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Improved solution to nonlinear generalized Benjamin–Bona–Mahony–Burgers (GBBMB) equation by a meshless RBFs method

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Abstract

In this paper, based on the RBF collocation method and finite differences, a numerical method is proposed to solve nonlinear generalized Benjamin–Bona–Mahony–Burgers (GBBMB) equation. First order finite differences and Crank-Nicolson method are applied to discretize the temporal parts. The spatial parts are approximated by MQ-RBF interpolation which results in a linear system of algebraic equations. Approximate solutions are determined by solving such a system. The proposed scheme is verified by solving some test problems and computing error norms L_∞, and L_2. Results show the efficiency of the suggested method and the error has been improved.

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INTRODUCTION

Most phenomena in applied sciences and engineering are modeled by nonlinear partial differential equations (PDEs). Since their exact solutions are difficult to obtain, studying these types of equations is often challenging. Consequently, numerical methods for approximating nonlinear PDEs have been widely regarded by researchers and have been successfully applied to numerous real-world problems (e.g., Patil & Maniyeri, 2019; Jiang et al., 2019; Rossi et al., 2019; Gao & Keyes, 2019; Jose et al., 2017). One well-known class of numerical methods is meshless methods, which have attracted significant attention in recent decades. They have been applied as powerful tools, especially for problems in computational mechanics. The advantage of meshless methods over traditional numerical techniques such as the finite difference method (FDM), finite element method (FEM), and finite volume method (FVM) is that they do not require mesh generation or domain/surface discretization.

Several types of meshless methods exist, including the moving least square meshless method (Dabboura et al., 2016), the meshless local Petrov-Galerkin method (MLPG; Atluri & Zhu, 2000), smooth-particle hydrodynamics (Wang et al., 2016), the reproduced kernel particle method (RKPM; Liu et al., 1995), the finite point method (Onate et al., 1996), the mesh-free weak-strong form (MWS; Liu & Gu, 2003), the diffuse element method (DEM; Nayroles et al., 1992), and the radial basis functions (RBFs) method. Each of these approaches has specific advantages for certain classes of problems. Among them, the RBFs method is considered the simplest and most efficient.

The RBFs method was introduced by Ronald Hardy, an Iowa State geodesist, in 1971 (Hardy, 1971). He proposed this method to efficiently interpolate scattered data on topographic surfaces. In RBFs interpolation, a set of N distinct points, referred to as centers, is used. There are no constraints on the geometry of the problem domain or the position of the centers (Sarra, 2017). Various types of RBFs exist, including multiquadric (MQ), thin plate spline (TPS), Gaussian, linear, inverse quadric (IQ), and inverse multiquadric (IMQ) functions. Hardy used the MQ function in his interpolation scheme. Later, Duchon (1977) proposed the use of TPS for data interpolation. The use of RBFs was initially limited to scattered data interpolation until 1990, when Edward Kansa, a physicist, applied them to solve PDEs (Kansa, 1990a, 1990b). His method, however, produced ill-conditioned matrices for a large number of nodes due to the asymmetrical nature of the interpolation matrix. To address this issue, Fasshauer (1997) proposed a Hermite-based approach in which the collocation matrices are symmetric and have smaller condition numbers. Since then, significant efforts have been made by numerous researchers to improve the method and develop new versions of it (e.g., Rosales & La Rocca, 2006; Li & Chen, 2003; Wendland, 2002; Fornberg & Piret, 2007; Šarler & Vertnik, 2006; Ling & Kansa, 2005).

In recent years, the RBFs method has been considered an efficient tool for solving various problems, including PDEs (Kazem et al., 2012; Siraj-ul-Islam et al., 2013; Kadalbajoo et al., 2015; González Casanova et al., 2019), integral equations (Dastjerdi & Ahmadabadi, 2017; Assari & Dehghan, 2018), and fractional equations (Chandhini et al., 2018; Piret & Hanert, 2013). The present study focuses on the numerical solution of a two-dimensional nonlinear PDE using the RBFs method.

$$u_t - \Delta u_t - \Delta u + (1,1).\nabla u = \nabla.\left(F(u)\right) + f(x,y,t),$$

$$t > 0, \quad (x,y) \in \Omega, \tag{1}$$

with following initial and boundary conditions.

$$\begin{cases} u(x,y,0) = p(x,y), & (x,y) \in \Omega, \\ u(x,y,t) = q(x,y,t), & t > 0, \ (x,y) \in \partial\Omega \end{cases} \tag{2}$$

where $u = u(x,y,t)$, $F(u)$ is the vector valued function, $\Omega \subseteq R^2$, ∇, Δ are gradient and Laplacian operators respectively. This equation is known as the nonlinear generalized Benjamin–Bona–Mahony–Burgers (GBBMB) equation.

The GBBMB equation is applied in various scientific fields, including the analysis of surface waves with long wavelengths in liquids, hydromagnetic waves in a cold plasma, acoustic gravity waves in incompressible fluids, and acoustic waves in harmonic crystals. Its modified or generalized forms have been widely studied both numerically and analytically by many researchers (Gomez et al., 2010; Guo & Fang, 2012; Kadri et al., 2008; Noor et al., 2011; Abbasbandy & Shirzadi, 2010; Abdollahzadeh et al., 2011; Omrani & Ayadi, 2008; Omrani, 2006; Qinghua & Zheng, 2012; Xiao & Zhao, 2013; Yin & Hu, 2010; Dehghan et al., 2015).

Several analytical and numerical methods have been developed to solve the nonlinear GBBMB equation. Tari and Ganji (2007) utilized He's methods to approximate analytical solutions for the nonlinear GBBMB equation. Additionally, Ganji et al. (2009) proposed an exponential function method to address a specific type of the nonlinear GBBMB equation. In another study, Dehghan et al. (2014) developed a meshless approach based on radial basis functions (RBFs) to solve the nonlinear GBBMB equation. Haq et al. (2019) introduced a numerical technique that combined Haar wavelets and finite difference methods to solve the nonlinear GBBMB equation. Furthermore, Hajiketabi et al. (2018) presented a new numerical approach to solve the nonlinear GBBMB equation using the Lie-group method with RBFs. Recently, Ali Ebrahimijahan and Dehghan (2019) proposed a method based on integrated radial basis functions (IRBFs) to solve the nonlinear GBBMB and regularized long-wave equations.

This study aims to develop a numerical method that combines the RBF collocation method with finite differences to solve the nonlinear GBBMB equation. The temporal discretization is achieved using finite differences and the Crank-Nicolson scheme, while the spatial parts are approximated using a two-dimensional RBF interpolation. The multiquadric RBF (MQ-RBF) is selected due to its widespread application and superior approximation properties.

The manuscript is organized as follows. Section 2 introduces the RBF method by presenting its basic concepts and definitions. Section 3 details the implementation of the RBF collocation method for the time-discretized nonlinear GBBMB equation. In Section 4, the proposed method is applied to several test problems, and the results are presented. Finally, Section 5 provides a conclusion summarizing the key findings of the study.

A BRIEF REVIEW OF RBFS METHOD

In this section, some basic concepts and definitions are expressed for the radial basis functions interpolation.

Definition 1. Let $\mathbb{R}^d$ be d-dimensional Euclidean space and $x^* \in \mathbb{R}^d$. A *radial basis function* is a function which is both continuous and multivariable like $\varphi : \mathbb{R}^d \to \mathbb{R}$ that its value at any point $x \in \mathbb{R}^d$ is dependent on the distance from a certain point $x^* \in \mathbb{R}^d$ This function could be written as $\varphi(r)$ where $r = \|x - x^*\|$ and $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$. The function φ is an univariable function in r and x^* is a center of RBF φ.

Definition 2. Given the data (x_i, f_i), with $i = 1, \ldots, N$, $x_i \in \mathbb{R}^d$, and $f_i \in \mathbb{R}$, the *scattered data interpolation problem* is defined as finding a smooth function s such that $s(x_i) = f_i$, for $i = 1, \ldots, N$. Function s is called an interpolant.

A radial basis function interpolant u at centers $X = \{x_1^c, x_2^c, \ldots, x_N^c\} \subseteq \mathbb{R}^d$ assumes the following form.

$$u(\mathrm{x}) = \sum_{j=1}^{N} \alpha_j \psi_j(\mathrm{x}) \tag{3}$$

Where $\psi_j(\mathbf{x}) = \varphi\left(\left\|\mathbf{x} - \mathbf{x}_j^c\right\|_2\right) = \varphi(r_j)$, φ is a radial basis function, coefficients $\alpha_j, j = 1, \ldots, N,$ are constants to be determined such that the following interpolation condition at the set of N centers, X is hold.

$$u(x_i^c) = f_i, \quad i = 1, \ldots, N \tag{4}$$

Imposing the interpolation condition 4 to 3 leads to a linear system as follows.

$$u = B\mathrm{a} \tag{5}$$

where $\mathbf{a} = (\alpha_1, \ldots, \alpha_N)^T$, $u = (u_1, \ldots, u_N)^T$, and B is a $N \times N$ matrix called the *interpolation matrix* or the *system matrix* defined as follows.

$$B = \begin{bmatrix} \varphi_1(r_1) & \varphi_2(r_1) & \cdots & \varphi_N(r_1) \\ \vdots & \vdots & & \vdots \\ \varphi_1(r_N) & \varphi_2(r_N) & \cdots & \varphi_N(r_N) \end{bmatrix}$$

where $r_j = \left(\left\|\mathbf{x}_i^c - \mathbf{x}_j^c\right\|_2\right),\ i, j = 1, \ldots, N.$

According to Definition 1, a radial basis function (RBF) is independent of the spatial dimension. This unique property allows for the transformation of a multivariable problem into a one-variable problem, thereby simplifying the computational process. This is a significant advantage of the RBF interpolation scheme compared to other classical methods.

RBFs are generally categorized into two main types: infinitely smooth and piecewise smooth functions. Table 1 lists some of the most well-known RBFs. Infinitely smooth RBFs include a free parameter, known as the *shape parameter* (often denoted by ε\varepsilonε). While this parameter can be selected arbitrarily, its proper choice is crucial, as it directly affects the accuracy of the RBF interpolation scheme. This is particularly important in infinitely smooth RBF interpolation, where the value of the shape parameter significantly influences the scheme's precision (Fasshauer, 2007). In contrast, piecewise smooth

RBFs exhibit algebraic convergence rates, whereas infinitely smooth RBFs achieve higher rates of convergence, either spectral or exponential (Fornberg & Flyer, 2015; Buhmann, 2003).

This classification highlights the superiority of infinitely smooth RBFs in terms of convergence rates. However, the sensitivity of infinitely smooth RBFs to the shape parameter demands careful consideration, as an improper choice can negatively impact the overall performance and accuracy of the scheme.

Table 1: Some well-known RBFs

Category	Name of the function	Definition
Infinitely smooth RBFs	Multiquadric (MQ)	$\sqrt{1+\varepsilon^2 r^2}$
	Inverse Multiquadric (IMQ)	$\frac{1}{\sqrt{1+\varepsilon^2 r^2}}$
	Inverse Quadric (IQ)	$\frac{1}{\left(1+\varepsilon^2 r^2\right)}$
	Gaussian	$e^{-\varepsilon^2 r^2}$
Piecewise smooth RBFs	Linear	r
	Cubic	r^3
	Thin Plate Spline (TPS)	$r^2 \log(r)$

TIME DISCRETIZATION OF NONLINEAR GBBM EQUATION

In this section, Let $L^2(\Omega)$ be an arbitrary interval in R^2. by applying forward finite differences and also Crank-Nicolson scheme, the time variable is discretized for the first-order time derivative as follows.

$$\begin{aligned}
&U^{k+1} - U^k - \left(U_{xx}^{k+1} + U_{yy}^{k+1}\right) \\
&+\left(U_{xx}^{k} + U_{yy}^{k}\right) - \frac{\Delta t}{2}\left(U_{xx}^{k+1} + U_{yy}^{k+1}\right) \\
&-\frac{\Delta t}{2}\left(U_{xx}^{k} + U_{yy}^{k}\right) + \frac{\Delta t}{2}\left(U_{x}^{k+1} + U_{y}^{k+1}\right) \\
&+\frac{\Delta t}{2}\left(U_{x}^{k} + U_{y}^{k}\right) = \frac{\Delta t}{2}\left(F_{x}^{k} + F_{x}^{k+1}\right) \\
&+\frac{\Delta t}{2}\left(F_{y}^{k} + F_{y}^{k+1}\right) + \Delta t f^{k}.
\end{aligned} \tag{6}$$

Now, choose $F(u) = \frac{1}{2}u^2$. Then, $F_x = UU_x$, $F_y = UU_y$.

Approximated non-linear terms by the following formulas

$$\begin{aligned}
&\left(uu_x\right)^{n+1} + \left(uu_x\right)^{n} = u^{n+1}u_x^{n} + u^{n}u_x^{n+1}, \\
&\left(uu_y\right)^{n+1} + \left(uu_y\right)^{n} = u^{n+1}u_y^{n} + u^{n}u_y^{n+1}.
\end{aligned} \tag{7}$$

yields to

$$\begin{aligned}
&U^{k+1} - U^k - \left(U_{xx}^{k+1} + U_{yy}^{k+1}\right) \\
&+\left(U_{xx}^{k} + U_{yy}^{k}\right) - \frac{\Delta t}{2}\left(U_{xx}^{k+1} + U_{yy}^{k+1}\right) \\
&-\frac{\Delta t}{2}\left(U_{xx}^{k} + U_{yy}^{k}\right) + \frac{\Delta t}{2}\left(U_{x}^{k+1} + U_{y}^{k+1}\right) \\
&+\frac{\Delta t}{2}\left(U_{x}^{k} + U_{y}^{k}\right) = \frac{\Delta t}{2}\left(U^{k+1}U_{x}^{k} + U^{k}U_{x}^{k+1}\right) \\
&+\frac{\Delta t}{2}\left(U^{k+1}U_{y}^{k} + U^{k}U_{y}^{k+1}\right) + \Delta t f^{k}.
\end{aligned}$$

This equation can be simplified as follows.

$$U^{k+1}-\left(1+\frac{\Delta t}{2}\right)\left(U_{xx}^{k+1}+U_{yy}^{k+1}\right)$$
$$+\frac{\Delta t}{2}(1-U^{k})\left(U_{x}^{k+1}+U_{y}^{k+1}\right)$$
$$-\frac{\Delta t}{2}\left(U_{x}^{k}+U_{y}^{k}\right)U^{k+1}= \quad (8)$$
$$U^{k}-\left(1-\frac{\Delta t}{2}\right)\left(U_{xx}^{k}+U_{yy}^{k}\right)$$
$$-\frac{\Delta t}{2}\left(U_{x}^{k}+U_{y}^{k}\right)+\Delta t f^{k}.$$

Discretizing Eq. 8 in space by RBF expansion (3) results in

$$\sum_{j=1}^{N}\alpha_j^{k+1}\varphi\left(r_j\right)-\left(1+\frac{\Delta t}{2}\right)\sum_{j=1}^{N}\alpha_j^{k+1}\Delta\varphi\left(r_j\right)$$
$$+\frac{\Delta t}{2}\left(1-\sum_{j=1}^{N}\alpha_j^{k}\varphi\left(r_j\right)\right)\sum_{j=1}^{N}\alpha_j^{k+1}\nabla\varphi\left(r_j\right)$$
$$-\frac{\Delta t}{2}\sum_{j=1}^{N}\alpha_j^{k}\nabla\varphi\left(r_j\right)\sum_{j=1}^{N}\alpha_j^{k+1}\varphi\left(r_j\right)=$$
$$\sum_{j=1}^{N}\alpha_j^{k}\varphi\left(r_j\right)-\left(1-\frac{\Delta t}{2}\right)\sum_{j=1}^{N}\alpha_j^{k}\Delta\varphi\left(r_j\right)$$
$$-\frac{\Delta t}{2}\sum_{j=1}^{N}\alpha_j^{k+1}\Delta\varphi\left(r_j\right)+\Delta t f^{k}.$$

Considering *N* collocation points $\{x_i\}_{i=1}^{N}$ in Ω leads to the following linear system.

$$\left(B-\left(1+\frac{\Delta t}{2}\right)M+\frac{\Delta t}{2}\left(I-D^{k}\right)N\right.$$
$$\left.-\frac{\Delta t}{2}\left(D_x^{k}+D_y^{k}\right)B\right)\mathbf{a}^{k+1}=$$
$$\left(B-\left(1-\frac{\Delta t}{2}\right)M-\frac{\Delta t}{2}N\right)\mathbf{a}^{k}+\Delta t\mathbf{f}^{k},$$

Where $\mathbf{f}^{k}=\left(f\left(x_1,t^{k}\right),\dots,f\left(x_N,t^{k}\right)\right)^{T}$, $\mathbf{a}^{k}=\left(\alpha_1^{k},\dots,\alpha_N^{k}\right)$, $D^{k}=diag(U^{k})$, $D_x^{k}=diag(U_x^{k})$, $D_y^{k}=diag(U_y^{k})$, and $M=B_{xx}+B_{yy}$. B_{xx} and B_{yy} are matrices of second derivative of the system matrix, *B*, respectively in x, and y. B_x and B_y are matrices of first derivative of the system matrix, *B*, respectively in x, and y.

Let

$$T_L=B-\left(1+\frac{\Delta t}{2}\right)M+$$
$$\frac{\Delta t}{2}\left(I-D^{k}\right)N-\frac{\Delta t}{2}\left(D_x^{k}+D_y^{k}\right)B,$$

and

$$T_R=B-\left(1-\frac{\Delta t}{2}\right)M-\frac{\Delta t}{2}N.$$

By the assumption T_L is non-singular, a^{k+1} obtained as follows

$$\mathbf{a}^{k+1}=T_L^{-1}T_R\mathbf{a}^{k}+\Delta t T_L^{-1}\mathbf{f}^{k}.$$

Recalling that $\mathbf{a}=B^{-1}u$, the approximate PDE solution at t^{k+1} is obtained as follows.

$$u^{n+1}=Au^{n}+F \quad (9)$$

Where $A=BT_L^{-1}T_RB^{-1}$ and $F=\Delta t T_L^{-1}\mathbf{f}^{k}$.

NUMERICAL EXPERIMENTS

In this section, the following some test problems are numerically solved for the purpose of verifying the ability of the proposed method with regards to the nonlinear GBBMB equations. Among all of the RBFs, MQ, the most popular RBF, is used in computations due to the rapid convergent rate. Here, following MQ radial basis function is used.

$$\varphi(r)=\sqrt{1+\varepsilon^2r^2},$$

Where ε is the shape parameter.

Meshless methods which are based on radial basis functions (RBFs) contain a free shape parameter that plays an important role for the accuracy and condition number of the coefficient matrix of the method. Most authors use the trial and error method for obtaining a good shape parameter that results in best accuracy. Here, the shape parameter is chosen by trial and error method.

The domain Ω is chosen as the unit region, i.e. $\Omega=[0,1]^2$. In order to test the accuracy, two error norms, L_∞ and L_2 defined as follows are computed.

$$L_2 = \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\tilde{u}_{i,j} - u_{i,j}\right)^2},$$

$$L_\infty = \max_{1\le i,j\le N}\left|\tilde{u}_{i,j} - u_{i,j}\right|,$$

where $\tilde{u}$ and u denote the approximate and exact solutions, respectively.

Test Problem 1

Consider the nonlinear GBBMB equation as follows.

$$u_t - \Delta u_t - \Delta u + (1,1).\nabla u = uu_x$$

$$+uu_y + f(x,y,t),\ t>0,\ \ (x,y)\in\Omega,$$

with conditions

$$\begin{cases} u(x,y,0)=0, \ \ (x,y)\in\Omega, \\ u(x,y,t)=t\sin(x+y),\ (x,y)\in\partial\Omega,\ 0<\mathrm{t}\le \mathrm{T}, \end{cases}$$

and the source term

$$f(x,y,t) = 2t\cos(x+y)$$

$$+\sin(x+y)(3+2t-2t^2\cos(x+y))$$

The exact solution of the problem is $u(x,y,t)=t\sin(x+y)$.

Numerical solutions are calculated at various values of time variable $T=0.1, 0.2, 0.5, 1$ with $N=10$ and time step $\Delta t = 0.001$ $\Delta t = 0.001$ and shape parameter= 1.4. Consequently, the results are provided in Table 2, showing that the proposed method is accurate sufficiently. The numerical results at $N=4$, $T=1$ and different time steps $\Delta t = \dfrac{1}{10(2^n)} (n=0,\dots,8)$ and the numerical results presented by Haq et al. (2019) are all provided in Table 3. Values of shape parameter are derived by trial and error method. Comparison the results show that better approximations are obtained by the proposed scheme. Approximate and exact solutions and also absolute error are illustrated in Fig. 1. According to this figure, it can be seen that approximate solutions are very close to the exact ones.

Table 2: Error norms of test problem 1 at $N=10$ $\Delta t = 0.001$

T	L_∞	L_2
0.1	6.7949e-06	3.4022e-05
0.2	1.2265e-05	6.1129e-05
0.5	2.1897e-05	1.0869e-04
1	2.9094e-05	1.2306e-04

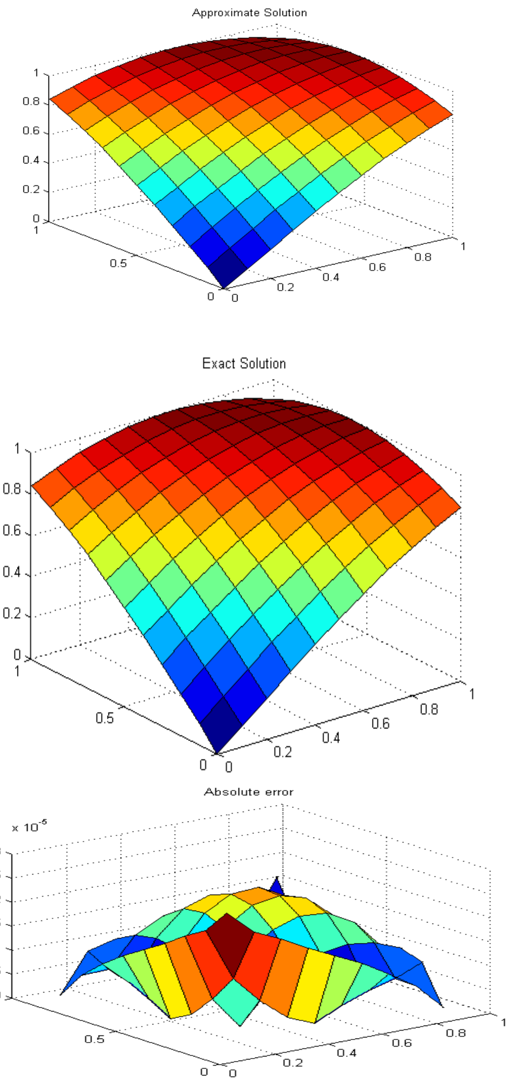

Fig. 1. Graphs of approximate and exact solutions and absolute error of test problem 1 at $T=1$, $N=10$, $\Delta t = 0.001$

Table 3: Error norms of test problem 1 at $T=1$, $N=4$

Δt	L_∞	L_2	L_∞ [54]	L_2 [54]	CPU time	Shape parameter

1/10	1.9000e-3	3.4000e-3	4.1369e - 03	1.0243e - 02	0.1550	1.5
1/20	9.9446e-04	1.8000e-3	2.4115e - 03	5.8698e - 03	0.1718	1.1
1/40	5.7353e-04	9.8716e-04	1.5207e - 03	3.6346e - 03	0.1747	0.8
1/80	3.1700e-04	5.5014e-04	1.0682e - 03	2.5188 - 03	0.1958	0.6
1/160	1.2022e-04	1.7277e-04	8.4022e - 04	1.9708 - 03	0.2301	0.6
1/320	5.0058e-05	8.2924e-05	7.2574e - 04	1.7034 - 03	0.3083	0.5
1/640	2.0853e-05	3.9993e-05	6.7980e - 04	1.5724 - 03	0.47338	0.4
1/1280	3.2936e-05	5.0085e-05	6.6576e - 04	1.5079 - 03	0.7473	0.3
1/2560	2.4328e-05	4.8605e-05	6.5874e - 04	1.4759 - 03	1.3567	0.3

Test Problem 2

Consider the nonlinear GBBMB equation as follows.

$$u_t - \Delta u_t - \Delta u + (1,1).\nabla u$$

$$= uu_x + uu_y + f(x,y,t), \quad t>0, \quad (x,y)\in \Omega,$$

With initial and boundary conditions

$$\begin{cases} u(x,y,0)=1, \quad (x,y)\in\Omega, \\ u(x,y,t)=1+t\sin(x+y), \ (x,y)\in\partial\Omega, 0<\text{t}\leq \text{T}, \end{cases}$$

and the source term

$$f(x,y,t) = sin(x+y)\left(3+2t-2t^2cos(x+y)\right),$$

with the exact solution $u(x,y,t)=1+t\sin(x+y)$.

In table 4, Numerical solutions are calculated at various values of time variable $t=0.1,0.2,0.5,1$ with $N=10$ and time step $\Delta t = 0.001$ and shape parameter= 1.4. The results are provided in Table 4 show that the proposed method is accurate sufficiently.

Table 4: Error norms of test problem 2 at $N=10$ $\Delta t = 0.001$

T	L_∞	L_2
0.1	3.7096e-06	1.5701e-05
0.2	7.4830e-05	2.8961e-05
0.5	1.9129e-05	6.5406e-05
1	3.9390e-05	1.5772e-04

CONCLUSION

In this study, an PDE called nonlinear generalized Benjamin–Bona–Mahony–Burgers (GBBMB) equation was studied numerically. The finite difference formula and Crank Nicolson technique were implemented to discretized the temporal parts. As a result, a time semi-discrete formula was obtained. After that, a fully discrete formula was achieved by approximating the spatial terms using RBF interpolation. Numerical results show that the suggested method has better accuracy and the error has been improved.

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