Optimization
Iranian Journal of Optimization
Volume 14, Issue 2, 2022, 93-100
Research Paper

Online version is available on: www.ijo.rasht.iau.ir

Islamic Azad University
Rasht Branch
ISSN: 2588-5723
E-ISSN:2008-5427

A Constructive Scheme for Tripled Fixed Point Problems in Hilbert Space

Samuel Adamariko Aniki*

*Department of Mathematics, Faculty of Science, Confluence University of Science and Technology, Osara, Kogi State, Nigeria

Revise Date 18 May 2022
Accept Date: 19 July 2022

Keywords:
Mann iterative scheme
tripled fixed point
mapping
Hilbert space

Abstract

Tripled fixed point is an extension to coupled fixed point theory. The idea of tripled fixed point has largely become a focus of research interest in the area of mathematical analysis, especially for their vast application. This research presents a common tripled fixed point iteration for approximating tripled fixed points in linear spaces which is in the context of a Hilbert space. Here, a tripled Mann iterative scheme is defined and applied to resolve the problem of common tripled fixed points of certain mappings. Hence, this work is an extension to recent research in the literature.

***Correspondence E-mail**: smlaniki@yahoo.com

INTRODUCTION

The notion of fixed point theory has gained a lot of momentum recently in the area of mathematical analysis and its applications. The existence of a fixed point for a contractive-type mapping in partially ordered metric spaces has been recently worked on by several authors, Ran and Reurings (2004), Bhaskar and Lakshmikantham (2006), Lakshmikantham and Ciric (2009), Loung and Thuan (2011). After the presentation of the earlier works in this sense, research interest in this subject matter has expanded significantly.

In order to ensure the existence and uniqueness of a solution of periodic boundary value problems, Bhaskar and Lakshmikantham (2006) proved the existence and uniqueness of a coupled fixed point in the setting of partially ordered metric spaces. Consequently, so many researches has been done on tripled fixed point, for its existence and uniqueness, and also the analysis of fixed point properties via mixed monotone mappings in a complete metric spaces (Abbas, Aydi & Karapınar, 2011; Aydi, Karapınar & Shatanawi, 2012; Aydi & Karapınar, 2012; Aydi, Karapınar & Radenovic, 2013; Karapınar, Aydi & Mustafa, 2013).

The Mann iterative procedure is the earliest known iterative procedures examined in linear spaces except the most widely used Picard iteration. Some of the most recent references on Mann iteration can be found in (Dehaish, Khamsi & Khan, 2013; Kim, 2019).

Recently, in the work of Choudhury and Kundu (2016), the authors initiated the study of coupled fixed point iteration by introducing a coupled Mann iterative scheme and applied the same to the context of Hilbert space of approximate coupled fixed points of certain mappings. Coupled and tripled fixed points research have become the focus of interest in recent times, particularly for their potential applications. Very recently, Kim (2020) extensively worked on a constructive scheme for common coupled fixed point problems in Hilbert space and based on this, our aim is to generalize this work to tripled fixed point for Mann pair iterative scheme in the context of Hilbert space.

PRELIMINARIES

In this section, we will consider some definitions that will be relevant in the course of demonstrating our findings.

Definition 1 (Cheng & Ross, 2015) The Parallelogram law states that for any vector κ and λ in a Hilbert space H, we have

$$\|\kappa+\lambda\|^2+\|\kappa-\lambda\|^2=2\|\kappa\|^2+2\|\lambda\|^2.$$

Definition 2 (Kim, 2020) The Mann iteration is as follows: Let λ be a closed convex subset of a Hilbert space H and $T:\lambda\to\lambda$ be a self-mapping. Then for $\theta_0\in\lambda$,

$$\theta_{n+1}=(1-\eta_n)\theta_n+\eta_n T\theta_n,\ \ n\geq 0.$$

where $\{\eta_n\}\subset(0,1)$ satisfying suitable control conditions.

Definition 3 (Kim, 2020) For a non empty set X and mappings $\Psi,\Omega:\ X^2\to X,\ (\kappa,\lambda)\in X^2$ is a common couple fixed point of ψ and Ω, if $\Psi(\kappa,\lambda)=\lambda,\ \Psi(\lambda,\kappa)=\lambda,\ \Omega(\kappa,\lambda)=\kappa,$ $\Omega(\lambda,\kappa)=\lambda.$

The following contractive inequality conditions are used on Ψ and Ω which are subdivided into two conditions.

Definition 4 (Kim, 2020) Let H be a Hilbert Space and C a nonempty Closed convex subset of H. Then $\Psi,\Omega: C^2\to C$ be any mappings.

(1) (Ψ,Ω) satisfies contractive inequality condition I if $\forall\ \kappa,\lambda,\theta_1,\theta_2\ \in C,$

$$\|\Psi(\kappa,\lambda)-\Psi(\theta_1,\theta_2)\|^2+\|\Omega(\kappa,\lambda)-\Omega(\theta_1,\theta_2)\|^2\leq\beta_1(\|\kappa-\theta_1\|^2+\|\lambda-\theta_2\|^2)+\beta_2\{(\|\theta_1-\Omega(\theta_2,\theta_1)\|^2+\|\theta_2-\Psi(\theta_1,\theta_2)\|^2)(1+\|\kappa-\Psi(\kappa,\lambda)\|^2+\|\lambda-\Omega(\lambda,\kappa)\|^2)+(\|\kappa-\Psi(\kappa,\lambda)\|^2+$$

$\|\lambda - \Omega(\lambda,\kappa)\|^2)(1 + \|\theta_1 - \Psi(\theta_1,\theta_2)\|^2 + \|\theta_2 - \Omega(\theta_2,\theta_1)\|^2)\},$

(2) (Ψ,Ω) satisfies contractive inequality condition II if $\forall\ \kappa,\lambda,\theta_1,\theta_2\ \in C,$

$$\|\Omega(\kappa,\lambda) - \Psi(\theta_1,\theta_2)\|^2 + \|\Psi(\lambda,\kappa) - \Omega(\theta_2,\theta_1)\|^2 \le \beta_1(\|\kappa-\theta_1\|^2 + \|\lambda-\theta_2\|^2) + \beta_2\{(\|\theta_1 - \Psi(\theta_1,\theta_2)\|^2 + \|\theta_2 - \Omega(\theta_2,\theta_1)\|^2)(1 + \|\kappa - \Omega(\kappa,\lambda)\|^2 + \|\lambda - \Psi(\lambda,\kappa)\|^2) + (\|\kappa - \Omega(\kappa,\lambda)\|^2 + \|\lambda - \Psi(\lambda,\kappa)\|^2)(1 + \|\theta_1 - \Psi(\theta_1,\theta_2)\|^2 + \|\theta_2 - \Omega(\theta_2,\theta_1)\|^2)\}$$

where $\beta_1,\beta_2 > 0$ and $\beta_2 < \frac{1}{4}$

Definition 5 (Kim, 2020) Let H be a Hilbert space and C a nonempty closed convex subset of H. Let $\Psi,\Omega: C^2 \to C$ be a mapping. Also, let $\{\kappa_n\}$ and $\{\lambda_n\}$ be sequences in C. Then, the coupled Mann pair iterative scheme is as follows:

$\kappa_{n+1} = (1-\eta_n)\kappa_n + \eta_n\Psi(\kappa_n,\lambda_n),$

$\lambda_{n+1} = (1-\eta_n)\lambda_n + \eta_n\Omega(\lambda_n,\kappa_n),\ n \ge 0,$

Where $0 < \eta_n < 1,\ n \ge 0$ and $0 < \lim_{n\to\infty}\eta_n = \delta$.

MAIN RESULTS

In order to show our main results, firstly we must define some useful terms.

Definition 6 For a nonempty set X and mappings $\Pi,\Psi,\Omega: X^3 \to X$, with $(\kappa,\lambda,\mu) \in X^3$ is a common tripled fixed point of Π, Ψ and Ω, if $\Pi(\kappa,\lambda,\mu) = \kappa,\ \Psi(\lambda,\mu,\kappa) = \lambda,\ \Omega(\mu,\kappa,\lambda) = \mu,\ \ \Omega(\kappa,\lambda,\mu) = \kappa,\ \Psi(\lambda,\mu,\kappa) = \lambda, \Pi(\mu,\kappa,\lambda) = \mu$

Definition 7 Let H be a Hilbert space and C a nonempty closed convex subset of H. Then, $\Pi,\Psi,\Omega: C^3 \to C$ be any mappings which satisfies any of the following contractive inequality conditions

(1) (Π,Ψ,Ω) satisfies contractive inequality condition I if $\forall\ \kappa,\lambda,\mu,\theta_1,\theta_2,\theta_3 \in C,$

$$\|\Pi(\kappa,\lambda,\mu) - \Pi(\theta_1,\theta_2,\theta_3)\|^2 + \|\Psi(\lambda,\mu,\kappa) - \Psi(\theta_2,\theta_3,\theta_1)\|^2 + \|\Omega(\mu,\kappa,\lambda) - \Omega(\theta_3,\theta_1,\theta_2)\|^2 \le \beta_1(\|\kappa-\theta_1\|^2 + \|\lambda-\theta_2\|^2 + \|\mu-\theta_3\|^2) + \beta_2\{(\|\theta_1 - \Pi(\theta_1,\theta_2,\theta_3)\|^2 + \|\theta_2 - \Psi(\theta_2,\theta_3,\theta_1)\|^2 + \|\theta_3 - \Omega(\theta_3,\theta_1,\theta_2)\|^2)(1 + \|\kappa - \Pi(\kappa,\lambda,\mu)\|^2 + \|\lambda - \Psi(\lambda,\mu,\kappa)\|^2 + \|\mu - \Omega(\mu,\kappa,\lambda)\|^2) + (\|\kappa - \Pi(\kappa,\lambda,\mu)\|^2 + \|\lambda - \Psi(\lambda,\mu,\kappa)\|^2 + \|\mu - \Omega(\mu,\kappa,\lambda)\|^2)(1 + \|\theta_1 - \Pi(\theta_1,\theta_2,\theta_3)\|^2 + \|\theta_2 - \Psi(\theta_2,\theta_3,\theta_1)\|^2 + \|\theta_3 - \Omega(\theta_3,\theta_1,\theta_2)\|^2)\},$$

(2) (Π,Ψ,Ω) satisfies contractive inequality condition II if $\forall\ \kappa,\lambda,\mu,\theta_1,\theta_2,\theta_3 \in$

Definition 8 Let H be a Hilbert space and C a nonempty closed convex subset of H. then, let $\Pi,\Psi,\Omega : C^3 \to C$ be a mapping. Also, let $\{\kappa_n\},\{\lambda_n\}$ and $\{\mu_n\}$ be sequences in C. Then, the tripled Mann pair iterative scheme is as follows:

$$\begin{cases}\kappa_{n+1} = (1-\eta_n)\kappa_n + \eta_n\Pi(\kappa_n,\lambda_n,\mu_n),\\ \lambda_{n+1} = (1-\eta_n)\lambda_n + \eta_n\Psi(\lambda_n,\mu_n,\kappa_n),\\ \mu_{n+1} = (1-\eta_n)\mu_n + \eta_n\Omega(\mu_n,\kappa_n,\lambda_n)\end{cases} \quad (1)$$

Where $0 < \eta_n < 1,\ n \ge 0$ (2)

$0 < \lim_{n\to\infty}\eta_n = \delta$ (3)

Theorem 1 Let $\Pi,\Psi,\Omega : C^3 \to C$ be m such that mappings define on a closed nonempty convex subset C of a Hilbert space H, such that (Π,Ψ,Ω) satisfies contractive inequality conditions I and II. Therefore, the tripled Mann pair iterative scheme which is constructed in$(1)-(3)$, where if is convergent, it δ satisfies $\frac{7}{4(2-\beta_2)} < \delta < 1$, and converges to a common tripled fixed point of Π,Ψ,Ω.

Proof. Let $(\kappa_n,\lambda_n,\mu_n) \to (\kappa,\lambda,\mu)$ as $n \to \infty$.

$$\|\Omega(\kappa,\lambda,\mu)-\Psi(\theta_1,\theta_2,\theta_3)\|^2 + \|\Psi(\lambda,\mu,\kappa) - \Pi(\theta_2,\theta_3,\theta_1)\|^2 + \|\Pi(\mu,\kappa,\lambda) - \Omega(\theta_3,\theta_1,\theta_2)\|^2 \le \beta_1(\|\kappa-\theta_1\|^2 + \|\lambda-\theta_2\|^2 + \|\mu-\theta_3\|^2) + \beta_2\{(\|\theta_1-\Pi(\theta_1,\theta_2,\theta_3)\|^2 + \|\theta_2-\Psi(\theta_2,\theta_3,\theta_1)\|^2 + \|\theta_3-\Omega(\theta_3,\theta_1,\theta_2)\|^2)(1 + \|\kappa-\Omega(\kappa,\lambda,\mu)\|^2 + \|\lambda-\Psi(\lambda,\mu,\kappa)\|^2 + \|\mu-\Pi(\mu,\kappa,\lambda)\|^2) + (\|\kappa-\Omega(\kappa,\lambda,\mu)\|^2 + \|\lambda-\Psi(\lambda,\mu,\kappa)\|^2 + \|\mu-\Pi(\mu,\kappa,\lambda)\|^2)(1 + \|\theta_1-\Pi(\theta_1,\theta_2,\theta_3)\|^2 + \|\theta_2-\Psi(\theta_2,\theta_3,\theta_1)\|^2 + \|\theta_3-\Omega(\theta_3,\theta_1,\theta_2)\|^2)\}$$

Where $\beta_1, \beta_2 > 0$ with $\beta_2 < \frac{1}{4}$

Contractive Inequality Condition I

On utilizing the parallelogram law, we obtain;

$$\|\Pi(\kappa,\lambda,\mu)-\kappa\|^2 + \|\Psi(\lambda,\mu,\kappa)-\lambda\|^2 + \|\Omega(\mu,\kappa,\lambda)-\mu\|^2 = \|\Pi(\kappa,\lambda,\mu)-\kappa_{n+1}+\kappa_{n+1}-\kappa\|^2 + \|\Psi(\lambda,\mu,\kappa)-\lambda_{n+1}+\lambda_{n+1}-\lambda\|^2 + \|\Omega(\mu,\kappa,\lambda)-\mu_{n+1}+\mu_{n+1}-\mu\|^2 \le 2\|\Pi(\kappa,\lambda,\mu)-\kappa_{n+1}\|^2 + 2\|\kappa_{n+1}-\kappa\|^2 + 2\|\Psi(\lambda,\mu,\kappa)-\lambda_{n+1}\|^2 + 2\|\lambda_{n+1}-\lambda\|^2 + 2\|\Omega(\mu,\kappa,\lambda)-\mu_{n+1}\|^2 + 2\|\mu_{n+1}-\mu\|^2 \quad (4)$$

$\forall\ \kappa, \lambda, \mu \in C$. Since by (Kim, 2020),

$$\|\Pi(\kappa,\lambda,\mu)-\kappa_{n+1}\|^2 = \left\|(1-\eta_n)(\Pi(\kappa,\lambda,\mu)-\kappa_n) + \eta_n\big(\Pi(\kappa,\lambda,\mu)-\Pi(\kappa_n,\lambda_n,\mu_n)\big)\right\|^2 \le 2(1-\eta_n)^2\|\Pi(\kappa,\lambda,\mu)-\kappa_n\|^2 + 2{\eta_n}^2\|\Pi(\kappa,\lambda,\mu) - \Pi(\kappa_n,\lambda_n,\mu_n)\|^2 \quad (5)$$

Similarly,

$$\|\Psi(\lambda,\mu,\kappa)-\lambda_{n+1}\|^2 \le 2(1-\eta_n)^2\|\Psi(\lambda,\mu,\kappa)-\lambda_n\|^2 + 2{\eta_n}^2\|\Psi(\lambda,\mu,\kappa) - \Psi(\lambda_n,\mu_n,\kappa_n)\|^2 \quad (6)$$

and

$$\|\Omega(\mu,\kappa,\lambda)-\mu_{n+1}\|^2 \le 2(1-\eta_n)^2\|\Omega(\mu,\kappa,\lambda)-\mu_n\|^2 + 2{\eta_n}^2\|\Omega(\mu,\kappa,\lambda) - \Omega(\mu_n,\kappa_n,\lambda_n)\|^2 \quad (7)$$

Employing condition I, (5), (6) and (7) in (4), we obtain;

$$\|\Pi(\kappa,\lambda,\mu)-\kappa\|^2 + \|\Psi(\lambda,\mu,\kappa)-\lambda\|^2 + \|\Omega(\mu,\kappa,\lambda)-\mu\|^2 \le 2\|\kappa_{n+1}-\kappa\|^2 + 2\|\lambda_{n+1}-\lambda\|^2 + 2\|\mu_{n+1}-\mu\|^2 + 4(1-\eta_n)^2\|\Pi(\kappa,\lambda,\mu)-\kappa_n\|^2 + 4{\eta_n}^2\|\Pi(\kappa,\lambda,\mu)-\Pi(\kappa_n,\lambda_n,\mu_n)\|^2 + 4(1-\eta_n)^2\|\Psi(\lambda,\mu,\kappa)-\lambda_n\|^2 + 4{\eta_n}^2\|\Psi(\lambda,\mu,\kappa)-\Psi(\lambda_n,\mu_n,\kappa_n)\|^2 + 4(1-\eta_n)^2\|\Omega(\mu,\kappa,\lambda)-\mu_n\|^2 + 4{\eta_n}^2\|\Omega(\mu,\kappa,\lambda)-\Omega(\mu_n,\kappa_n,\lambda_n)\|^2 \le 2\|\kappa_{n+1}-\kappa\|^2 + 2\|\lambda_{n+1}-\lambda\|^2 + 2\|\mu_{n+1}-\mu\|^2 + 4{\eta_n}^2(\|\Pi(\kappa,\lambda,\mu)-\Pi(\kappa_n,\lambda_n,\mu_n)\|^2 + \|\Psi(\lambda,\mu,\kappa)-\Psi(\lambda_n,\mu_n,\kappa_n)\|^2 + \|\Omega(\mu,\kappa,\lambda)-\Omega(\mu_n,\kappa_n,\lambda_n)\|^2) + 4(1-\eta_n)^2[2\|\Pi(\kappa,\lambda,\mu)-\kappa\|^2 + 2\|\kappa-\kappa_n\|^2] + 4(1-\eta_n)^2[2\|\Psi(\lambda,\mu,\kappa)-\lambda\|^2 + 2\|\lambda-\lambda_n\|^2] + 4(1-\eta_n)^2[2\|\Omega(\mu,\kappa,\lambda)-\mu\|^2 + 2\|\mu-\mu_n\|^2] \quad (8)$$

$$\leq 2\|\kappa_{n+1}-\kappa\|^2+2\|\lambda_{n+1}-\lambda\|^2 + 2\|\mu_{n+1}-\mu\|^2 + 4{\eta_n}^2[\beta_1(\|\kappa-\kappa_n\|^2 + \|\lambda-\lambda_n\|^2+\|\mu-\mu_n\|^2) + \beta_2\{(\|\kappa_n-\Pi(\kappa_n,\lambda_n,\mu_n)\|^2 + \|\lambda_n-\Psi(\lambda_n,\mu_n,\kappa_n)\|^2 + \|\mu_n-\Omega(\mu_n,\kappa_n,\lambda_n)\|^2)(1 + \|\kappa-\Pi(\kappa,\lambda,\mu)\|^2 + \|\lambda-\Psi(\lambda,\mu,\kappa)\|^2 + \|\mu-\Omega(\mu,\kappa,\lambda)\|^2) + (\|\kappa-\Pi(\kappa,\lambda,\mu)\|^2 + \|\lambda-\Psi(\lambda,\mu,\kappa)\|^2 + \|\mu-\Omega(\mu,\kappa,\lambda)\|^2)(1 + \|\kappa_n-\Pi(\kappa_n,\lambda_n,\mu_n)\|^2 + \|\lambda_n-\Psi(\lambda_n,\mu_n,\kappa_n)\|^2 + \|\mu_n-\Omega(\mu_n,\kappa_n,\lambda_n)\|^2)\}] + 8(1-\eta_n)^2[\|\Pi(\kappa,\lambda,\mu)-\kappa\|^2 + \|\Psi(\lambda,\mu,\kappa)-\lambda\|^2 + \|\Omega(\mu,\kappa,\lambda)-\mu\|^2+\|\kappa-\kappa_n\|^2 + \|\lambda-\lambda_n\|^2+\|\mu-\mu_n\|^2]$$

Since

$$\|\kappa_n-\Pi(\kappa_n,\lambda_n,\mu_n)\|^2 = \frac{1}{\eta_n^2}\|\kappa_n - \kappa_{n+1}\|^2 \qquad (9)$$

$$\|\lambda_n-\Pi(\lambda_n,\mu_n,\kappa_n)\|^2 = \frac{1}{\eta_n^2}\|\lambda_n - \lambda_{n+1}\|^2 \qquad (10)$$

and

$$\|\mu_n-\Omega(\mu_n,\kappa_n,\lambda_n)\|^2 = \frac{1}{\eta_n^2}\|\mu_n-\mu_{n+1}\|^2 \qquad (11)$$

From (8), we have;

$$\|\Pi(\kappa,\lambda,\mu)-\kappa\|^2+\|\Psi(\lambda,\mu,\kappa)-\lambda\|^2 + \|\Omega(\mu,\kappa,\lambda)-\mu\|^2 \leq 2\|\kappa_{n+1}-\kappa\|^2 + \iota 2\|\lambda_{n+1}-\lambda\|^2 + 2\|\mu_{n+1}-\mu\|^2 + 8(1-\eta_n)^2[\|\Pi(\kappa,\lambda,\mu)-\kappa\|^2 + \|\Psi(\lambda,\mu,\kappa)-\lambda\|^2 + \|\Omega(\mu,\kappa,\lambda)-\mu\|^2+\|\kappa-\kappa_n\|^2 + \|\lambda-\lambda_n\|^2+\|\mu-\mu_n\|^2] + 4{\eta_n}^2\left[\beta_1(\|\kappa-\kappa_n\|^2 + \|\lambda-\lambda_n\|^2+\|\mu-\mu_n\|^2) + \beta_2\left\{\frac{1}{\eta_n^2}(\|\kappa_n-\kappa_{n+1}\|^2 + \|\lambda_n-\lambda_{n+1}\|^2 + \|\mu_n-\mu_{n+1}\|^2)(1 + \|\kappa-\Pi(\kappa,\lambda,\mu)\|^2 + \|\lambda-\Psi(\lambda,\mu,\kappa)\|^2 + \|\mu-\Omega(\mu,\kappa,\lambda)\|^2)\left(1 + \frac{1}{\eta_n^2}(\|\kappa_n-\kappa_{n+1}\|^2 + \|\lambda_n-\lambda_{n+1}\|^2 + \|\mu_n-\mu_{n+1}\|^2)\right)\right\}\right] \qquad (12)$$

aking $n\to\infty$ in (12), by (3), we have;

$$\|\Pi(\kappa,\lambda,\mu)-\kappa\|^2+\|\Psi(\lambda,\mu,\kappa)-\lambda\|^2 + \|\Omega(\mu,\kappa,\lambda)-\mu\|^2 \leq 8(1-\delta)^2[\|\Pi(\kappa,\lambda,\mu)-\kappa\|^2+\|\Psi(\lambda,\mu,\kappa)-\lambda\|^2 + \|\Omega(\mu,\kappa,\lambda)-\mu\|^2] + 4\delta^2\beta_2\{\|\kappa-\Pi(\kappa,\lambda,\mu)\|^2+\|\lambda-\Psi(\lambda,\mu,\kappa)\|^2 + \|\mu-\Omega(\mu,\kappa,\lambda)\|^2\} = 4(2(1-\delta)^2+\beta_2\delta^2)(\|\Pi(\kappa,\lambda,\mu)-\kappa\|^2 + \|\Psi(\lambda,\mu,\kappa)-\lambda\|^2 + \|\Omega(\mu,\kappa,\lambda) - \mu\|^2) \qquad (13)$$

From $\beta_2<\frac{1}{4}$ and $0<\delta<1$, we have;

$$2(1-\delta)^2+\beta_2\delta^2 = 2-4\delta = 2\delta^2+\beta_2\delta^2 \leq 2-4\delta+2\delta+\beta_2\delta = 2-(2-\beta_2)\delta$$

Since $\frac{7}{4(2-\beta_2)} < \delta$, we obtain;

$$(1-\delta)^2 + \beta_2\delta^2 = 2 - 4\delta = 2\delta^2 + \beta_2\delta^2$$
$$\leq 2 - 4\delta + 2\delta + \beta_2\delta$$
$$= 2 - (2-\beta_2)\delta$$

Since $\frac{7}{4(2-\beta_2)} < \delta$, we obtain;

$$2(1-\delta)^2 + \beta_2\delta^2 < \frac{1}{4} \quad (14)$$

Going by (13) and (14), we get;

$$\|\Pi(\kappa,\lambda,\mu) - \kappa\|^2 + \|\Psi(\lambda,\mu,\kappa) - \lambda\|^2 + \|\Omega(\mu,\kappa,\lambda) - \mu\|^2 = 0$$

$\|\Pi(\kappa,\lambda,\mu) - \kappa\|^2 = 0, \|\Psi(\lambda,\mu,\kappa) - \lambda\|^2$ and $\|\Omega(\mu,\kappa,\lambda) - \mu\|^2$.

Therefore,

$\Pi(\kappa,\lambda,\mu) = \kappa, \Psi(\lambda,\mu,\kappa) = \lambda$ and $\Omega(\mu,\kappa,\lambda) = \mu$ (15)

Contractive Inequality Condition II

On using the parallelogram law, we have;

$$\begin{aligned}
&\|\Omega(\kappa,\lambda,\mu) - \kappa\|^2 \\
&+ \|\Psi(\lambda,\mu,\kappa) - \lambda\|^2 \\
&+ \|\Pi(\mu,\kappa,\lambda) - \kappa\|^2 \\
&= \|\Omega(\kappa,\lambda,\mu) - \kappa_{n+1} + \kappa_{n+1} - \kappa\|^2 \\
&+ \|\Psi(\lambda,\mu,\kappa) - \lambda_{n+1} + \lambda_{n+1} - \lambda\|^2 \\
&+ \|\Pi(\mu,\kappa,\lambda) - \mu_{n+1} + \mu_{n+1} - \mu\|^2 \\
&\leq 2\|\Omega(\kappa,\lambda,\mu) - \kappa_{n+1}\|^2 \\
&+ 2\|\kappa_{n+1} - \kappa\|^2 \\
&+ 2\|\Psi(\lambda,\mu,\kappa) - \lambda_{n+1}\|^2 \\
&+ 2\|\lambda_{n+1} - \lambda\|^2 \\
&+ 2\|\Pi(\mu,\kappa,\lambda) - \mu_{n+1}\|^2 \\
&+ 2\|\mu_{n+1} \\
&- \mu\|^2 \qquad (16)
\end{aligned}$$

∀ κ, λ, μ ∈ C. since;

$$\begin{aligned}
&\|\Omega(\kappa,\lambda,\mu) - \kappa_{n+1}\|^2 \\
&= \big\|(1-\eta_n)(\Omega(\kappa,\lambda,\mu) - \kappa_n) \\
&+ \eta_n\big(\Omega(\kappa,\lambda,\mu) - \Psi(\kappa_n.\lambda_n,\mu_n)\big)\big\|^2 \\
&\leq 2(1-\eta_n)^2\|\Omega(\kappa,\lambda,\mu) - \kappa_n\|^2 \\
&+ 2\eta_n^2\|\Omega(\kappa,\lambda,\mu) \\
&- \Psi(\kappa_n,\lambda_n,\mu_n)\|^2 \qquad (17)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\|\Psi(\lambda,\mu,\kappa) - \lambda_{n+1}\|^2 \\
&\leq 2(1-\eta_n)^2\|\Psi(\lambda,\mu,\kappa) - \lambda_n\|^2 \\
&+ 2\eta_n^2\|\Psi(\lambda,\mu,\kappa) \\
&- \Pi(\lambda_n,\mu_n,\kappa_n)\|^2 \qquad (18)
\end{aligned}$$

and

$$\begin{aligned}
&\|\Pi(\mu,\kappa,\lambda) - \mu_{n+1}\|^2 \\
&\leq 2(1-\eta_n)^2\|\Pi(\mu,\kappa,\lambda) - \mu\|^2 \\
&+ 2\eta_n^2\|\Pi(\mu,\kappa,\lambda) \\
&- \Omega(\mu,\kappa,\lambda)\|^2 \qquad (19)
\end{aligned}$$

Hence, using condition II, (17), (18) and (19) in (16) yields;

$$\begin{aligned}
&\|\Omega(\kappa,\lambda,\mu) - \kappa\|^2 + \|\Psi(\lambda,\mu,\kappa) - \lambda\|^2 \\
&+ \|\Pi(\mu,\kappa,\lambda) - \mu\|^2 \\
&\leq 2\|\kappa_{n+1} - \kappa\|^2 + 2\|\lambda_{n+1} - \lambda\|^2 \\
&+ 2\|\mu_{n+1} - \mu\|^2 \\
&+ 4(1-\eta_n)^2\|\Omega(\kappa,\lambda,\mu) - \kappa_n\|^2 \\
&+ 4\eta_n^2\|\Omega(\kappa,\lambda,\mu) - \Psi(\kappa_n,\lambda_n,\mu_n)\|^2 \\
&+ 4(1-\eta_n)^2\|\Psi(\lambda,\mu,\kappa) - \lambda_n\|^2 \\
&+ 4\eta_n^2\|\Psi(\lambda,\mu,\kappa) - \Pi(\lambda_n,\mu_n,\kappa_n)\|^2 \\
&+ 4(1-\eta_n)^2\|\Pi(\mu,\kappa,\lambda) - \mu\|^2 \\
&+ 4\eta_n^2\|\Pi(\mu,\kappa,\lambda) \\
&- \Omega(\mu,\kappa,\lambda)\|^2 \qquad (20)
\end{aligned}$$

$$\begin{aligned}
&\leq 2\|\kappa_{n+1} - \kappa\|^2 + 2\|\lambda_{n+1} - \lambda\|^2 \\
&+ 2\|\mu_{n+1} - \mu\|^2 \\
&+ 4(1-\eta_n)^2[2\|\Omega(\kappa,\lambda,\mu) - \kappa\|^2 \\
&+ 2\|\kappa - \kappa_n\|^2] \\
&+ 4(1-\eta_n)^2[2\|\Psi(\lambda,\mu,\kappa) - \lambda\|^2 \\
&+ 2\|\lambda - \lambda_n\|^2] \\
&+ 4(1-\eta_n)^2[2\|\Pi(\mu,\kappa,\lambda) - \mu\|^2 \\
&+ 2\|\mu - \mu_n\|^2] \\
&+ 4\eta_n^2(\|\Omega(\kappa,\lambda,\mu) - \Psi(\kappa_n,\lambda_n,\mu_n)\|^2 \\
&+ \|\Psi(\lambda,\mu,\kappa) - \Pi(\lambda_n,\mu_n,\kappa_n)\|^2 \\
&+ \|\Pi(\mu,\kappa,\lambda) \\
&- \Psi(\mu,\kappa,\lambda)\|^2) \qquad (21)
\end{aligned}$$

$$\leq 2\|\kappa_{n+1}-\kappa\|^2+2\|\lambda_{n+1}-\lambda\|^2+2\|\mu_{n+1}-\mu\|^2$$
$$+8(1-\eta_n)^2[\|\Omega(\kappa,\lambda,\mu)-\kappa\|^2$$
$$+\|\Psi(\lambda,\mu,\kappa)-\lambda\|^2+\|\Pi(\mu,\kappa,\lambda)-\mu\|^2$$
$$+\|\kappa-\kappa_n\|^2+\|\lambda-\lambda_n\|^2+\|\mu-\mu_n\|^2]$$
$$+4\eta_n^2[\beta_1(\|\kappa-\kappa_n\|^2+\|\lambda-\lambda_n\|^2+\|\mu-\mu_n\|^2)$$
$$+\beta_2\{(\|\kappa_n-\Omega(\kappa_n,\lambda_n,\mu_n)\|^2$$
$$+\|\lambda_n-\Psi(\lambda_n,\mu_n,\kappa_n)\|^2$$
$$+\|\mu_n-\Pi(\mu_n,\kappa_n,\lambda_n)\|^2)(1+\|\kappa-\Omega(\kappa,\lambda,\mu)\|^2$$
$$+\|\lambda-\Psi(\lambda,\mu,\kappa)\|^2+\|\mu-\Pi(\mu,\kappa,\lambda)\|^2)(1$$
$$+\|\kappa_n-\Omega(\kappa_n,\lambda_n,\mu_n)\|^2+\|\lambda_n-\Psi(\lambda_n,\mu_n,\kappa_n)\|^2$$
$$+\|\mu_n-\Pi(\mu_n,\kappa_n,\lambda_n)\|^2)\}] \quad (22)$$

nce,

$$\|\kappa_n-\Omega(\kappa_n,\lambda_n,\mu_n)\|^2=\frac{1}{\eta_n^2}\|\kappa_n-\kappa_{n+1}\|^2 \quad (23)$$

$$\|\lambda_n-\Psi(\lambda_n,\mu_n,\kappa_n)\|^2=\frac{1}{\eta_n^2}\|\lambda_n-\lambda_{n+1}\|^2 \quad (24)$$

$$\|\mu_n-\Pi(\mu_n,\kappa_n,\lambda_n)\|^2=\frac{1}{\eta_n^2}\|\mu_n-\mu_{n+1}\|^2 \quad (25)$$

Using (23), (24), (25) in (22), we have;

Taking $n\to\infty$. In (26) by (3), we have;

$$\|\Omega(\kappa,\lambda,\mu)-\kappa\|^2+\|\Psi(\lambda,\mu,\kappa)-\lambda\|^2+\|\Pi(\mu,\kappa,\lambda)-\mu\|^2$$
$$\leq 8(1-\delta)^2[\|\Omega(\kappa,\lambda,\mu)-\kappa\|^2+\|\Psi(\lambda,\mu,\kappa)-\lambda\|^2+\|\Pi(\mu,\kappa,\lambda)-\mu\|^2]$$
$$+4\delta^2\beta_2\{\|\kappa-\Omega(\kappa,\lambda,\mu)\|^2+\|\lambda-\Psi(\lambda,\mu,\kappa)\|^2+\|\mu-\Pi(\mu,\kappa,\lambda)\|^2\}$$
$$=4(2(1-\delta)^2+\beta_2\delta^2)(\|\Omega(\kappa,\lambda,\mu)-\kappa\|^2+\|\Psi(\lambda,\mu,\kappa)-\lambda\|^2+\|\Pi(\mu,\kappa,\lambda)-\mu\|^2) \quad (27)$$

Since $\beta_2<\frac{1}{4}$ and $\frac{7}{2(2-\beta_2)}<\delta<1$, we obtain

$$2(1-\delta)^2+\beta_2\delta^2<\frac{1}{4} \quad (28)$$

From (270 and (28), we have;

$$\|\Omega(\kappa,\lambda,\mu)-\kappa\|^2+\|\Psi(\lambda,\mu,\kappa)-\lambda\|^2+\|\Pi(\mu,\kappa,\lambda)-\mu\|^2=0$$

Then,

$\|\Omega(\kappa,\lambda,\mu)-\kappa\|^2=0, \|\Psi(\lambda,\mu,\kappa)-\lambda\|^2=0$ and $\|\Pi(\mu,\kappa,\lambda)-\mu\|^2=0$.

Therefore,

$\Omega(\kappa,\lambda,\mu)=\kappa,\ \Psi(\lambda,\mu,\kappa)=\lambda$ and $\Pi(\mu,\kappa,\lambda)=\mu$.

Then, by Conditions I and II, (κ,λ,μ) is a common tripled fixed point of Π,Ψ and Ω. Hence, this completes the proof.

CONCLUSION

This work has shown that tripled Mann iterative scheme can be applied to resolve the problem of common tripled fixed points of certain mappings. Hence, the work can further be extended to fixed point theory via mixed monotone mappings.

ACKNOWLEDGEMENTS

We are very grateful to all whose contributions had enriched this work.

REFERENCES

Abbas, M., Aydi, H. & Karapınar, E. (2011). Tripled fixed points of multi-valued nonlinear contraction mappings in partially ordered metric spaces. *Abstract and Applied Analysis*,1-13. Doi:10.1155/2011/812690

Aydi, H. & Karapınar, E. (2012). New Meir-Keeler type tripled fixed point theorems on ordered partial metric spaces. *Mathematical Problems in Engineering*, 1-17. Doi: 10.1155/2012/409872

Aydi, H., Karapınar, E. & Shatanawi, W. (2012). Tripled fixed point results in generalized metric spaces. *Journal of Applied Mathematics*, 1-10. Doi:10.1155/2012/314279

Aydi, H., Karapınar, E. & Radenovic, S. (2013). Tripled coincidence fixed point results for Boyd-Wong and Matkowski type contractions. *Journal of the Spanish Royal Academy of Sciences, Series A Mathematics*, **107**, 339-353.

Bhaskar, T.G. & Lakshmikantham, V. (2006). Fixed point theorems in partially ordered metric spaces and applications. *Nonlinear Analysis*, 65, 1379–1393.

Cheng, R. & Ross, W.T. (2015). Weak parallelogram laws on Banach spaces and applications to prediction. *Periodica*

Mathematica Hungarica,**71**(1), 45-58. Doi: 10.1007/s10998-014-0078-4.

Choudhury, B.S. & Kundu, S. (2016). A constructive algorithm for a coupled fixed point problem. Nonlinear Funct. Anal. Appl., **21**, 567–572.

Dehaish, B.A.I., Khamsi, M.A. & Khan, A.R. (2013). Mann iteration process for asymptotic pointwise nonexpansive mappings in metric spaces. *Journal of Mathematical Analysis and Applications*, 397, 861–868.

Karapınar, E., Aydi, H. & Mustafa, Z. (2013). Some tripled coincidence point theorems for almost generalized contractions in ordered metric spaces. *Tamkang Journal of Mathematics*, 44, 233-251.

Kim, K.S. (2019). Convergence theorems of variational inequality for asymptotically nonexpansive nonself mapping in CAT(0) spaces. *Mathematics*, 7(12), 1234.

Kim, K.S. (2020). A constructive scheme for a common coupled fixed point problems in Hilbert Space. *Mathematics*. 8(10), 1717.

Lakshmikantham, V. & Ciric, L. (2009). Coupled fixed point theorems for non linear contractions in partially ordered metric spaces. *Nonlinear Analysis*, **70**, 4341–4349.

Luong, N.V. & Thuan, N.X. (2011). Coupled fixed points in partially ordered metric spaces and application. *Nonlinear Analysis*, 74, 983–992.

Ran, A.C.M. & Reurings, M.C.B. (2004). A fixed point theorem in partially ordered sets and some applications to matrix equations. *Proceedings of the American Mathematical Society*, 132, 1435–1443.