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Numerical solution of integro-differential equations via pertubed-Gegenbauer, Jacobi polynomials and Galerkin method

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Abstract

In this paper, we proposed perturbed Galerkin method for solving integro-diffrential equations via shifted Gegenbauer and shifted Jacobi polynomials as approximating polynomials. We use Galerkin method to transform the perturbed integro-differential equation to system of linear algebraic equations and obtained N + 1 linear equations with N + m + 2 unknowns. Moreover, with m+1 boundary conditions we obtained N+m+2 algebraic equations which was then solved to obtain the approximate solutions at various values of α and β depending on the orthogonal polynomials, that's shifted Gegenbauer or shifted Jacobi polynomials. The proposed method was implemented on some selected problems in the literature to validate the effectiveness and the accuracy of the proposed method.

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INTRODUCTION

In the last decade, attention have been giving to integro-differential equations due to its application in Mathematical sciences especially in modeling (such as biological application and engineering), Mathematical finance and control theory. Most of the problems that arise in these areas are very difficult to solve analytically. Researchers have employed different approaches in finding the numerical solution to integro-differential equations using different numerical methods. Finite element method for solving non-linear integro-differential model was proposed in Jangveladze et al. (2011), Biazar & Salehi (2016) employed second kind shifted Chebyshev Galerkin method to solve integro-differential equations, Mohamed et al. (2014) reported Legendre-Galerkin method for solving Fredholm form of integro-differential equations, Yalcinbas & Sezer (2000, 2006); Huang & Li (2009) employed Taylor polynomials to approximate the solution of high-order linear Volterra-Fredholm integro-differential equations, Tau method were employed in Hosseini & Shahmorad (2003a,b); Shahmorad (2005) to estimate the error of Fredholm-Volterra integro-differential equations. Bildik et al. (2010) compared the accuracy of Legendre and variational iteration method while Jimoh & Issa (2014) compared variational iteration method and homotopy's method for solving general linear Fredholm integro-differential equations, Khater et al. (2007) employed Legendre polynomials to solve integro-differential equations while Shahsavara (2010) investigate numerical solution of linear Volterra and Fredholm integro-differential equations using Haar wavelets. Convergence and stability of Galerkin's method was reported in Chen et al. (2015), Demir et al. (2021) proposed Pell-Lucas matrix-collocation method for solving Fredholm-type delay integro-differential equations with variable delays, Issa et al. (2019) employed perturbed Galerkin method to solve delay Fredholm and Volterra integro-differential equations using first kind shifted Chebyshev polynomial as approximating polynomial, Golbabai & Seifollahi (2007) employed radial basis function networks in the numerical solution of linear integro-differential equations; Gumgum et al. (2018) investigated Lucas polynomial together with standard and Chebyshev-Lobatto collocation points to solve functional integro-differential equations involving variable delays, analytical properties and asymptotic behaviour of solutions for system of integro-diferential equations was reported in Smarda & Khan (2012), Issa, Biazar, et al. (2022) investigated perturbed Galerkin method via fourth kind shifted Chebyshev polynomials. The main feature in this paper is to extend the work reported in Issa, Biazar, et al. (2022) by introducing shifted Gegenbauer and Jacobi polynomials as approximating polynomial $\Phi_N(u)$ for solving integro-differential equations, since its solution generalizes the results of some other orthogonal polynomials such as Legendre, shifted Chebyshev polynomials of certain kinds and many more.

This paper is organized as follows. In section 2, we review some notable orthogonal polynomials like Legendre polynomial $P_j(u)$ in 2.1.1, shifted Chebyshev polynomials in 2.1.2, shifted Gegenbauer polynomial $C_i^{(\beta)}(u)$ in 2.1.3 and in section 2.1.4, we present shifted Jacobi polynomial $C_n^{(\alpha,\beta)}(u)$ 2.1.4. We present the formulation of the scheme for the proposed method in section 3, numerical examples are presented in section 4 with concluding remarks given in section 5.

PRELIMINARIES

Some notable orthogonal polynomials

Some of the notable orthogonal polynomials $\psi(t)$ are defined here:

$\psi(t)$ is an orthogonal polynomial with respect to the weight function $\omega(t)$ in an interval [a, b] with the inner product of $\psi(t)$ given as:

$$\langle \psi_m(t), \psi_n(t) \rangle = \int_a^b \omega(t)\psi_m(t)\psi_n(t)dt = \begin{cases} 0, & m \neq n \\ \lambda_n, & m = n \end{cases} \quad (1)$$

Some of these orthogonal polynomials $\psi_m(t)$ m(t) are:

Legendre polynomials

Legendre polynomial is an orthogonal polynomial $P_m(u)$ defined in an interval [-1, 1] with weight function $\omega(u)$ and recurrence relation:

$$P_{k+1}(u) = \frac{2k+1}{k+1}uP_k(u) - \frac{k}{k+1}P_{k-1}(u), \; k \geq 1, \quad (2)$$

With $P_0(x) = 1, \; P_1(u) = u.$

Chebyshev polynomials

Chebyshev polynomials are of different kinds, the prominent ones are given here with their respective weight functions $\omega(u)$ in the interval [-1, 1] as:

$$\psi_m = \begin{cases} T_m(u) = \cos(mn), & \omega(m) = \dfrac{1}{\sqrt{1-u^2}} \\ U_m(u) = \dfrac{\sin(m+1)}{\sin(u)}, & \omega(m) = \sqrt{1-u^2} \\ V_m(u) = \dfrac{\cos(m+\frac{1}{2})u}{\cos(\frac{u}{2})}, & \omega(m) = \sqrt{\dfrac{1+u}{1-u}} \\ W_m(u) = \dfrac{\sin(m+\frac{1}{2})u}{\sin(\frac{u}{2})}, & \omega(m) = \sqrt{\dfrac{1-u}{1+u}} \end{cases} \quad (3)$$

The inner product for the third and fourth kinds Chebyshev polynomials in the interval [-1, 1] are defined as:

$$\begin{aligned} \langle \psi_m(t), \psi_n(t) \rangle &= \langle V_m(u), V_n(u) \rangle \\ &= \int_{-1}^{1} \sqrt{\frac{1+u}{1-u}} V_m(u) V_n(u) du \\ &= \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}, \\ \langle \psi_m(t), \psi_n(t) \rangle &= \langle W_m(u), W_n(u) \rangle \\ &= \int_{-1}^{1} \sqrt{\frac{1-u}{1+u}} W_m(u) W_n(u) du \\ &= \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}, \end{aligned} \quad (4)$$

See Mason & Handscomb (2003) for details.

Shifted Gegenbauer polynomials

Gegenbauer polynomials $C_i^{(\beta)}(u)$ defined in the interval [-1, 1] with respect to weight function $\omega(u) = (1-u^2)^{\left(\beta - \frac{1}{2}\right)}$ can be determined using

$$C_i^{(\beta)}(u) = \sum_{j=0}^{i} \frac{(-1)^j \Gamma(2\beta+2i-j)\Gamma\left(\beta+\frac{1}{2}\right)}{(i-j)!\Gamma(2\beta)\Gamma(j+1)\Gamma\left(i-j+\beta+\frac{1}{2}\right)} u^{i-j} \quad (5)$$

The recurrence relation is given as:

$$C_i^{(\beta)}(u) = \frac{1}{i}\left[\begin{matrix} 2(i+\beta-1)uC_{i-1}^{(\beta)}(u) \\ -(i+2\beta-2)C_{i-2}^{(\beta)}(u) \end{matrix}\right], \; i \geq 2, \quad (6)$$

where $C_0^{(\beta)}(u) = 1,\; C_1^{(\beta)}(u) = 2\beta u$ .This recurrence relation can be transform to another interval [a, b] by introducing the variable $\lambda = \dfrac{2u-(a+b)}{b-a}$. Hence, the shifted Gegenbauer polynomial in term of u is obtained as:

$$C_i^{(\beta)*}(u) = \frac{1}{i}\left[\begin{matrix} 2(i+\beta-1)\left(\dfrac{2u-(a+b)}{b-a}\right)C_{i-1}^{(\beta)*}(u) \\ -(i+2\beta-2)C_{i-2}^{(\beta)*}(u) \end{matrix}\right], \quad i \geq 2, \quad (7)$$

where

$$C_0^{(\beta)*}(u) = 1,\; C_1^{(\beta)}(u) = 2\beta\left(\frac{2u-(a+b)}{b-a}\right).$$

The analytic form of the shifted Gegenbauer polynomial $C_i^{(\beta)*}(\lambda)$ is given as:

$$C_i^{(\beta)*}(\lambda) = \sum_{j=0}^{i} \frac{(-1)^j \Gamma(2\beta+2i-j)\Gamma\left(\beta+\frac{1}{2}\right)}{(i-j)!\Gamma(2\beta)\Gamma(j+1)\Gamma\left(i-j+\beta+\frac{1}{2}\right)} \lambda^{i-j} \quad (8)$$

The orthogonality condition is

$$\begin{aligned} &\left\langle C_m^{(\beta)*}(u), C_n^{(\beta)*}(u) \right\rangle \\ &= \int_0^1 (u-u^2)^{\left(\alpha-\frac{1}{2}\right)} C_m^{(\beta)*}(u) C_n^{(\beta)*}(u) du \quad (9) \\ &= \begin{cases} 0, & \text{for } m \neq n \\ \dfrac{\pi 2^{1-4\alpha}\Gamma(n+2\alpha)}{n![\Gamma(\alpha)]^2(n+\alpha)}, & \text{for } m = n \end{cases} \end{aligned}$$

see Issa, Yisa, & Biazar (2022); Izadkhah & Saberi-Nadja_ (2015) for details.

Jacobi polynomials

The well-known Jacobi polynomials $P_n^{(\alpha,\beta)}(u),\; n = 0,\, 1, \ldots,$ with parameters $\alpha,\; \beta > -1$ and weight function $\omega(u) = (1-u)^{\alpha}(1+u)^{\beta}$. The explicit form of Jacobi polynomials that was used in Szego (1975) takes the form:

$$\begin{aligned} P_n^{(\alpha,\beta)}(u) &= \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)} \\ &\times \sum_{m=0}^{n} \frac{n!\Gamma(\alpha+\beta+m+n+1)}{(n-m)!m!\Gamma(\alpha+m+1)}\left(\frac{u-1}{2}\right)^m \end{aligned} \quad (10)$$

where $P_0^{(\alpha,\beta)}(u) = 1$, $P_1^{(\alpha,\beta)}(u) = (\alpha+1) + (\alpha+\beta+2)\left(\dfrac{u-1}{2}\right)$. When $\alpha = \beta = 0$, then its reduces to Legendre polynomials, while choosing $\alpha = \beta = \dfrac{1}{2}$, gives Chebyshev polynomials of first kind and also when $\alpha = \beta = 1$, gives Gegenbauer polynomials of the form , that's $P_n^{(1,1)}(u)=C_n^{\frac{3}{2}}(u)$ and so on.

FORMULATION OF THE SCHEME FOR PERTURBED GALERKIN METHOD

Consider integro-differential equations of the form:

$$\sum_{j=0}^{m} P_j(u)\frac{d^j}{du^j}\Phi(u)+f(u) +\lambda_1\int_a^b \nu_1(u,t)\Phi(u)du+\lambda_2\int_a^b \nu_2(u,t)\Phi(u)du \quad (11)$$

subject to the initial condition:

$$\Phi^j(a)=\rho_j,\ j=0,\ 1,\dots,\ (m-1) \quad (12)$$

where $\Phi(u)$ is an unknown function, $P_j(u),\ f(u),\ \nu_1(u,t)$ and $\nu_2(u,t)$ are known functions, m is the order of Eq. (11), λ_1,λ_2 are real numbers and u is the independent variable defined in the interval
[a, b] except stated otherwise.

Suppose $\Phi_N(u)$ is an approximant of degree N to the function $\Phi(u)$, then we write

$$\Phi_N(u)=\sum_{i=0}^{N}\delta_i\varphi\left(\frac{2u-(a+b)}{b-a}\right). \quad (13)$$

where δ_I are unknowns to be determined and $\varphi_I(u),\ i=0,\ 1,\ \dots,N$ are orthogonal polynomials (which is either Gegenbauer or Jacobi polynomials).

Perturbing Eq. (11), we obtain

$$\sum_{j=0}^{m} P_j(u)\frac{d^j}{du^j}\Phi(u)+f(u)+\lambda_1\int_a^b \nu_1(u,t)\Phi(u)du +\lambda_2\int_a^b \nu_2(u,t)\Phi(u)du+\Gamma_N(u) \quad (14)$$

Multiplying Eq. (14) by $\varphi_k\left(\frac{2u-(a+b)}{b-a}\right),\ k=m,\ m+1,\ M+m+1$ and integrate the resulting equation with respect to the independent variable, in the intervals [a, b], we obtain:

$$\int_a^b\left[\sum_{j=0}^{m}P_j(u)\sum_{i=0}^{N}\left(\frac{2}{b-a}\right)^j\delta_i\frac{d^j}{du^j}\varphi_i\left(\frac{2u-(a+b)}{b-a}\right) -\lambda_1\int_a^b \nu_1(u,t)\sum_{i=0}^{N}\delta_i\varphi_i\left(\frac{2u-(a+b)}{b-a}\right)du -\lambda_2\int_a^u \nu_2(u,t)\sum_{i=0}^{N}\delta_i\varphi_i\left(\frac{2u-(a+b)}{b-a}\right)du-\Gamma_N(u)\right] \varphi_k\left(\frac{2u-(a+b)}{b-a}\right)du =\int_a^b\left[f(u)\varphi_k\left(\frac{2u-(a+b)}{b-a}\right)\right]du,k=m,m+1,\dots,N+m+1, \quad (15)$$

where

$$\Gamma_N(u)=\sum_{r=0}^{m}\tau_{r+1}\varphi_{N-m+r+1}\left(\frac{2u-(a+b)}{b-a}\right), \quad (16)$$

N is the degree of approximation.

Eq. (15) in matrix form becomes

$$\chi\alpha=G, \quad (17)$$

The remaining equations are obtained using the attached conditions (12), that is

$$\frac{d^j}{du^j}\Phi_N(u)=\sum_{i=0}^{N}\left(\frac{2}{b-a}\right)^j\frac{d^j}{du^j}\varphi\left(\frac{2u-(a+b)}{b-a}\right)\Bigg|_{u=a} =\rho_j,\ j=0,\ 1,\ m-1. \quad (18)$$

From Eqs. (15) and (18), we obtain the values of the unknowns $\delta_i,\ i=0,\ 1,\ \dots,N,$ then substitute in Eq. (13) to obtain the approximate solution of degree N.

NUMERICAL EXAMPLES

In this section, we implement the proposed method (PM) on selected problems from the literature and compare the results with existing results by computing the absolute maximum error ξ_N, where

$$\xi_N=\max_{0\le i\le 100}\left|\Phi(u_i)-\Phi_N(u_i)\right|,\ u_i=a+ih \quad (19)$$

Example 1

Consider the following delay Volterra integro-differential equation Issa et al. (2019)

$$\Phi'(u)-\Phi(u-1)+4\Phi(u)+3\int_{u-1}^{u}\Phi(t)dt =2\exp(1-u),\ u\ge 0 \quad (20)$$

with initial condition $\Phi(0)=1,$ and the exact solution $\Phi(u)=\exp(-u)$.

Replacing $\Phi(u)$ in Eq. (20) with the corresponding approximate solution $\Phi_N(u)$ and add perturbation terms $\Gamma_N(u)$, we obtain

$$\Phi'_N(u)-\Phi_N(u-1)+4\Phi_N(u)$$
$$+3\int_{u-1}^{u}\Phi_N(t)dt=2\exp(1-u) \quad (21)$$
$$+\sum_{r=0}^{m}\tau_{r+1}\varphi_{N-m+r+1}\left(\frac{2u-(a+b)}{b-a}\right).$$

Multiply Eq. (21) by

$$\varphi_k\left(\frac{2u-(a+b)}{b-a}\right),\ k=1,\ 2,\dots N+2$$

then integrate the resulting equation over [a, b], we get:

$$\int_a^b\left[\Phi'_N(u)-\Phi_N(u-1)+4\Phi_N(u)+3\int_{u-1}^{u}\Phi_N(t)dt\right]$$
$$\times\varphi_k\left(\frac{2u-(a+b)}{b-a}\right)du$$
$$=\int_a^b\left[2\exp(1-u)+\sum_{r=0}^{m}\tau_{r+1}\varphi_{N-m+r+1}\left(\frac{2u-(a+b)}{b-a}\right)\right]$$
$$\times\varphi_k\left(\frac{2u-(a+b)}{b-a}\right)du \quad (22)$$

Solving Eq. (22) together with the initial condition (expressing in terms of shifted Gegenbauer polynomial or Jacobi polynomial), that's firstly, in terms of shifted Gegenbauer polynomial, we have:

$$\Phi(0)=1\Rightarrow\sum_{r=0}^{N}\frac{(-1)^r\,\Gamma(r+2\beta)}{r!\,\Gamma(2\beta)}\delta_r=1 \quad (23)$$

To obtain the approximate solution N (u) of degree N. Table 1 and 2 shows the maximum errors at different values of N relative to the existing results in the literature. Figure 1a display the exact solution and its corresponding approximate solutions at various values of N while the errors graphs is display in Fig. 1. Fig. 2 is the comparison of the errors.

Table 1: Gegenbauer approximation: Maximum absolute errors for Example 1 at various values of N

N	$\beta=1$	$\beta=2$	$\beta=3$	**Issa et al. (2019)**
4	5.69×10^{-4}	4.18×10^{-4}	4.78×10^{-4}	6.96×10^{-4}
8	8.07×10^{-7}	7.98×10^{-7}	7.89×10^{-7}	8.19×10^{-7}
12	1.28×10^{-9}	1.27×10^{-9}	1.26×10^{-9}	1.30×10^{-9}

Table 2: Jacobi approximation: Maximum absolute errors for Example 1 at various values of N

N	$P^{(\frac{1}{2},\frac{1}{2})}(u)$	$P^{(\frac{3}{2},\frac{3}{2})}(u)$	$P^{(\frac{5}{2},\frac{5}{2})}(u)$
4	5.69×10^{-4}	4.18×10^{-4}	4.78×10^{-4}
8	8.07×10^{-7}	7.98×10^{-7}	7.89×10^{-7}
12	1.28×10^{-9}	1.27×10^{-9}	1.26×10^{-9}

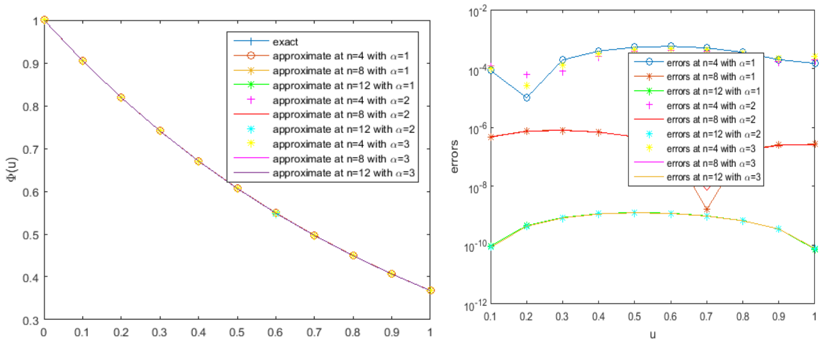

Fig. 1. Example 1: Approximate solutions (left-side) and its corresponding absolute errors (right-side) with exact solution $\Phi(u)=u\cos(u)$.

Example 2

Consider the Volterra integro-differential equation in Issa & Salehi (2017); Biazar & Salehi (2016); Wazwaz (2011); Issa, Biazar, et al. (2022):

$$\Phi'(u)-\int_0^u\Phi(t)dt=1-2u\sin(u),\ 0\le u\le1,\Phi(0)=0.$$

Comparison of the maximum absolute errors at various values of N relative to the existing ones in the literature is shown in Table 3. We have the exact solution and its corresponding approximate solutions in Fig. 2.

Table 3: Maximum absolute errors with Gegenbauer as approximating polynomial

N	$\beta=1$	$\beta=2$	$\beta=3$	**Issa et al. (2022)**	**Biazar and Salahi (2016)**
4	4.49×10^{-4}	3.69×10^{-4}	1.00×10^{-3}	8.12×10^{-4}	2.30×10^{-3}
8	1.66×10^{-10}	8.36×10^{-10}	3.13×10^{-10}	7.49×10^{-10}	1.38×10^{-8}
12	7.77×10^{-16}	9.55×10^{-15}	6.64×10^{-14}	9.40×10^{-16}	8.09×10^{-15}

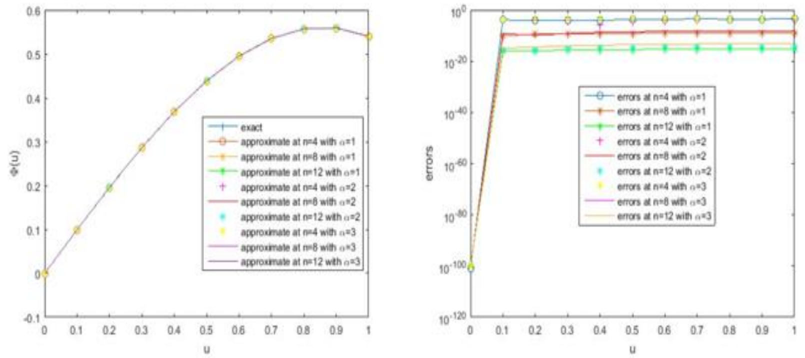

Fig. 2. Example 2: Approximate solutions and its absolute errors with $C_N^{\beta}(u)$ and approximating polynomial

Example 3

Considering the Volterra integro-differential equation Issa & Salehi (2017); Biazar & Salehi (2016); Wazwaz (2011); Issa, Biazar, et al. (2022):

$$\Phi'(u)+\int_0^u t\Phi(t)dt+1-\frac{1}{2}u^2+u\exp(u)=0,$$

$0\le u\le 1,\ \Phi(0)=0,$

with the exact solution $\Phi(u)=1-\exp(u)$.

Table 4 displays the maximum absolute errors at different values of N and β using Geqenbauer as approximating polynomial while Table 5 is the corresponding maximum absolute errors using Jacobi polynomial as approximation. Fig. 3 exhibit the graphs of the exact and the corresponding approximate solutions at various values of N and using Gegenbauer polynomial as approximating polynomial.

Table 4: Maximum absolute errors using Gegenbauer as approximating polynomial

N	$\beta=1$	$\beta=2$	$\beta=3$	**Issa et al. (2019)**	**Biazar and Salahi (2016)**
4	1.78×10^{-5}	2.81×10^{-5}	1.88×10^{-5}	7.38×10^{-5}	7.20×10^{-3}
8	1.16×10^{-11}	1.50×10^{-11}	2.95×10^{-9}	8.59×10^{-11}	2.31×10^{-9}
12	2.22×10^{-17}	1.33×10^{-17}	5.11×10^{-17}	8.91×10^{-17}	9.24×10^{-16}

Table 5: Maximum absolute errors using Jacobi as approximating polynomial

N	$P^{(\frac{1}{2},\frac{1}{2})}(u)$	$P^{(\frac{3}{2},\frac{3}{2})}(u)$	$P^{(\frac{5}{2},\frac{5}{2})}(u)$
4	1.78×10^{-5}	2.81×10^{-5}	1.88×10^{-4}
8	1.16×10^{-11}	1.50×10^{-11}	2.95×10^{-9}
12	2.22×10^{-17}	1.33×10^{-17}	5.11×10^{-15}

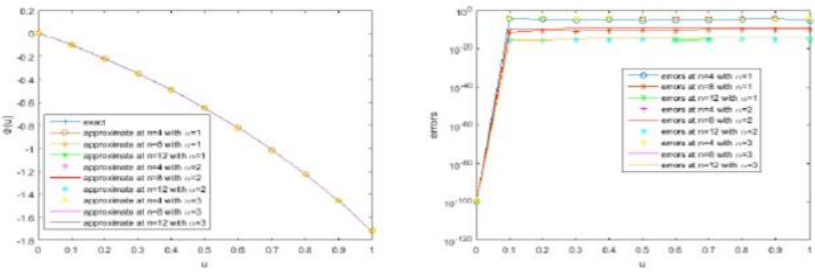

Fig. 3. Example 3: Approximate solutions and its absolute errors with $C_N^{\beta}(u)$ and approximating polynomial

Example 4

Considering Fredholm-Volterra integro-differential equation (Issa & Salehi (2017); Biazar & Salehi (2016); Yalcinbas & Sezer (2000); Shahmorad (2005); Issa, Biazar, et al. (2022)):

$$\Phi''(u)+u\Phi'(u)-u\Phi(u)-\int_{-1}^{1}\sin(u)\exp(t)\Phi(t)dt$$
$$=\exp(u)-2\sin(u), -1\le u\le 1, \Phi(0)=1,\ \Phi'(0)=1,$$

with the exact solution $\Phi(u)=\exp(u)$.

Table 6 shows the maximum errors at various values of N and β. Fig. 4 displays the exact solution and the approximate solution $\Phi_N(u)$ at different values of N and β using Gegenbauer as approximating polynomial.

Table 4: Maximum absolute errors using Gegenbauer as approximating polynomial

N	$\beta=1$	$\beta=2$	$\beta=3$	**Issa et al. (2022)**	**Shahmorad (2005)**
5	2.18×10^{-4}	2.41×10^{-4}	1.30×10^{-4}	4.66×10^{-4}	3.19×10^{-3}
10	4.61×10^{-10}	1.83×10^{-10}	6.85×10^{-10}	5.49×10^{-10}	2.10×10^{-6}
15	9.10×10^{-18}	7.42×10^{-17}	4.42×10^{-17}	1.27×10^{-17}	7.50×10^{-11}

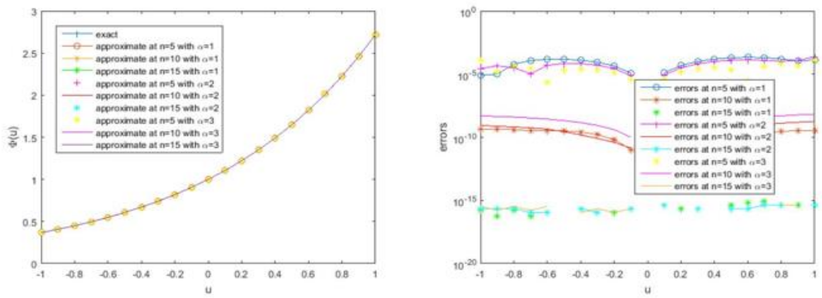

Fig. 4. Example 4: Approximate solutions and its absolute errors with $C_N^{\beta}(u)$ and approximating polynomial

DISCUSSION OF RESULTS AND CONCLUSION

Discussion of Results

Table 1-6 depict maximum errors obtained for the four selected problems from the literature. It was observed that the proposed method is more effective and accurate compared to the existing ones, although it gives the same degree of accuracy when compare with Issa, Biazar, et al. (2022) but the proposed method is more robust and more effective than Issa, Biazar, et al. (2022) because the proposed method generalize the results of other orthogonal polynomials such as Legendre polynomial, second kind shifted Chebyshev polynomial and some other orthogonal polynomials. Moreover, the accuracy improves as values of N changes. Figures 1-4 are the exact solutions and the corresponding approximant at various values of N while 1b-4b are the errors graphs.

Conclusion

In this paper, we propose a perturbed Galerkin method for solving integro-differential equations using shifted Gegenbauer and shifted Jacobi polynomials as approximating polynomials. We introduce $m+1$ perturbation terms into the integro-differential equation, which is then transformed into a system of algebraic linear equations using the Galerkin method. The resulting system is solved to obtain the unknown coefficients. For experimentation, we use $\beta=1, 2, 3$ for the shifted Gegenbauer polynomials $C_i^{(\beta)}(u)$ and $\alpha=\beta=\frac{1}{2}$, $\alpha=\beta=\frac{3}{2}$, $\alpha=\beta=\frac{5}{2}$ (with the same values for α and β) for shifted Jacobi polynomial $P_i^{(\alpha,\beta)}(u)$. The method is applied to selected problems from the literature, and the results are compared with existing solutions. The proposed method shows better accuracy compared to the Galerkin method, Tau method, and radial basis method, and agrees well with the perturbed Chebyshev Galerkin method. However, it is more robust and effective as it generates results using other orthogonal polynomials. In summary, the method is both more effective and accurate. In conclusion, the proposed method is more effective and accurate than existing methods.

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