencies of all DMUs cannot be less than θ^*_{ADMU} such as θ^*_{ADMU} is the optimal value of objective function model (6). With respect to this idea the following DEA bounded model is introduced:

$$\begin{aligned} max \quad & \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}} \\ s.t. \quad & \theta^*_{ADMU} \le \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1, \quad j = 1,\dots,n, \\ & u_r, v_i \ge 0, \quad r = 1,\dots,s, \quad i = 1,\dots,m. \end{aligned} \tag{7}$$

The model (7) can be transformed into the following LP models:

$$\begin{aligned} max/min \quad & \sum_{r=1}^{s} u_r y_{ro} \\ s.t. \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j = 1,\dots,n, \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i (\theta^*_{ADMU} x_{ij}) \ge 0, \quad j = 1,\dots,n, \\ & \sum_{i=1}^{m} v_i x_{io} = 1, \\ & u_r, v_i \ge 0, \quad r = 1,\dots,s, \quad i = 1,\dots,m. \end{aligned} \tag{8}$$

After solving two models (8), the efficiency interval is denoted by $[\theta_o^{L*}, \theta_o^{U*}]$ where θ_o^{U*} and θ_o^{L*} are maximum and minimum of the above objective function, respectively.

Definition. If $\theta_o^{U*} = 1$ in maximization form of model (8), then DMU_o is evaluated efficient; and DMU_o is evaluated inefficient in minimization form of model (8), when $\theta_o^{L*} = \theta_{ADMU}^*$, otherwise DMU_o is called DEA unspecified.

2.3. Malmquist productivity index

The Malmquist index was defined by Fare et al., (1992) as follows. Consider two time periods t and $t+1$ and suppose we have production function in time period t as well as $t+1$. Malmquist index calculation requires two single period and two mixed period measures. The two single period measures can be obtained by using the CCR DEA model (Charnes et al. 1978):

$$
\begin{aligned}
D_o^t(x_o^t, y_o^t) = \min \quad & \theta \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij}^t \le \theta x_{io}^t, \quad i = 1,\dots,m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^t \ge y_{ro}^t, \quad r = 1,\dots,s, \\
& \lambda_j \ge 0, \quad j = 1,\dots,n.
\end{aligned}
\tag{9}
$$

Where x_{io}^t is the ith input and y_{ro}^t is the rth output for DMU_o in time period t. The efficiency ($D_o^t(x_o^t, y_o^t) = \theta_o^*$) determines the amount by which observed inputs can be proportionally reduced, while still producing the given output level. Using $t+1$ instead of t for the above model, we get the other technical efficiency score for DMU_o in time period $t+1$ where it is defined as $D_o^{t+1}(x_o^{t+1}, y_o^{t+1})$.

The first of the mixed period measures, which is defined as $D_o^t(x_o^{t+1}, y_o^{t+1})$ is calculated as optimal value to the following linear programming problem:

$$
\begin{aligned}
D_o^t(x_o^{t+1}, y_o^{t+1}) = \min \ & \theta \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij}^t \le \theta x_{io}^{t+1}, \quad i = 1,\dots,m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^t \ge y_{ro}^{t+1}, \quad r = 1,\dots,s, \\
& \lambda_j \ge 0, \quad j = 1,\dots,n.
\end{aligned}
\tag{10}
$$

Using $t+1$ instead t and vice versa, the other mixed period is obtained as follows:

$$
\begin{aligned}
D_o^{t+1}(x_o^t, y_o^t) = \min\ & \theta \\
s.t.\quad & \sum_{j=1}^{n} \lambda_j x_{ij}^{t+1} \le \theta x_{io}^t, \quad i = 1,\dots,m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^{t+1} \ge y_{ro}^t, \quad r = 1,\dots,s, \\
& \lambda_j \ge 0, \quad j = 1,\dots,n.
\end{aligned} \tag{11}
$$

Then Malmquist productivity index is defined as

$$
MI_o = \left[\frac{D_o^t(x_o^{t+1}, y_o^{t+1})}{D_o^t(x_o^t, y_o^t)} \frac{D_o^{t+1}(x_o^{t+1}, y_o^{t+1})}{D_o^{t+1}(x_o^t, y_o^t)} \right]^{1/2}
$$

If $MI_o > 1$, then productivity gain and if $MI_o < 1$, then productivity loss. Productivity is unchanged if $MI_o = 1$.

The Malmquist index can be decomposed into two components as follows (Fare et al. (1992)):

$$
\begin{aligned}
MI_o &= \left[\frac{D_o^t(x_o^{t+1}, y_o^{t+1})}{D_o^t(x_o^t, y_o^t)} \frac{D_o^{t+1}(x_o^{t+1}, y_o^{t+1})}{D_o^{t+1}(x_o^t, y_o^t)} \right]^{1/2} \\
&= \frac{D_o^{t+1}(x_o^{t+1}, y_o^{t+1})}{D_o^t(x_o^t, y_o^t)} \left[\frac{D_o^t(x_o^{t+1}, y_o^{t+1})}{D_o^{t+1}(x_o^{t+1}, y_o^{t+1})} \frac{D_o^t(x_o^t, y_o^t)}{D_o^{t+1}(x_o^t, y_o^t)} \right]^{1/2}
\end{aligned}
$$

where the first component measures the technical efficiency change between two time periods and the next component measures the technology frontier shift between period t and $t+1$.

3. *The interval Malmquist productivity index*

In section using the bounded DEA model, we present an interval Malmquist productivity index. At first we should evaluate the anti-ideal DMU efficiency scores for each of time periods t and $t+1$. Taking time period t as the reference period, the θ_{ADMU}^{t*} is used for measures the $D_o^t(x_o^t, y_o^t)$ and $D_o^t(x_o^{t+1}, y_o^{t+1})$ which obtain the following model:

$$\begin{aligned} max \quad & \theta^t_{ADMU} = \sum_{r=1}^{s} u_r y_r^{\min,t} \\ s.t. \quad & \sum_{r=1}^{s} u_r y_{rj}^t - \sum_{i=1}^{m} v_i x_{ij}^t \le 0, \quad j = 1,\dots,n \\ & \sum_{i=1}^{m} v_i x_i^{\max,t} = 1 \\ & u_r, v_i \ge 0, \quad r = 1,\dots,s, \quad i = 1,\dots,m. \end{aligned} \tag{12}$$

The $x_i^{\max,t}, y_r^{\min,t}$ are the inputs and outputs of anti-ideal DMU in time period t which determined as follow:

$$x_i^{\max t} = \max_j \{x_{ij}^t\}, \qquad i = 1,\dots, m,$$
$$y_r^{\min,t} = \min_j \{y_{rj}^t\}, \qquad r = 1,\dots, s.$$

Similarly, it can be seen that we can obtain the anti-ideal DMU relative efficiency for time period $t+1$ as θ^{t+1*}_{ADMU} which is utilized to measures the interval efficiencies of $D_o^{t+1}(x_o^{t+1}, y_o^{t+1})$ and $D_o^{t+1}(x_o^t, y_o^t)$. The following model measures the interval efficiency:

$$\begin{aligned} D_o^t(x_o^t, y_o^t) = max/min \quad & \sum_{r=1}^{s} u_r y_{ro}^t \\ s.t. \quad & \sum_{r=1}^{s} u_r y_{rj}^t - \sum_{i=1}^{m} v_i x_{ij}^t \le 0, \quad j = 1,\dots,n, \\ & \sum_{r=1}^{s} u_r y_{rj}^t - \sum_{i=1}^{m} v_i (\theta^{t*}_{ADMU} x_{ij}^t) \ge 0, \quad j = 1,\dots,n, \\ & \sum_{i=1}^{m} v_i x_{io}^t = 1, \\ & u_r, v_i \ge 0, \quad r = 1,\dots,s, \quad i = 1,\dots,m. \end{aligned} \tag{13}$$

Suppose $D_o^{t,U}(x_o^t, y_o^t)$ and $D_o^{t,L}(x_o^t, y_o^t)$ be the upper and lower bound of the interval efficiency $D_o^t(x_o^t, y_o^t)$, which get by the maximization and minimization problems (13), respectively. Then efficiency interval is denoted by $[D_o^{t,L}(x_o^t, y_o^t), D_o^{t,U}(x_o^t, y_o^t)]$.

Similarly, Using $t+1$ instead of t, we get the interval efficiency $D_o^{t+1}(x_o^{t+1}, y_o^{t+1})$ as $[D_o^{t+1,L}(x_o^{t+1}, y_o^{t+1}), D_o^{t+1,U}(x_o^{t+1}, y_o^{t+1})]$.

The interval efficiency for mixed period measures is computed as follows:

$$
\begin{aligned}
D_o^t(x_o^{t+1}, y_o^{t+1}) = max/min \quad & \sum_{r=1}^{s} u_r y_{ro}^{t+1} \\
s.t. \quad & \sum_{r=1}^{s} u_r y_{rj}^t - \sum_{i=1}^{m} v_i x_{ij}^t \le 0, \quad j = 1,...,n, \\
& \sum_{r=1}^{s} u_r y_{rj}^t - \sum_{i=1}^{m} v_i (\theta_{ADMU}^{t*} x_{ij}^t) \ge 0, \quad j = 1,...,n, \\
& \sum_{i=1}^{m} v_i x_{io}^{t+1} = 1, \\
& u_r, v_i \ge 0, \quad r = 1,...,s, \quad i = 1,...,m.
\end{aligned} \tag{14}
$$

The interval efficiency for $D_o^t(x_o^{t+1}, y_o^{t+1})$ is given as $[D_o^{t,L}(x_o^{t+1}, y_o^{t+1}), D_o^{t,U}(x_o^{t+1}, y_o^{t+1})]$ and similarly, using $t+1$ instead t and vice versa for the model (14), we get the interval efficiency for $D_o^{t+1}(x_o^t, y_o^t)$ as defined $[D_o^{t+1,L}(x_o^t, y_o^t), D_o^{t+1,U}(x_o^t, y_o^t)]$. As above mentioned, we know that the Malmquist productivity index is defined as follows:

$$
MI_o = \left[\frac{D_o^t(x_o^{t+1}, y_o^{t+1})}{D_o^t(x_o^t, y_o^t)} \frac{D_o^{t+1}(x_o^{t+1}, y_o^{t+1})}{D_o^{t+1}(x_o^t, y_o^t)} \right]^{1/2}
$$

Note, we obtain an interval for each of factors in above formula. Measuring the Malmquist index as an interval for any DMU_o, $(o \in \{1, 2, ..., n\})$, the lower and upper bound of Malmquist productivity index are given as follows:

$$
MI_o^L = \left[\frac{D_o^{t,L}(x_o^{t+1}, y_o^{t+1})}{D_o^{t,U}(x_o^t, y_o^t)} \frac{D_o^{t+1,L}(x_o^{t+1}, y_o^{t+1})}{D_o^{t+1,U}(x_o^t, y_o^t)} \right]^{1/2} \tag{15}
$$

$$
MI_o^U = \left[\frac{D_o^{t,U}(x_o^{t+1}, y_o^{t+1})}{D_o^{t,L}(x_o^t, y_o^t)} \frac{D_o^{t+1,U}(x_o^{t+1}, y_o^{t+1})}{D_o^{t+1,L}(x_o^t, y_o^t)} \right]^{1/2}
$$

Thus the interval Malmquist productivity index is denoted as $[MI_o^L, MI_o^U]$. With respect to obtained Malmquist productivity index, we cluster decision making units in two periods time t and $t+1$, for evaluation progress and regress as follows:

$$
M^{++} = \{DMU_j \mid MI_j^L > 1\}
$$

$$
M^{+} = \{DMU_j \mid MI_j^U < 1\}
$$

$$
M^{-} = \{DMU_j \mid MI_j^L \le 1, MI_j^U \ge 1\}
$$

Where M^{++} and M^{+} sets contains the units which have progress and regress from period t to $t+1$, respectively. Consider the below cases for determining the progress and the regress of units which belong to set M^{-}:

(a) If $MI_o^L = 1$ and $MI_o^U = 1$, then there is not progress and regress for DMU_o.

(b) If $MI_o^L = 1$ and $MI_o^U > 1$, then DMU_o has only progress.

(c) If $MI_o^L < 1$ and $MI_o^U = 1$, then DMU_o has only regress.

(d) If $MI_o^L < 1$ and $MI_o^U > 1$, the following index is proposed for evaluating ratio of progress or regress for DMU_o:

$$\rho = \frac{MI_o^U - 1}{1 - MI_o^L}$$

It is travail that $0 \leq \rho \leq +\infty$. $\rho > 1$ Indicates more percent of progress relative to regress and $\rho < 1$ indicates more percent of regress relative to progress for DMU_o.

4. *Numerical example*

We now illustrate a numerical example. Consider 7 DMUs, each DMU use single inputs to produce two outputs where the input and outputs set in two time periods are shown in Table1. The Malmquist productivity index measures for each DMU and illustrated in Table 2. For measuring the interval Malmquist productivity index by formulas (12),(13),and (14), we obtain the lower and the upper bound of IMPI factors for all DMUs which are given in Table 3. The reports and results and also the lower and upper bound of IMPI for each of DMUs are come in the Table 4. The upper and lower bound of Malmquist productivity for units A-F are not less and greater than 1 respectively, therefore these units not belong to sets M^{++} and M^{+}.

5. *Conclusions*

The classical DEA models have been before used for measuring the Malmquist productivity index. This paper is presented a method for obtaining interval Malmquist productivity index by using the bounded DEA models. However, we utilized the bounded DEA models as are introduced by Wang et al. (2007), for this purpose. Since these models determine DMUs efficiency in a reasonable manner as an interval so that we can obtain the best and the worst relative efficiencies of DMUs and also the range of their efficiencies. This paper is used the bounded DEA model for crisp data for measuring interval Malmquist productivity index. One can be extended it in order to finding interval Malmquist productivity index with fuzzy, interval and ordinal data.

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