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Variational Iteration Method for Fredholm integral equations of the second kind

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Abstract

In this paper, He‘s variational iteration method is applied to Fredholm integral equations of the second kind. To illustrate the ability and simplicity of the method, some examples are provided. The results reveal that the proposed method is very effective and simple and for first fourth examples leads to the exact solution.

Keywords: Variational iteration method; Fredholm integral equation; Lagrange multiplier; Restricted variation.

1 Introduction

Variational iteration method [1-2] is a power device for solving various kinds of linear and non-linear functional equations. It was introduced by Ji-Huan He in 1998 and has been used by many mathematicians and engineers to solve various kinds of functional equations, such as wave equation [9], hyperbolic differential equations [10], Telegraph equation [11], nonlinear chemistry problems [12], Cauchy reaction diffusion problem [13], quadratic Riccati differential equation [14], and many other equations. Fredholm integral equation has been solved by some other methods, such as Adomian decomposition method [3, 7] and Homotopy perturbation method [3-6]. In this study, we use variational iteration method for fredholm integral equations of the second kind. The general form of this integral equation is given by

$$u(x) = f(x) + \int_a^b k(x,t)\, F(u(t))\, dt, \qquad a \le x \le b. \qquad (1)$$

Where $k(x,t)$ is the kernel of the integral equation, f and F are known functions and $u(x)$ is the unknown solution of integral equation, which we are going to find, via variational iteration method.

2 Application of Variational iteration method

For solving equation (1) by variational iteration method, first we differentiate once from both sides of equation (1) with respect to x:

$$u'(x) = f'(x) + \int_a^b \frac{\partial k(x,t)}{\partial x} F(u(t))\,dt. \qquad (2)$$

Now, we apply variational iteration method for equation (2). According to this method correction functional can be written in the following form:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\left(u'_n(s) - f'(s) - \int_a^b \frac{\partial k(s,t)}{\partial s} F(\tilde{u}_n(t))\,dt \right) ds,$$

(3)

where $\lambda(s)$ is a general Lagrange multiplier. To make the above correction functional stationary with respect to u_n, we have:

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta \int_0^x \lambda(s)\left(u'_n(s) - f'(s) - \int_a^b \frac{\partial k(s,t)}{\partial s} F(\tilde{u}_n(t))\,dt \right) ds$$

$$= \delta u_n(x) + \int_0^x \lambda(s)\delta\left(u'_n(s)\right) ds = \delta u_n(x) + \lambda(x)\delta u_n(x) + \int_0^x \lambda'(s)\delta u_n(s)\,ds = 0.$$

From the above relation for any δu_n, we obtain the Euler-Lagrange equation

$$\lambda'(s) = 0. \qquad (4)$$

With the following natural boundary condition:

$$\lambda(x) + 1 = 0. \qquad (5)$$

By using equations (4) and (5), Lagrange multiplier can be identified optimally as follows:

$$\lambda(s) = -1. \qquad (6)$$

Substituting the identified Lagrange multiplier into equation (3), results in the following iterative formula:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\left(u'_n(s) - f'(s) - \int_a^b \frac{\partial k(s,t)}{\partial s} F(u_n(t))\,dt \right) ds. \qquad (7)$$

By starting from $u_0(x)$, we can obtain the exact solution or an approximate solution of the equation (1). In some Fredholm integral equations by differentiating from the integral equation, we obtain a differential equation and we can solve this differential equation by variational iterative method. (See Example 2)

3 Numerical examples

To illustrate the ability and simplicity of the proposed technique, some examples are provided here.

Example 1. Consider the following linear Fredholm integral equation

$$u(x)=e^{3x}-\frac{1}{9}(2e^3+1)x+\int_0^1 x\,t\,u(t)dt, \qquad 0\le x\le 1, \tag{8}$$

with the exact solution, $u(x)=e^{3x}$ [3, 4].

The corresponding iterative formula (7) for this example can be constructed as follows:

$$u_{n+1}(x)=u_n(x)-\int_0^x\left(u'_n(s)-3e^{3s}-\frac{1}{9}(2e^3+1)-\int_0^1 t\,u_n(t)dt\right)ds. \tag{9}$$

By taking $u_0(x)=e^{3x}-\frac{1}{9}(2e^3+1)x$, we derive the following results:

$$u_1(x)=e^{3x}-\frac{x}{3^3}(2e^3+1),$$

$$u_2(x)=e^{3x}-\frac{x}{3^4}(2e^3+1),$$

$$u_3(x)=e^{3x}-\frac{x}{3^5}(2e^3+1),$$

$$\vdots$$

$$u_n(x)=e^{3x}-\frac{x}{3^{n+2}}(2e^3+1),$$

$$\vdots$$

Thus, we have $u(x)=\lim_{n\to\infty}u_n(x)=e^{3x}$, which is the exact solution.

Example 2. Consider the following fredholm integral equation with the exact solution $u(x) = \sin x + \cos x$ [4].

$$u(x) = (1-2\pi)\cos x + \sin x + 4\int_0^{\pi} \cos x \cos t\, u(t)\, dt. \qquad (10)$$

By differentiating twice from integral equation (10), the following differential equation will be obtained:

$$u''(x) + u(x) = 0. \qquad (11)$$

By applying variational iterative method for (11), we derive the following iterative formula:

$$u_{n+1}(x) = u_n(x) + \int_0^x \sin(s-x)(u''_n(s) + u_n(s))\, dt. \qquad (12)$$

Let's start with $u_0(x) = A\sin x + B\cos x$, as an initial approximation, and using the iterative formula (12) we get:

$u_1(x) = A\sin x + B\cos x,$

$u_2(x) = A\sin x + B\cos x,$

$\vdots$

For determineing A and B, we substitute the solution in the integral equation (10) and we obtain $A = B = 1$.

Example 3. Consider the following integral equation with the exact solution, $u(x) = x$[8].

$$u(x) = \frac{7}{8}x + \frac{1}{2}\int_0^1 x\, t\, u^2(t)\, dt. \qquad (13)$$

The iterative formula would be as:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u'_n(s) - \frac{7}{8} + \frac{1}{2}\int_0^1 t\, u_n{}^2(t)\, dt \right) ds. \qquad (14)$$

Consider the initial approximation $u_0(x) = \frac{7}{8}x$. Therefore other terms of the sequence are computed as follows:

$u_1(x) = \frac{497}{512}x = 0.9707031250\, x,$

$$u_2(x) = \frac{2082017}{2097152}x = 0.9927830696x,$$

$$u_3(x) = 0.9982022779x,$$

$$u_4(x) = 0.9995509734x,$$

$$u_5(x) = 0.9998877686x,$$

$$\vdots$$

The sequence tends to x, as $n \to +\infty$.

Example 4. Consider the following equation

$$u(x) = \cos x - x + \int_0^1 x\left(u^2(t) - \sin^2 t\right)dt, \quad (15)$$

with the exact solution, $u(x) = \cos x$ [6].

In the same procedure, the iterative formula can be expressed as the following:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u'_n(s) + \sin s + 1 - \int_0^1 \left(u_n{}^2(t) - \sin^2 t\right)dt \right)ds. \quad (16)$$

By using this iterative formula and taking $u_0(x) = \cos x$, we have:

$$u_1(x) = \cos x,$$

$$u_2(x) = \cos x,$$

$$\vdots$$

Therefore, the exact solution can be recognized easily.

Example 5. For the last example consider the following non-linear Fredholm integral equation

$$u(x) = -\sin(4x) - x^3\left(-\frac{367}{4096}\cos(4)\sin(4) + \frac{11357}{98304} - \frac{2095}{32768}\cos^2(4)\right) + \int_0^1 x^3 t^5 u^2(t)dt \quad (17)$$

Where the exact solution is, $u(t) = \sin(-4x)$ [5].

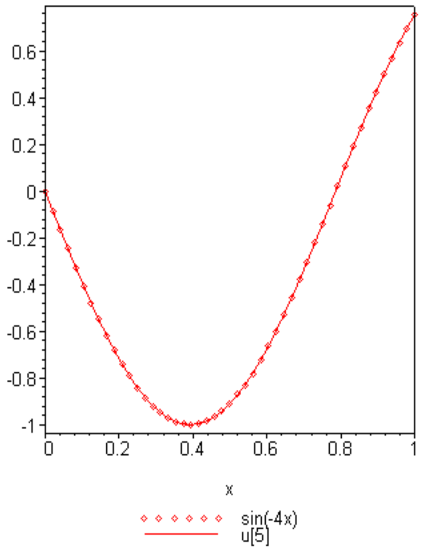
By using variational iteration method for equation (17) and considering $u_0(x) = -\sin(4x) - x^3\left(-\frac{367}{4096}\cos(4)\sin(4) + \frac{11357}{98304} - \frac{2095}{32768}\cos^2(4)\right)$, we obtain the following results:

$$u_1 = -0.003982967666\ x^3 - 8.\ \sin(x)\ \cos(x)^3 + 4.\ \sin(x)\ \cos(x),$$

$$u_2 = -0.000374692464\ x^3 - 1.\ \sin(4.\ x),$$
$$u_3 = -0.0000353613617\ x^3 - 1.\ \sin(4.\ x),$$
$$u_4 = -0.333819270\ 10^{-5}\, x^3 - 1.\ \sin(4.\ x),$$
$$u_5 = -0.315123700\ 10^{-6}\, x^3 - 1.\ \sin(4.\ x),$$
$\vdots$

Suppose $u(x) \approx u_5(x)$. Plots of approximated solution, $u_5(x)$, and the exact solution are presented in Fig.1.

Fig. 1: Plots of the exact and approximate solutions for Example 5



4 Conclusion and Discussion

In this work, variational iterative method has been successfully applied to find the solution of Fredholm integral equations of the second kind. It can be concluded that the method is very powerful and efficient technique for finding exact solutions for wide classes of problems. In this work, we have used the Maple 11 package to carry out the computations.

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