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Using Nonstandard Finite Difference Methods for Solving Converted Schrodinger Equation to an ODE

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Abstract

In this work, by introducing a transformation, the nonlinear Schrodinger equation is converted to an ordinary differential equation (ODE). Then, two nonstandard finite difference (NSFD) schemes are constructed for studying the reduced equation. It is shown that the methods preserve the positivity and boundedness properties of the original equation and are stable conditionally and consistence. Finally, the results of the methods are compared with each other and also with the results of the standard finite difference scheme at some points. The graphs of the errors of numerical solutions for these schemes are plotted and compared with the exact solutions.

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INTRODUCTION

In quantum mechanics, the Schrodinger equation is a partial differential equation (PDE) that describes how the quantum state of a physical system changes with time. It was formulated in late 1925 and published in 1926 by the Hustrain physicist Erwin Schrodinger. (Schrödinger, 1926) Solution of the Schrodinger's equation describes not only molecular, atomic, and subatomic systems, but also macroscopic systems possibility even the whole universe (Laloë, 2019)

It is used in physics and most of chemistry to deal with problems about the atomic structure of matter. In theoretical physics the nonlinear Schrodinger equation (NLSE) is a nonlinear variation of the Schrodinger equation. It is a classical field equation whose principal applications are to the propagation of light in nonlinear optical fibers and planer wave guides (Boris, 2005).

Unlike the linear Schrodinger equation, the NSLE never describes the time evolution of a quantum state. This equation with different versions is one of the most important models of mathematical physics with several applications to different fields such as nonlinear optic (Agrawal, 2001), models of protein dynamics (Fordy, 1990), plasma physics (Stenflo and Yu, 1997), self-focusing in laser pulses (Sulem and Sulem, 1999) and many other fields. In the present research various numerical scheme will be developed and compared for solving this equation (Bao, 2004; Bao and Jaksch, 2003; Dehghan and Taleei, 2010).

We consider the following nonlinear Schrodinger equation:

$$iq_t + q_{xx} + 2|q^2||q| = 0, \qquad (1)$$

With the initial and boundary conditions:

$$q(x,0) = e^{2it}, \quad q_x(x,0) = e^{2it}. \qquad (2)$$

The exact solution of this equation, which obtained by the Adomian method (Wazwaz, 2010), is:

$$q(x,t) = e^{i(x+t)}. \qquad (3)$$

Assume that $q(x,t)$ is the solution to the nonlinear Schrodinger Eq. 1, we use the following transformations:

$$q(x,t) = e^{i\theta}u(\zeta). \qquad (4)$$

Where

$$\theta = \alpha x + \delta t, \qquad (5)$$

$$\zeta = k(x - \lambda t), \qquad (6)$$

Where, α, δ, k and λ are real constants((Wazwaz, 2010)).

Substituting 4, 5, 6 in 1, we obtain $\lambda = 2\alpha$ and the following ODE

equation:

$$k^2 u''(\zeta) - (\delta + \alpha^2)u(\zeta) + 2u^3(\zeta) = 0. \quad (7)$$

Rewrite this second-order ordinary differential equation as follows:

$$u'' - k_1 u + k_2 u^3(\zeta) = 0, \quad (8)$$

Where

$$k_1 = \frac{\delta + \alpha^2}{k^2}, \qquad k_2 = \frac{2}{k^2}. \quad (9)$$

By setting $\delta = -1, k = 1, \lambda = 0$, we have the following equation:

$$u'' = -u - 2u^3. \quad (10)$$

In this paper we construct two NSFD schemes and one SFD scheme for the Eq. 10.

THE FINITE-DIFFERENCE SCHEMES

The main idea behind the finite-difference methods is to approximate the derivatives appearing in the partial differential equation by use of Taylor series. The solution domain of the problem is covered by a mesh of grid-lines

$$x_i = i\Delta x, \qquad i = 0,1,\dots,M, \quad (11)$$

$$t_n = n\Delta t, \qquad n = 0,1,\dots,N. \quad (12)$$

Parallel to the space and time coordinate axes, respectively. Approximations u_i^n to $u(i\Delta x, n\Delta)$ are calculated at the point of intersection of these lines, namely $(i\Delta x, n\Delta)$ which is referred to as the (i,n) grid-point, the constant spatial and temporal grid-spacing are $\Delta x = \frac{1}{M}, \Delta t = \frac{1}{N}$ respectively.

General finite difference schemes

The most general finite difference model for equation

$$\frac{d^2u}{dt^2} = f(u, \lambda), \quad (13)$$

Where λ is the system parameters, that is of second-order in the discrete derivative takes the following form

$$\frac{u_{k+1} - 2u_k + u_{k-1}}{\varphi(h,\lambda)} = F(u_k, u_{k+1}, u_{k-1}, \lambda, h). \quad (14)$$

The discrete derivative, on the left-side, is a generalization of that which is normally used. Namely

$$\frac{d^2u}{dt^2} \to \frac{u_{k+1} - 2u_k + u_{k-1}}{h^2}. \quad (15)$$

from Eq. 15 we have

$$\frac{d^2u}{dt^2} \to \frac{u_{k+1} - 2u_k + u_{k-1}}{\varphi(h,\lambda)}. \quad (16)$$

Where the denominator function $\varphi(h,\lambda)$ has the property

$$\varphi(h,\lambda) = h + o(h^2),$$

$$\lambda = fixed \qquad h \to 0. \quad (17)$$

This form the discrete derivative is based on the traditional definition of the derivative which can be generalized as follows:

$$\frac{d^2u}{dt^2} = \lim_{h\to 0} \frac{u[t + \psi_1(h)] - 2u(t) + u[t - \psi_1(h)]}{\psi_2(h)}, \quad (18)$$

Where

$$\psi_i(h) = h + o(h^2), \qquad h \to 0, \quad i = 1,2. \quad (19)$$

EXACT FINITE DIFFERENCE SCHEMES

Consider the general second-order differential equation

$$\frac{d^2u}{dt^2} = f(u,t,\lambda), \qquad u(t_0) = u_0, \qquad (20)$$

Where λ is the system parameters and $f(u,t,\lambda)$ is such that Eq. 20 has a unique solution exists for $0 \leq t < T$.

Let the solution to Eq. 20 be

$$u(t) = \varphi(\lambda, u_0, t_0, t), \qquad (21)$$

With

$$\varphi(\lambda, u_0, t_0, t_0) = u_0. \qquad (22)$$

Now consider a finite difference model for Eq. 20

$$u_{k+1} = F(\lambda, h, u_k, u_{k-1}, t_k), \qquad t_k = hk, \qquad h = \Delta t. \qquad (23)$$

Let the solution of 24 can be expressed in the form

$$u_k = \varphi(\lambda, h, u_0, u_{-1}, t_0, t_k), \qquad (24)$$

With

$$\varphi(\lambda, h, u_0, u_{-1}, t_0, t_0) = u_0\,. \qquad (25)$$

Definition 3-1

Eq. 20 and Eq. 24 are said to have same general solution if and only if

$u_k = u(t_k)$,

for $h > 0$.

Definition 3.2

An exact difference scheme is one for which the solution of the difference equation has the same general solution as the associated differential equation.

By using these two definitions, the following theorem can be stated.

Theorem 3.1

The second order differential equation

$$\frac{d^2u}{dt^2} = f(u,t,\lambda), \qquad u(t_0) = u_0, \qquad (26)$$

has an exact finite difference scheme that is given by

$$u_k = \varphi(\lambda, u_k, u_{k-1}, t_k, t_{k+1}), \quad , \qquad (27)$$

The function φ is the same that in Eq. 21

Proof:

The group property of the solutions to Eq. 26 gives:

$$u(t+h) = \varphi[\lambda, u(t+h), u(t-h), t, t+h]. \qquad (28)$$

Making the following identifications

$$t \to t_k, \qquad u(t) \to u_k, \qquad (29)$$

In Eq. 28 we obtained:

$$u_{k+1} = \varphi[\lambda, u_k, u_{k-1}, t_k, t_{k+1}].$$

This is the requirement for ordinary difference equation which has the same general solution as Eq. 20.

Note that this theorem is only an existence theorem.

It basically says that if a differential equation has a solution, then an exact finite-difference scheme exists.

A major implication of the theorem is that the solution of the difference equation is exactly equal to the solution of the ordinary differential equation on the computational grid for fixed, but, arbitrary step-size h.

Definition 3.3

Consider the second order differential Eq. 20 with the finite-difference equation of the form 23.

The method 25 is called elementary stable, if for any value of the step-size h, the linear stability of each fixed point $y*$ of system 20 is the same as the stability of $y*$ as a fixed point of the discrete method 25.

Theorem 3.2

If the difference scheme 25 satisfies

$$\frac{\partial F}{\partial t}(h,u) \geq 0, \qquad for\ h > 0, \qquad (30)$$

And for every $h > 0$ the equations $u = F(h,u)$ in u have the same roots with their multiplicity, then the difference scheme 23. is:

(i) Elementary stable and (ii) Stable with respect to monotonicity of solutions (Anguelov and Lubuma, 2003).

NONSTANDARD MODELING RULES

The genesis of nonstandard finite difference (NSFD) modeling procedures developed in 1989 by Mickens. Extensions and a summary of the known results up to 1994 are given in (Mickens, 1994).

For constructing nonstandard schemes; we concentrate on the exact finite difference scheme for the general logistic differential equation. The following observations are important

i) Exact finite difference schemes generally require that nonlinear terms be modeled non- locally. Thus, for the logistic equation the u^2 term is evaluated at two different grid points

$$u^2 \rightarrow u_k u_{k+1}, \qquad or \qquad u^2 \rightarrow 2u_k^2 - u_k u_{k+1},$$

However, for finite fixed, nonzero values of step-size, the two representations of the squared terms are not equal, i.e,

$$u_{k+1}u_k \neq (u_k)^2 \qquad and \qquad 2u_k^2 - u_k u_{k+1} \neq (u_k)^2,$$

So a seemingly trivial modification in the modeling nonlinear terms can lead to major changes in the solution behaviors of the difference equations.

ii) The discrete derivatives for both differential equations have denominator functions that are more complicated than those used in the standard modeling procedure, in fact we have :

$$\frac{du}{dt} \rightarrow \frac{u_{k+1} - u_k}{\varphi(h,\lambda)}, \qquad (31)$$

where $t_k = (\Delta t) = hk,\ x_k$ is an approximation to $x(t_k)$ and the denominator function satisfies the condition

$$\varphi(h,\lambda) = h + o(h^2). \quad (32)$$

In Eq. 32, λ represents various parameters appearing in the differential equation. This way of constructing discrete derivatives can be easily extended to partial derivatives(Mickens,1989؛ Mickens,1994).

For example, the time-derivative in the logistic equation is replaced by the following discrete representation

$$\frac{du}{dt} \to \frac{u_{k+1} - u_k}{\frac{e^{\lambda h} - 1}{\lambda}}. \quad (33)$$

iii) The order of discrete derivatives in the exact finite difference schemes is always equal to the corresponding order of derivatives of the differential equation.

The rules for construction the discrete models

Rule 1: The order of the discrete derivatives must be exactly equal to the orders of the corresponding derivatives of the differential equations.

Rule 2: Denominator function for the discrete derivatives must, in general, be expressed in terms of more complicated functions of the step sizes than those conventionally used.

Rule 3: Nonlinear terms must, in general, be modeled nonlocally on the computational grid or lattice.

Rule 4: Special solutions of the differential equations should be special (discrete) solutions of the finite difference models.

Rule 5: The finite-difference equations should not have solutions that do not correspond exactly to solutions of the differential equations.

General NSFD method for Eq. 10

In this subsection, we construct general NSFD schemes for the differential Eq. 10.

The scheme is given as follows:

$$\frac{\Delta^2 u_k}{\varphi} \to -u_k - 2u_k^2((1+\theta)u_{k+1} - \theta u_{k-1}), \quad (34)$$

Where

$\varphi = h^2 + o(h^4)$ and $\theta \geq 0$.

We are able to find sufficient condition such that the scheme 34 has the stability stated in (i), (ii), of theorem 3.1, the main result is stated in the following theorem.

Theorem 4.2.1

If $\theta \geq \frac{1}{2}$ then the difference scheme 34 is stable with respect to monotonicity of solutions and is elementary stable on the positively invariant interval $[0,\infty)$.

Proof:

It is not difficult to see that the difference Eq.34 and the differential Eq.10 share the same fixed-points.

The difference Eq.34 can be written as

$$u_{k+1} = F(\varphi, u_k, u_{k-1}),$$

where

$$F(\varphi,u)=\frac{u+2\varphi\theta u^3-\varphi u}{1+2\varphi u^2(1+\theta)}, \qquad (35)$$

The partial derivative of $F(\varphi,u)$ is

$$\frac{\partial F}{\partial u} = \frac{1-\varphi+\varphi u^2(6\theta-2(1+\theta))+4\varphi^2u^4\theta(1}{(1+2\varphi u^2(1+\theta))^2} \qquad (36)$$

Since all parameters and variables are assumed nonnegative and $0\le\varphi\le 1$.

We see from the Eq.36, the condition $\frac{\partial F}{\partial u}\ge 0$ is satisfied if $\left(u\theta-2(1+\theta)\right)\ge 0$, ie $\theta\ge\frac{1}{2}$.

TWO NONSTANDARD FINITE-DIFFERENCE SCHEMES FOR EQ. 10

The first NSFD scheme

The first (NSFD) scheme selected for Eq.10 in this paper is

$$\frac{u_{m+1}-2u_m+u_{m-1}}{4\mathrm{Sin}^2\, h/2} = -u_m-2u_m^2\frac{(3u_{m+1}-u_{m-1})}{2}. \qquad (37)$$

This particular discretization has several important features:

1. The nonlinear term is modeled bye following representation:

$$u_m^3\to\left(\frac{3u_{m+1}-u_{m-1}}{2}\right)u_m^2. \qquad (38)$$

2. The denominator function in this scheme is :

$$\varphi(\Delta x)=4\mathrm{Sin}^2\, h/2. \qquad (39)$$

Since the expression given by Eq. 37 is linear in u_{m+1}, it can be solved to give:

$$u_{m+1} = \frac{u_m(2-4S)+u_{m-1}(4Su_m^2-1)}{1+12u_m^2S}, \qquad (40)$$

Where $\mathrm{S}=\mathrm{Sin}^2\, h/2$

Analysis of nonstandard finite difference approximation for the first scheme:

stability

consider the Eq.40 and show the stability of this equation. We write homogeneous case of Eq.40. The characteristic equation for this equation is:

$$12rs+r+(4s-2)=0, \qquad (41)$$

We know that if the characteristic roots less or equal to 1 then the equation is stable. So, in this equation if $0.625\le s\le 1$ then the equation is stable.

Consistency

A finite-difference is said to be consistent with an equation if, in limit as the grid spacing tends to zero, the finite-difference formula is identical to the equation at each point in the solution domain.

To researching the consistency of Eq.10 by first method, consider Eq. 40

written in the form:

$$\ell_\Delta(u_j)=u_{j+1}(1+12u_j^2s)-u_j(2-4s)-u_{j-1}(4su_j^2-1)=0. \qquad (42)$$

Replacing u_j in 42 by the exact solution $\hat{u}_j$ of Eq.10 and after rearranging, we have:

$$\ell_{\Delta}\{\hat{u}_j\} = (u_{j+1} - 2u_j + u_{j-1}) + 4su_j + 4su_j^2(3u_{j+1} - u_{j-1}). \quad (43)$$

Since $\hat{u}_j$ is continuously differential, the terms of 43 may be replaced by their Taylor expansions about the $(j\Delta x)$, this gives:

$$\ell_{\Delta}(\hat{u}_j) = \left(\frac{2\Delta x^2}{2!}\frac{\partial^2 u}{\partial x^2}\bigg|_j + 2\frac{\Delta x^4}{4!}\frac{\partial^4 u}{\partial x^4}\bigg|_j + \cdots\right) + 4su_j + 4su_j^2\left(2u_j + 4\Delta x\frac{\partial u}{\partial x}\bigg|_j + \frac{2\Delta x^2}{2!}\frac{\partial^2 u}{\partial x^2} + 4\frac{\Delta x^3}{3!}\frac{\partial^3 u}{\partial x^3} + \cdots\right), \quad (44)$$

or

$$\ell_{\Delta}(\hat{u}_j) = 2u_j^3 + u_j + \frac{(\Delta x)^2}{4s}\left(\frac{\partial^2 u}{\partial x^2}\bigg|_j + \frac{2(\Delta x)^4}{4!}\frac{\partial^4 u}{\partial x^4}\bigg|_j + \frac{2(\Delta x)^4}{6!}\frac{\partial^6 u}{\partial x^6}\bigg|_j + \cdots\right), \quad (45)$$

$$\ell_{\Delta}(\hat{u}_j) = 2u_j^3 + u_j + \frac{(\Delta x)^2}{4s}\left(\frac{\partial^2 u}{\partial x^2}\bigg|_j + E(\hat{u}),\right), \quad (46)$$

where

$$E(\hat{u}) = \frac{2(\Delta x)^2}{4s}\bigg|_j \frac{\partial^4 u}{\partial x^4} + \frac{2(\Delta x)^4}{6!}\frac{\partial^6 u}{\partial x^6}\bigg|_j + \cdots \quad (47)$$

is the truncation error of the second order accurate in space.

As previously seen , when the grid spacing get smaller and smaller with the first method the truncation error gets smaller and smaller at a fix point in the solution domain.

In the limit as $\Delta x \to 0$, the nonstandard finite difference formula 40 is equivalent to the Eq.10, so this method is consistent.

Preserving the positivity and boundary conditions

According to the nonstandard finite difference rules, the Eq. 40 has to satisfy two criteria, positivity and boundary condition:

The boundary condition is (Mickens, 2000;)

$$0 \le u_m \le 1 \to 0 \le u_{m+1} \le 1, \quad (48)$$

and the positivity condition is

$$0 \le u_m \to 0 \le u_{m+1},$$

It can easily be shown by the following argument that the Eq. 40 include this conditions:

$$0 \le u_m \le 1 \to 0 \le u_m\left(2 - 4Sin^2 {}^h/_2\right) \le 2, \quad (50)$$

$$0 \le u_{m-1} \le 1 \to 0 \le u_{m-1}\left(4Sin^2 {}^h/_2\, u_m^2 - 1\right) \le 3, \quad (51)$$

and

$$0 \le 1 + 12{u_m}^2 Sin^2 {}^h/_2 \le 13, \quad (52)$$

So from 49 ,50, 51 we have

$$0 \le u_{m+1} \le \frac{5}{13} \le 1.$$

The second NSFD scheme

We use the Mickens NSFD model (Mickens, 2002), for construct the second scheme (NSFD) to Eq.10.

In this model, denote the fixed – point of equation

$$\frac{d^2u}{dx^2} = f(u)\,, \quad (53)$$

By

$$\{\bar{u}^{(i)}, i = 1,2, \ldots I\}\,, \quad (54)$$

That, (I) may be unbounded.

The fixed-points are the solution to the equation

$$f(\bar{u}) = 0, \quad (55)$$

Define $R_i$as,

$$R_i = \frac{df[\bar{u}^{(i)}]}{du}, \quad (56)$$

and R^* as $R^* = Max|R_i|, \quad i = 1 \dots I;$

Linear stability analysis applied to the i-th fixed-point gives the following results:

i) If R_i>0 the fixed-point $u(t) = \bar{u}^{(i)}$ is linearity unstable

ii) If R_i<0 the fixed-point $u(t) = \bar{u}^{(i)}$ is linearity unstable

So; the nonstandard finite-difference scheme for Eq. 53 is:

$$\frac{u_{k+1} - 2u_k + u_{k-1}}{\frac{\varphi(hR^*)}{R^*}} = f(u_k), \quad (57)$$

where

$$\varphi(z) = Z^2 + o(z^4), \qquad z \to 0,$$
$$0 < \varphi(z) < 1 \quad , \qquad z > 0. \quad (58)$$

Theorem 5.2.1

The nonstandard finite difference scheme, Eq.57 has fixed-points with exactly the same linear stability properties as the differential Eq. 53 (Mickens, 2002; Mickens, 2005).

So, in the Eq. 10, we have :

$f(u) = -u - 2u^3$, and the fixed points are $\bar{u}^{(1)} = 0$ and $\bar{u}^{(2)} = \frac{+i\sqrt{2}}{2}, \bar{u}^{(3)} = -i\frac{\sqrt{2}}{2},$

Therefore,

$$Ri = -1 - 6u_i^2 \to R_0 = -1$$
$$and \ \ R_2 = R_3 = 2 \to R^* = 2, \quad (59)$$

points. The denominator function 57 is :

$$\varphi = \frac{1 - e^{-2h}}{2}. \quad (60)$$

The NSFD scheme for Eq.10, is:

$$\frac{u_{m+1} - 2u_m + u_{m-1}}{h\varphi} = -u_m - u_m^2(3u_{m+1} - u_{m-1}). \quad (61)$$

Since the expression 60 is linear in u_{m+1}, it can be solved for to give:

$$u_{m+1} = \frac{u_m(2 - h\varphi) + u_{m-1}(h\varphi u_m^2 - 1)}{1 + 3u_m^2 h\varphi}. \quad (62)$$

Analysis of (NSFD) approximation for the second scheme:

Stability

Consider the Eq. 61 that is second NSFD scheme for Eq. 10 for stability, we write homogeneous case of this equation. Then the characteristic equation is:

$$3r^2 h\varphi + r + (h\varphi - 2) = 0, \quad (63)$$

If the characteristic roots less or equal to 1,

We should have the following unequal relation:

$$h\varphi \geq {}^{1}/_{4}, \quad (64)$$

That is the stability condition.

Consistency

To show the consistency of Eq. 10 by the second Method; we consider Eq. 61 written in the following form:

$$\ell_\Delta(u_j) = u_{j+1}(1+3hsu_j) - u_j(2-hs) - u_{j-1}(hsu_j^2-1) = 0, \quad (65)$$

where $s = \varphi$.

Replacing u_j in 64 by the exact solution $\hat{u}_j$ of Eq. 10 and after rearranging, we have

$$\ell_\Delta(\hat{u}_j) = (u_{j+1} - 2u_j + u_{j-1}) + u_j hs + hsu_j^2(3u_{j+1} - u_{j-1}) = 0, \quad (66)$$

By replacing the Taylor expansions in to the terms of above equation about the $j\Delta x$, we have:

$$\ell_\Delta(\hat{u}_j) = \left(\frac{2\Delta x^2}{2!}\frac{\partial^2 u}{\partial x^2}\Big|_j + \frac{2\Delta x^4}{4!}\frac{\partial^4 u}{\partial x^4}\Big|_j + \cdots\right) + u_j hs + hsu_j^2\left(2u_j + 4\Delta x\frac{\partial u}{\partial x}\Big|_j + \frac{2\Delta x^2}{2!}\frac{\partial^2 u}{\partial x^2} + \cdots\right), \quad (67)$$

Or

$$\ell_\Delta(\hat{u}_j) = u_j + 2u_j^3 + \frac{(\Delta x)^2}{hs}\left(\frac{\partial^2 u}{\partial x^2}\Big|_j + \frac{2(\Delta x)^2}{4!}\frac{\partial^4 u}{\partial x^4}\Big|_j + \cdots\right). \quad (68)$$

So

$$\ell_\Delta(\hat{u}_j) = 2u_j^3 + u_j + \frac{(\Delta x)^2}{hs}\left(\frac{\partial^2 u}{\partial x^2}\Big|_j + E(\hat{u})\right), \quad (69)$$

where

$$E(\hat{u}) = \frac{2(\Delta x)^2}{4!}\frac{\partial^4 u}{\partial x^4} + \frac{2(\Delta x)^4}{6!}\frac{\partial^6 u}{\partial x^6} + \cdots. \quad (70)$$

In the limit as $\Delta x \to 0$ the finite difference formula 70 is equivalent to the Eq. 10 So; the 61 equation is consistence too.

The positivity and boundary conditions

It can be shown that the Eq. 61 preserve the positivity and boundary conditions i.e, if

$$0 \le u_m \le 1 \to 0 \le u_{m+1} \le 1, \quad (71)$$

Because, we have

$$0 \le u_m \le 1 \to 0 \le (2-hs)u_m \le 2 - hs; \quad (72)$$

$$\begin{array}{l} 0 \le u_{m-1} \le 1 \\ 0 \le u_m^2 \le 1 \end{array} \to 0 \le u_{m-1}(hsu_m^2 - 1) \le hs - 1; \quad (73)$$

and

$$1 \le 1 + 3u_m^2 hs \le 3hs; \quad (74)$$

from 71,74 we have:

$$0 \le u_{m+1} \le \frac{1}{3hs} \le 1. \quad (75)$$

The standard finite difference scheme

In this section we construct (SFD) scheme by consider the following approximation of the derivative for solving the Eq. 10.

$$\frac{\partial^2 u}{\partial x^2}\Big|_j = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} - \frac{(\Delta x)^2}{12}\frac{\partial^4 u}{\partial x^4}\Big|_j + o(\Delta x)^4, \quad (76)$$

By replacing this term in Eq. 10, and omitting term of $\{o(\Delta x^2)\}$, we obtain

$$u_{i+1} = u_i(2-h^2) - u_{i-1} - 2h^2u_i^3, \quad (77)$$

Where $h = \Delta x$.

Clearly, the total of the truncation error, in using the 77 instead of 10 is:

$$E\{u_i\} = \left[-\frac{(\Delta x)^2}{12}\frac{\partial^4 u}{\partial t^4}\right] + o(\Delta x^4).$$

NUMERICAL TEST

In this Section we use two numerical schemes that were introduced, in Sections The first NSFD scheme and the second NSFD scheme for solving the nonlinear partial differential equation Schrodinger 1.

The accuracy of our proposed numerical methods is measured by computing the difference of the analytic and the numerical solutions at some points.

Then we compare the error of these two NSFD and SFD schemes at $t = 6$ and some Δx.

In this example, the analytical solution of the Eq.1 is (Wazwaz, 2010)

$$q(x,t) = e^{i(x+t)}, \quad (78)$$

The numerical solution for this equation is:

$$q(x,t) = u(x)e^{-it} \quad (79)$$

That $u(x)$ is the numerical solution of the ODE 10, with initial and boundary values,

$$u_1 = u(0) = e^{2it}$$
$$u_x(0) = e^{2it} \quad (80)$$

In the numerical schemes 40 and 61, when $m = 0$ we need u_{-1}, that is calculated as follows:

$$u_x(0) = \frac{u_1 - u_{-1}}{\Delta x} = e^{2it} \rightarrow \frac{u_1 - u_{-1}}{\Delta x} = u_1 \rightarrow u_{-1} = u_1(1 - \Delta x) \approx u_1 e^{i\Delta x}, \quad (81)$$

So

$$u_1 = e^{i(2t+\Delta x)}. \quad (82)$$

We use the first NSFD and second NSFD schemes at $t = 6$, by $\Delta x = 0.1$ and $\Delta x = 0.05, 0.02$. Then compare these methods by SFD scheme in same points by calculating the errors. The results are shown in the Tale 1 and the plots of errors are in Fig.1-Fig.6.Note that the Fig.4-Fig.6, are at polar coordinates.

Table 1: Shows the error of two NSFD schemes and SFD schemes for equation (1-10) a t =6 , Δx=0.02,0.05,0.1

	Error of NSFD 1	Error of NSFD 2	Error of SFD
$t = 6$ $\Delta x = 0.1$	0	0	0
	1.1444e – 16	1.1444e - 16	1.1444e - 16
	0.0199	0.0174	0.020
	0.508	0.0503	0.0584
	0.1100	0.0952	0.1118
$t = 6$ $\Delta x = 0.05$	0	0	0
	8.9681e - 16	8.9681e - 16	8.9681e - 16
	0.0050	0.0047	0.0050

	0.0149	0.0139	0.0149
	0.0249	0.0257	0.0295
$t = 6$ $\Delta x = 0.02$	0	0	0
	1.2413e - 16	1.2413e - 16	1.2413e – 16
	8.0001e – 04	7.8039e - 04	8.0001e - 04
	0.0024	0.0023	0.0024
	0.0048	0.0047	0.0048

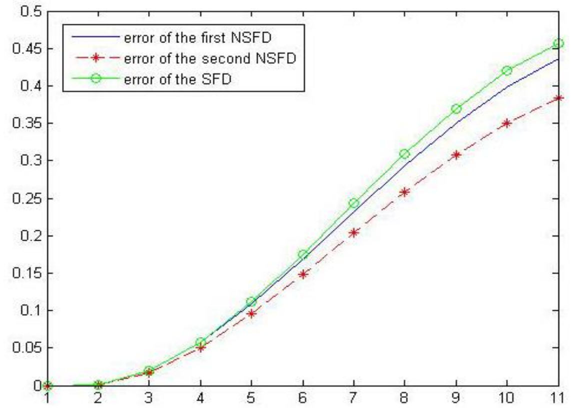

Fig 1. The plot error of two NSFD schemes and SFD scheme for equation (1-10) at t=6, $\Delta x = 0.1$

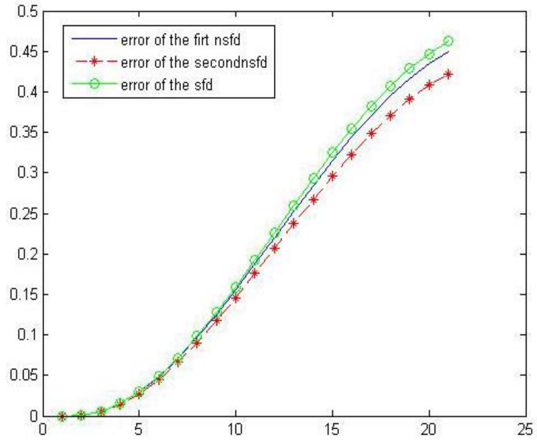

Fig 2. The plot error of two NSFD schemes and SFD scheme for equation (1-10) at t=6, $\Delta x = 0.05$

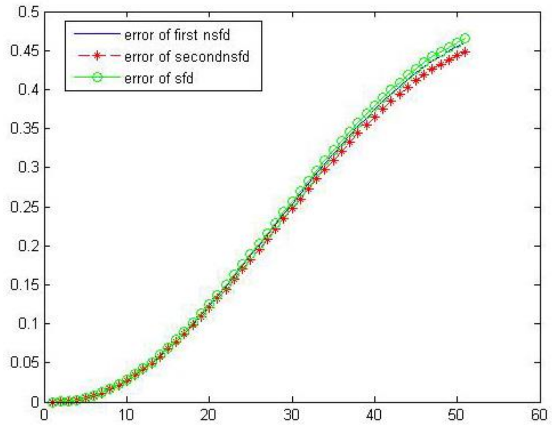

Fig. 3. The plot error of two NSFD schemes and SFD scheme for equation (1-10) at t=6, $\Delta x = 0.02$

From the above results we can see that the error of second NSFD scheme is lower than the other method, so this method works better than the other methods.

The following graphs are the graphs of the exact solution and numerical solutions that are obtained by the previous methods at $t = 2$ and $\Delta \boldsymbol{x} = \mathbf{0.02}$.

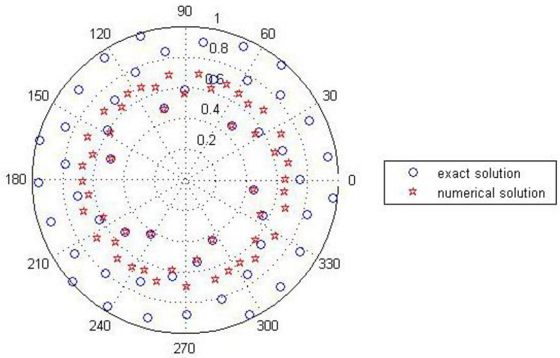

Fig. 4. The plot of exact and numerical solution by the first NSFD method at $t = 2$, $\Delta x = 0.02$

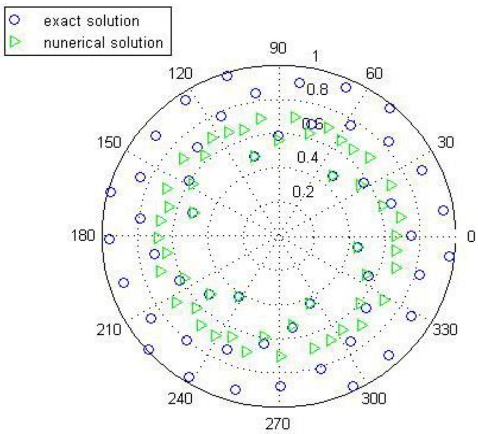

Fig. 5. The plot of exact and numerical solution by second NSFD method at $t = 2$, $\Delta x = 0.02$

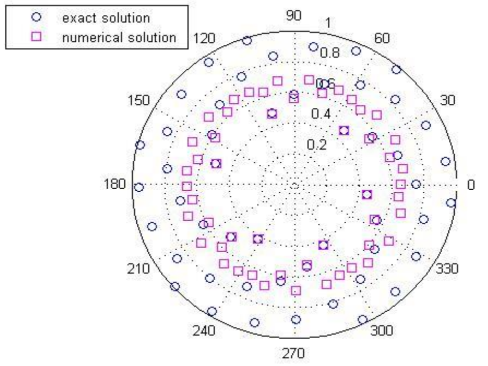

Fig. 6. The plot of exact and numerical solution by the SFD method at $t = 2$, $\Delta x = 0.02$

By attention to these graphs we can see that in Fig. 5,the number of points that exact and numerical solutions are coincident together are more than the others.

CONCLUSION

The nonlinear Schrodinger equation is one of the most important equations in quantum mechanics, chemistry and physics science. In this work, we studied two nonstandard finite difference (NSFD) and one explicit finite-difference (SFD) schemes to approximate the solution of this equation.

In Section 1, the nonlinear Schrodinger equation was converted into an ODE by a transformation. In Section 2, we have presented two NSFD schemes for reduced ODE and have shown that both schemes are conditionally stable and consistence of order $O\{(\Delta x)^4\}$, and both have the positivity and boundary property. Also in the same section, we constructed to SFD scheme with total error of order $O\{(\Delta x)^4\}$ to approximate the solution of this equation. In Section 6, we solved the Schrodinger equation with these numerical methods at some points.

Then we compared these solutions to the exact solution of this equation that is given by the Adomian method(Wazwaz,2010). By drawing the errors of these methods at $t = 6$ and divers Δx, it is observed that the plot error of the second NSFD method is lower than the others.

It shows that the second numerical scheme works better than the other one and SFD method. Finally, we have plotted the exact and numerical solutions of the NSFD and SFD methods at $t = 2$ and $\Delta x = 0.02$.

The results also show that the number of calculated points which are exactly the same as corresponding exact solutions, are more in the second NSFD method.

The errors of the second NSFD method were smallest than the others in all of the points. So this method is better than the others method.

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