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Using lexicographic parametric programming for identifying efficient hyperpalnes in DEA

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Abstract

This paper investigates a procedure for identifying all efficient hyperplanes of production possibility set (PPS). This procedure is based on a method which recommended by Pekka J. Korhonen[8]. He offered using of lexicographic parametric programming method for recognizing all efficient units in data envelopment analysis (DEA). In this paper we can find efficient hyperplanes, via using the parameterization of the right hand side vector of the envelopment problem of each efficient unit.

Keywords: Efficiency analysis; Data envelopment analysis; Lexicographic; Parametric programming; Efficient hyperplanes.

1. Introduction

Data Envelopment Analysis (DEA) is a non-parametric method which was originally proposed by Charnes, Cooper and Rhodes [2] and it has become one of the most widely used methods in operations research/management science.

DEA is a mathematical programming approach that focuses on an important task: to evaluate performance of the units called decision making units (DMUs). In order to reach this aim DEA assess the relative efficiency of comparable units (DMUs) with multiple inputs and outputs.

Based on different essential properties and information about existing data on the performance of the units, DEA forms efficient surfaces (frontiers). If a DMU lies on the surface, it is referred to as an efficient unit; otherwise, it is inefficient.

DEA also provides efficiency scores and reference units for inefficient DMUs. The reference set for inefficient units consists of efficient units and determines an exclusive hypothetical on the efficient frontier, which can be regarded as a target unit for those inefficient units. However finding all references of an inefficient DMU is not an easy job.

Although identifying efficient units has been initial task in DEA, but no many papers in DEA have been written on the subject of "finding efficient hyperplanes", such as: a) Jahanshahloo et al. [5] tried to establish a method to find all strong defining hyperplanes of production possibility set, b) Zohrebandian et al. [9] investigated a method to obtain efficient frontier by using 0-1 integer programming.

Korhonen [8] exhibited using of lexicographic parametric programming to classify the units as efficient, extreme efficient, weakly efficient and inefficient. Using that interactive technique, we may move from unit to unit along an efficient curve. The units entering to basis are recognized efficient and all units dominated by an efficient facet are inefficient.

In this paper, we suggest a procedure to find all efficient hyperplenes of production possibility set. Finding the PPS structural hyperplanes has a special significance, like obtaining the returns to scale of DMUs based on the position of the efficient hyperplanes of efficient frontier.

The sections of this paper are organized as follows. In the next section, section 2, necessary theory, some basic definitions, models and concepts are represented; the main principles of the procedure are described in section 3. Section 4, illustrates an example and section 5 concludes the paper with some remarks.

2. Backgrounds

2.1. Lexicographic parametric programming

Consider the MOLP problem:

$$
\begin{aligned}
&\max \quad c^1x\\
&\max \quad c^2x\\
&\quad \vdots\\
&\max \quad c^Px\\
&s.t \quad Ax=b\\
&\qquad x\geq 0
\end{aligned}
$$

(1)

where $A=\left[a_{ij}\right]^{m\times n}$, (m<n), b is a m-vector and $C=\left[c^i\right]^{p\times n}$, $(p\geq 2)$. Also assume that $\text{rank}(A)=m$ and $\text{rank}(C)=p$. One way to solve this kind of problems is "Lexicographic optimization" as follows:

$$
\begin{aligned}
&\text{lex max} \ \{c^1x, c^2x, ..., c^px\}\\
&s.t \quad Ax=b\\
&\qquad x\geq 0,
\end{aligned}
$$

(2)

In lexicographic optimization, we proceed in this manner: initially, solve the problem only with the first objective function. If the optimal solution is unique consider it for whole program, else the second objective is used with this additional restriction: $c^1x=q_1^*$, that q_1^* is the optimal value of objective function. If second problem had unique solution, then it is the optimal solution of whole program, else use the next objective function with this additional restriction: $c^2x=q_2^*$ and so on. Generally, k-th problem that we should solve is:

$$
\begin{aligned}
&\max \quad c^kx\\
&\text{s.t} \quad c^ix=q_i^*, \quad i=1,2,...,k-1\\
&\qquad Ax=b,\\
&\qquad x\geq 0
\end{aligned}
\tag{3}
$$

Consider x^* as the optimal basic feasible solution of lexicographic optimization problem (2). Let B be an optimal basis corresponding to the optimal solution x^* and let $A=[a_1,a_2,...,a_n]=[B,N]$, where N consists of the non-basic columns of A. We denote $Z_j^k=c_B^kB^{-1}a_j$.

Definition 1. The solution x^* is the optimal solution of lexicographic optimization problem (2) if it is feasible and the following optimal conditions are fulfilled:

$$
\begin{aligned}
&Z_j^1-c_j^1\geq 0, \quad j\in R\\
&\text{and if} \quad \exists i,\exists j: Z_j^i-c_j^i=0 \quad \text{then} \quad Z_j^{i+1}-c_j^{i+1}\geq 0, \ j\in R.
\end{aligned}
$$

where R is the index set of all non-basic variables.

2.2. DEA

Suppose that we have n DMUs, each of them consumes m inputs to produce s outputs. We denote by $x_j \in \mathbb{R}^m_+$, the vector of inputs consumed by DMU_j that produces output vector $y_j \in \mathbb{R}^s_+$, also denote by 1 the summation vector $(1,...,1)^T$.

In DEA, the Production Possibility Set (PPS) is defined as follows:

$$T = \{(x,y) \mid y \text{ can be produced by} x\}.$$

This can be defined by (in presence of constant returns to scale):

$$T_c = \left\{(x,y) \middle| x \geq \sum_{j=1}^{n} \lambda_j x_j, y \leq \sum_{j=1}^{n} \lambda_j y_j, \lambda = (\lambda_1,...,\lambda_n) \geq 0\right\},$$

where $\lambda \in \mathbb{R}^n_+$ is semipositive vector. Adding the constraint $1\lambda = 1$, to T_c, we obtain T_v as follows: $T_V = \left\{(x,y) \middle| x \geq \sum_{j=1}^{n} \lambda_j x_j, y \leq \sum_{j=1}^{n} \lambda_j y_j, 1\lambda = 1, \lambda \geq 0\right\}$.

As mentioned T_c is built on the assumption of constant returns to scale but in T_v we don't agree to any assumption of returns to scale of activities.

For calculating the efficiency of DMUs under assumption of constant returns to scale, Charnes et al. [2], [3] introduced the CCR models. Later Banker et al. [1] developed the so-called BCC models with variable returns to scale.

The CCR and BCC models can be presented in a primal or dual form, that we call them multiplier and envelopment form, respectively. The multiplier model provides information on the weights of inputs and outputs, wherever the envelopment models give us information on the lacks of outputs and surplus of inputs of a unit. Moreover, the envelopment model characterizes the reference set for inefficient units. References of an inefficient DMU_o (under evaluation DMU) are efficient DMUs that there is a combination of them such that dominates DMU_o. We say $DMU_p : (x_p, y_p)$ dominates $DMU_q : (x_q, y_q)$ if and only if $x_p \leq x_q$, $y_p \geq y_q$ and inequality be strict in at least one component. All references of a unit are efficient and there exist a defining hyperplane which contains all these DMUs, i.e. the references are always on efficient surfaces. [6]

Without loss of generality, we will consider a DEA model, that Halme et al. [4] called it a general combined model. This model uses a general directional vector $W = \begin{pmatrix} w^y \\ w^x \end{pmatrix} \geq 0,\ W \neq 0$. By different value of W in the following general DEA formulation, in the so-called envelopment form, we can obtain basic envelopment models of DEA:

$$\begin{aligned} \max \quad & z = \sigma + \varepsilon(1^T s^+ + 1^T s^-) \\ s.t \quad & \sum_{j=1}^{n} \lambda_j y_j - \sigma w^y - s^+ = y_o \\ & \sum_{j=1}^{n} \lambda_j x_j + \sigma w^x + s^- = x_o \\ & \lambda \in \Lambda, \\ & \lambda, s^+, s^- \geq 0, \\ & \varepsilon > 0 \quad \text{("Non-Archimedean")}, \end{aligned} \tag{4}$$

where x_o is the input vector and y_o is the output vector of a DMU_o under consideration.

$$\Lambda = \begin{cases} \{\lambda \mid 1^T \lambda = 1 \ , \lambda \geq 0\} & \text{for variable returns to scale model} \\ \{\lambda \mid 1^T \lambda \leq 1 \ , \lambda \geq 0\} & \text{for non-increasing returns to scale model} \\ \{\lambda \mid 1^T \lambda \geq 1 \ , \lambda \geq 0\} & \text{for non-decreasing returns to scale model} \\ \{\lambda \mid \lambda \geq 0\} & \text{for constant returns to scale model} \end{cases}$$

In (4), when $w^y = y_o$ and $w^x = x_o$ we have combined model. In the input oriented model $w^y = 0$ and $w^x = x_o$ and for output oriented model $w^x = 0$ and $w^y = y_o$. The value of σ is called an inefficiency score. A DMU is efficient if and only if the optimal value of model (4), σ^*, equal 0 and also all the slack variables s^-, s^+ are zero. Otherwise, the DMU is inefficient [7]. In the case $z^* > 0$ but $\sigma^* = 0$, we have weakly efficient solutions.

There is two ways to deal with the infinitesimal in the model (4): a) to replace it by a small number or b) to use a lexicographic approach. The second is a better way. Hence we use the lexicographic formulation model (4):

$$
\begin{aligned}
&\text{lex max} \quad \left\{\sigma, 1^T S\right\} \\
&s.t \quad \sum_{j=1}^{n} \lambda_j y_j - \sigma w^y - s^+ = y_o \\
&\qquad \sum_{j=1}^{n} \lambda_j x_j + \sigma w^x + s^- = x_o \\
&\qquad \lambda \in \Lambda, \\
&\qquad \lambda, s^+, s^- \geq 0,
\end{aligned}
\tag{5}
$$

As declared before we first solve (5), using σ as an objective function. If the optimal solution, σ^*, is not unique, we formulate a new model by adding the constraint $\sigma = \sigma^*$ into model (5) and solve it by using $1^T s$ as the objective function[1].

Korhonen and Siitari [8] formulated a lexicographic parametric programming approach to identifying efficient units in DEA:

$$
\begin{aligned}
&\text{lex max} \quad \left\{\sigma, 1^T S\right\} \\
&s.t \quad \sum_{j=1}^{n} \lambda_j y_j - \sigma w^y - s^+ = y_o + t\left(y_j - y_o\right) \\
&\qquad \sum_{j=1}^{n} \lambda_j x_j + \sigma w^x + s^- = x_o + t\left(x_j - x_o\right) \\
&\qquad \lambda \in \Lambda, \\
&\qquad \lambda, s^+, s^- \geq 0, \\
&\qquad t : 0 \to 1.
\end{aligned}
\tag{6}
$$

Let us to define the set

$K = \left\{ (x, y) \middle| x = \sum_{j=1}^{n} \lambda_j x_j, y = \sum_{j=1}^{n} \lambda_j y_j, (\lambda_1, ..., \lambda_n) \in \Lambda \right\}$ then we have the following definitions [8]:

Definition 2. A point $(x^*, y^*) \in K$ is *efficient* iff there dose not exist another $(x, y) \in K$ such that dominates (x^*, y^*).

[1] For details see [7].

Definition 3. A point $(x^*, y^*) \in K$ is *weakly efficient* iff there dose not exist another $(x, y) \in K$ such that $x < x^*$ and $y > y^*$,

When (x^*, y^*) is not efficient, it is *inefficient*. If (x^*, y^*) is not even weakly efficient, we call it strictly inefficient.

Efficient Frontier is the set of all points (actual or virtual) with efficiency score is equal to zero.

Definition 4. A *hyperplane* H in $\Re^n$ is a set of the form $\{x \mid p^T x = \alpha\}$ where p is a non-zero vector in $\Re^n$ and α is a scalar. Also, p is usually called the normal or gradient to the hyperplane [5]. A hyperplane is strong if none of component of p are zero; and it is weak if some component of p are zero.

A hyperplane divides $\Re^n$ into two regions, called *halfspaces*. Hence two halfspaces H^+ and H^- are two collection of points of the form $\{x \mid p^T x \geq \alpha\}$ and $\{x \mid p^T x \leq \alpha\}$, respectively, so that $H = H^+ \bigcap H^-$.

Definition 5. A hyperpane H is *supporting hyperplane* of X , if $H \bigcap X \neq \varnothing$ and ($X \subseteq H^+$ or $X \subseteq H^-$).

Definition 6. H is a *strong defining hyperplane* (*Efficient hyperplane*) of PPS if and only if it is supporting at least $m+s$ strong efficient DMUs of PPS lying on H and in its gradient components corresponding with output vector are nonnegative and components corresponding with input vector are nonpositive (in presence of variable returns to scale). Note that when we deal with T_c, one of these *m+s* DMUs can be origin, therefore only m+s-1 strong efficient DMUs of PPS necessarily lie on H .[2]

Suppose among all DMUs, L of them are strong efficient; then all of these DMUs are on some strong supporting hyperplane. The following theorem demonstrates which DMUs lie on the same supporting hyperplane.

Theorem 1. Let (x_p, y_p) and (x_q, y_q) be observed DMUs that lie on a strong supporting hyperplane, then each convex combination of them is on the same hyperplane (in T_v and T_c).

[2] For details and properties see [6].

Proof. See Jahanshahloo et al. [5].

Corollary. If (x_p, y_p) and (x_q, y_q) are two efficient DMUs lying on the same hyperplane then for $\mu \in (0,1)$, $\mu(x_p, y_p) + (1-\mu)(x_q, y_q)$ is efficient and on that hyperplane.

As stated in Definition 6; each efficient hyperplanes, at least consist of $m+s-1$ efficient extreme units (in presence of variable returns to scale). Now we use this fact to identifying all efficient hyperplanes.

Theorem 2. Suppose B be the optimal basis of following envelopment model and $\lambda_i \in x_B : i = 1,...,q$, then there exist a hyperplane which passes through DMUs corresponding to $\lambda_1, \lambda_2, ..., \lambda_q$ and they are efficient.

Proof. Consider following envelopment model:

$$\begin{aligned} \min \quad & \theta \\ s.t \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{ip} \quad , \quad i = 1,...,m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{rp} \quad , \quad r = 1,...,s \\ & \lambda_j \ge 0 \quad , \quad j = 1,...,n \end{aligned} \tag{7}$$

Let B be the optimal basis of (7) and $\lambda_t^* \in x_B$ so $\lambda_t^* \ge 0$. The dual form of (7) is:

$$\begin{aligned} \max \quad & \sum_{r=1}^{p} u_r y_{rp} \\ s.t \quad & \sum_{r=1}^{p} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \ , \ j = 1,...,n \\ & \sum_{i=1}^{m} v_i x_{ip} = 1 \\ & u_r \ge 0 \ , \ r = 1,...,s \\ & v_i \ge 0 \ , \ i = 1,...,m \end{aligned} \tag{8}$$

Assuming (u^*, v^*) as optimal solution of (8), we have:

$$u^* y_t - v^* x_t \le 0 \quad \text{or} \quad u^* y_t - v^* x_t + u_t^o = 0$$

First let $\lambda_t^* > 0$ then by Theorem of complementary slackness we get: $\lambda_t^* u_t^o = 0, \lambda_t^* > 0$ hence $u_t^o = 0$. This means that t-th constraint is binding:

$$u^* y_t - v^* x_t = 0$$

Now let $\lambda_t^* = 0$ then using Fundamental Theorem of Duality: $u_t^o \neq 0$ so $u^* y_t - v^* x_t < 0$. Since λ_t^* is in the optimal basis, degeneracy is occurred. To avoid this matter we can use a "Lexicographic cycling prevention rule", but instead of this rule we can formulate (7) as follows:

$$\begin{aligned} \min \quad & \theta \\ s.t \quad & -\sum_{j=1}^{n} \lambda_j x_{ij} + \theta x_{ip} \geq 0 + \varepsilon^i \quad , \quad i = 1,...,m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rp} + \varepsilon^{m+r} \quad , \quad r = 1,...,s \\ & \lambda_j \geq 0 \quad , \quad j = 1,...,n \\ & \varepsilon > 0 \quad \text{("Non-Archimedean")}, \end{aligned} \tag{9}$$

The dual of (9) is:

$$\begin{aligned} \max \quad & \sum_{r=1}^{s} u_r y_{rp} + \sum_{r=1}^{s} u_r \varepsilon^{m+r} + \sum_{i=1}^{m} v_i \varepsilon^i \\ s.t \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \ , \ j = 1,...,n \\ & \sum_{i=1}^{m} v_i x_{ip} = 1 \\ & u_r \geq 0 \ , \ r = 1,...,s \\ & v_i \geq 0 \ , \ i = 1,...,m \\ & \varepsilon > 0 \quad \text{("Non-Archimedean")}, \end{aligned} \tag{10}$$

Since (9) is non-degenerate, consequently for each $\lambda_t^* \in x_B$, corresponding constraint in (10) is tight, means: $u^* y_t - v^* x_t = 0$.

Subsequently, $u^* y_t - v^* x_t = 0$ is the hyperplane which passes through DMUs corresponding to $\lambda_1, \lambda_2, ..., \lambda_q$. □

Theorem 3. All efficient hyperplanes of a PPS can be identified with this procedure.

Proof. Since we consider all efficient extreme units of PPS and make the directions with them in every possible direction, thus by Theorem 2 and Definition 6 all efficient hyperplanes of a PPS are identified. □

Suppose we solve (6). If the entering variable is not uniquely defined, use the lexicographic rule as in [8]. If after using that rule the entering variable is not uniquely defined, yet, then select the non-basic variable as the entering variable, which corresponds to the maximum value of *t*-parameter.

3. Using lexicographic parametric programming for identifying efficient hyperplanes

Assume that we have n DMUs each consuming m inputs to produce s outputs, as declared in section 2. We are interested in spotting efficient hyperplanes. To achieve our aim we will suggest a procedure. This procedure is based on: a) the "lexicographic parametric programming for identifying efficient units in DEA", which proposed by Korhonen et al. [8], in his method we can identify all efficient extreme units. b) Also we use the attributes stated in Definition 6 and c) Theorem 2.

Step 0**.** Define the set**:** $J=\{j \mid j=1,...,n\}$**.**

Step 1. Choose an initial unit $DMU_o, o \in J$. Project (x_o, y_o) onto the efficient frontier by using the formula:

$$\begin{aligned} \text{lex max} \quad & \{\sigma, 1^T S\} \\ s.t \quad & \sum_{j=1}^{n} \lambda_j y_j - \sigma w^y - s^+ = y_o \\ & \sum_{j=1}^{n} \lambda_j x_j + \sigma w^x + s^- = x_o \\ & \lambda_j \geq 0 \ , \qquad j=1,...,n \\ & s^+, s^- \geq 0, \end{aligned} \tag{11}$$

After solve (11) let B be a square matrix which is the optimal basis of (11).

If $\lambda_o \notin x_B$, then drop index "o" from "J".

If $J \neq \varnothing$ then repeat step 1

Else

Stop

Else

If $\lambda_o \in x_B$ then define the set $J' = \{1,2,...,n\} - \{o\}$ and go to step 2.

Step 2.

Choose $j \in J'$ and consider the parametric programming formulation as follows:

$$\begin{aligned} &\text{lex max} \quad \{\sigma, \mathbf{1}^T S\} \\ &s.t \quad \sum_{j=1}^{n} \lambda_j y_j - \sigma w^y - s^+ = y_o + t(y_j - y_o) \\ &\sum_{j=1}^{n} \lambda_j x_j + \sigma w^x + s^- = x_o + t(x_j - x_o) \qquad (12) \\ &\lambda_j \geq 0 \ , \qquad j = 1,...,n \\ &s^+, s^- \geq 0, \end{aligned}$$

To solve (12) use lexicographic parametric programming pending to $\lambda_o \in x_B$. Whenever in iteration, λ_o selected as the leaving variable, or DMU_j selected as the entering variable, then stop.

Define: $E_j^o = \{i \,|\, \lambda_i \in x_B\}$.

Set $|E_j^o| = \text{cardinal } E_j^o$.

If $|E_j^o| = m + s - 1$ then the members of $\{DMU_r \,|\, r \in E_j^o\}$ are on a same hyperplane.

Set $J' = J' - E_j^o$.

IF $J' \neq \varnothing$ then go to step 2

Else

Set $J = J - \{o\}$

If $J \neq \varnothing$ then go to step 1

Else

Stop.

4. An illustrative numerical example

We formulate the algorithm by the general combined model (5) with lexicographic optimization in presence of constant returns to scale.

Example 1.

Assume we have 7 units which are evaluated with three outputs and one input as illustrated in Fig. 1. Data are available in Table 1.

Table 1. Illustrative example

	Unit 0(A)	Unit 1(H)	Unit 2(B)	Unit 3(C)	Unit 4(D)	Unit 5(E)	Unit 6(F)
Output 1	1	0.95	0.9	1.5	4	1.5	2.75
Output 2	0	1	2	3	1	2.5	2
Output 3	3	2.75	2.5	1.5	2	1.6	0.45
Input	1	1	1	1	1	1	1

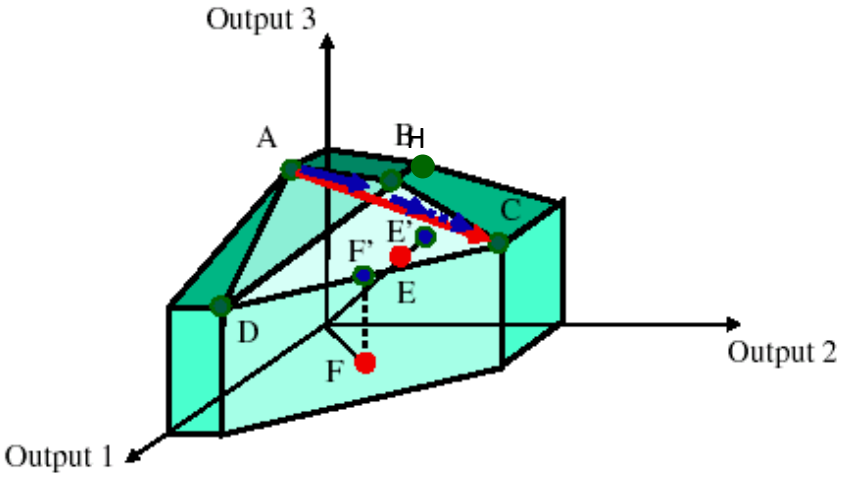

Fig. 1. Illustration of example

In this example we choose $w_{output\,1} = w_{output\,2} = w_{output\,3} = 1$ and $w_{input} = 0$.

Step 0: $J=\{0,1,2,3,4,5,6\}$

Step 1.

Start the procedure by investigating the efficiency of unit A. An optimal tableau is given in Table 2. Since $0\in x_B$, so unit A is efficient. As you can see in the optimal tableau we have $m+s-1=3$ basic variable associated with a unit, "4, 0 and 2" which are corresponding with "D, A and B" respectively. So the set points of D,A,B are on the same efficient hyperplane. Therefore $J'=\{1,3,5,6\}$

Table 2. An optimal tableau for unit A

Nonbasic variables	1	3	5	6	8	9	10	11	RHS$(u_\circ)$
z_j-c_j	0	391.304	411.957	1057.609	162.043	172.913	662.043	2151.174	0
7	0	0.391	0.412	1.058	0.163	0.174	0.663	2.152	0
4	0	0.348	0.347	0.971	-0.272	0.043	0.228	0.413	0
0	0.5	-0.87	-0.629	-1.014	0.054	0.391	-0.446	-0.283	1
2	0.5	1.522	1.283	1.043	0.217	-0.435	0.217	0.87	0

Step 2.

Choose $1\in J'$ next and define the parameter vector $\delta=u_H-u_A$. In the tableau, we subtract column u_0 from the column 1 as shown in Table 3.

Table 3. The problem formulation for moving from A to H

Nonbasic variables	1	3	5	6	8	9	10	11	$u_\circ$	$\delta=u_1-u_\circ$
z_j-c_j	0	391.304	411.957	1057.609	162.043	172.913	662.043	2151.174	0	0
6	0	0.391	0.412	1.058	0.163	0.174	0.663	2.152	0	0
3	0	0.348	0.347	0.971	-0.272	0.043	0.228	0.413	0	0
0	0.5	-0.87	-0.629	-1.014	0.054	0.391	-0.446	-0.283	1	-0.5
1	0.5	1.522	1.283	1.043	0.217	-0.435	0.217	0.87	0	0.5

When we desire to increase parameter *t*, the variable 0 is selected as the leaving variable, which corresponds to the unit under consideration. This means we leave

the hyperplane containing unit 0. Hence we stop and select another unit as our direction to move. Seeing that $J' = \{3,5,6\}$, with selecting 3 and defining the parameter vector $\delta = u_C - u_A$ in Table 2, the variable 0 is selected as the leaving variable. So we cannot move and select the next direction, 5. This direction doesn't give us permission to move on the existing hyperplane more, as in direction 4, also with the last direction, 6. Set $J = J - \{0\}$. Since $J' = \varnothing$ and $J \neq \varnothing$, stop and go to step 1.

Step1.Now consider unit 1. The optimal tableau is given in Table 4.

Table 4. An optimal tableau for unit H

Nonbasic variables	1	3	4	5	6	9	10	11	RHS
$z_j - c_j$	0	598.72	596.32	618.724	1636.428	198.84	79816	2397.48	0
0	0.5	-0.8	0.2	-0.56	-0.82	0.4	-0.4	-0.2	0.5
2	0.5	1.8	0.8	1.56	1.82	-0.4	0.4	1.2	0.5
7	0	0.6	0.6	0.62	1.64	0.2	0.8	2.4	0
8	0	-1.28	-3.68	-1.276	-3.572	-0.16	-0.84	-1.52	0

Since $1 \notin x_B$, so unit H is not efficient. Set $J = J - \{1\} = \{2,3,4,5,6\}$, repeat step 1.

Step 1.Now consider unit 2 as initial unit. The optimal Table is given in Table 5.

Table 5. An optimal tableau for unit B

Nonbasic variables	0	1	5	6	8	9	10	11	RHS
$z_j - c_j$	450	225	128.75	601.25	186.5	349	461.5	2024	0
4	0.4	0.2	0.095	0.565	-0.25	0.2	0.05	0.3	0
2	1.75	1.375	0.181	-0.731	0.313	0.25	-0.563	0.375	1
7	0.45	0.225	0.129	0.601	0.187	0.35	0.462	2.025	0
3	-1.15	-0.575	0.724	1.166	-0.063	-0.45	0.513	0.325	0

The preliminary units which are on the same hyperplane are "4, 2 and 3" means $\{D,B,C\}$. So $J' = \{0,1,5,6\}$.

Step2. With choosing $0 \in J'$, we obtain following tableau:

Table 6. The problem formulation for moving from B to A

Nonbasic variables	0	1	5	6	8	9	10	11	RHS	δ
$z_j - c_j$	450	225	128.75	601.25	186.5	349	461.5	2024	0	450
4	0.4	0.2	0.095	0.565	-0.25	0.2	0.05	0.3	0	0.4
2	1.75	1.375	0.181	-0.731	0.313	0.25	-0.563	0.375	1	0.75
7	0.45	0.225	0.129	0.601	0.187	0.35	0.462	2.025	0	0.45
3	-1.15	-0.575	0.724	1.166	-0.063	-0.45	0.513	0.325	0	-1.15

We cannot increase parameter *t* before we have the basis in which the positive values of the element of the parameter vector correspond to zeroes in the right-hand side, for this reason the leaving variable is 3, but the entering variable is not uniquely defined. The alternative entering variables are 0, 1. For both of them, the ratio is *450/(-0.563)= 225/(-.575)= 391.304*. Also when we use the second objective, we get the same ratio. Consequently we select the non-basic variable as the entering variable, which corresponds to the maximum value of *t*-parameter. The value of *t*-parameter which variable 0 gives us is equal to 1 and the variable 1 gives us 0.5. Thus the correct entering variable for guaranteeing the efficient basis is variable 0 (Table 7). Set $J' = J' - \{0,1\} = \{5,6\}$.

Table 7. Variable 3 is replaced by variable 0 in the basis

Nonbasic variables	1	3	5	6	8	9	10	11	RHS	δ
-1	0	391.304	411.957	1057.609	162.043	172.913	662.043	2151.174	0	0
4	0	0.348	0.347	0.971	-0.272	0.043	0.228	0.413	0	0
2	0.5	1.522	1.283	1.043	0.217	-0.435	0.217	0.87	1	-1
7	0	0.391	0.412	1.058	0.163	0.174	0.663	2.152	0	0
0	0.5	-0.87	-0.629	-1.014	0.054	0.391	-0.446	-0.283	0	1

The units $\{4,2,0\} = \{D,B,A\}$ are on the same hyperplane. Because the entering variable is 0, so stop moving on this direction. When we choose 5, 6 as the

direction, we don't achieve any efficient hyperplane because variable 2 leave the basis. Set $J=\{3,4,5,6\}$ and go to step 1.

When we repeat the process for unit 3, we realize the set points of C,B,D are on the same efficiant hyperplane. With using unit 4 (D) as the initial unit, we reach the sets $\{C,B,D\}$ and $\{A,B,D\}$, put on two different hyperplanes, in view of the fact that we desired. Choosing the units 5 (E) and 6 (F) as the initial unit and solve (11) we recognize that these units are not efficient.

5. Conclusion

This paper has presented a process to find efficient hyperplanes, by means of using the parameterization of the right hand side vector of the envelopment problem of each efficient unit. With considering all efficient extreme units of PPS and make the directions with them in every possible direction, all efficient hyperplanes of a PPS can be identified, so based on this fact we formulate an algorithm. This algorithm demonstrates which set of efficient units are on the same hyperplanes. Since we categorized all efficient units in the sets which containing m+s-1 member, all efficient hyperplanes can be identified. As future study, we propose using this method for the other models of DEA and trying to make this procedure better.

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