Optimization
Iranian Journal of Optimization
Volume 14, Issue 1, 2022, 39-49
Research Paper

Online version is available on: www.ijo.rasht.iau.ir

Islamic Azad University
Rasht Branch
ISSN: 2588-5723
E-ISSN:2008-5427

An Entropy Based Shapley Value for Ranking in Data Envelopment Analysis

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Revise Date 12 March 2022
Accept Date: 06 July 2022

Keywords:
Data envelopment analysis
Ranking
Cooperative game
Shapley value
Entropy

Abstract

In traditional DEA, DMUs are divided into Efficient and inefficient, but the score of all efficient units are equal to one and there is no discrimination between them. Thus many ranking methods are proposed to increase discrimination power. This paper proposes an integrated framework of cooperative games and entropy to rank efficient units by considering efficient units as players in a cooperative game, A subset of these players is defined as the coalition of S. The sum of the efficiency of inefficient DMUs with respect to the frontier of production possibility set contain inefficient DMUs and the member of coalition S is defined as the characteristic function of the coalition S, which is used to determine the marginal effect of efficient DMUs. Then, a new Shapley Value resulted from aggregating the marginal effects of efficient DMUs weighted by Shannon entropy is used for ranking efficient DMUs. For the first time, we use the entropy to create a Shapley value for calculating the rank of efficient units.

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INTRODUCTION

Data Envelopment Analysis (DEA) is a scientific non-parametric technique for evaluating Decision-Making Units(DMUs). This method evaluates the performance of homogenous DMUs that use some inputs to produce some outputs. Originally, Charnes et al. [6] (Charnes et al., 1978) introduced a linear programing method for evaluating the efficiency of DMUs without recognizing the production function, as CCR model. Later, Banker et al. [3], by adding a constraint corresponds to variable returns to scale, turned it into the BCC Model. In conventional DEA models, units are divided into efficient and inefficient, in which each efficient DMU has value one, without distinction among efficient DMUs. In order to rectify this problem, some models for ranking DMUs have been proposed [1].

CIn the new DEA literature, game theory and entropy are concepts to evaluate performance and ranking. situation in which there is a set of competitive players. Game theory can be used in such a frame work with conflicting interests.

The players of the game can act in the competitive and non-competitive circumstances. Considering the inherent competitive nature in the process of evaluating the performance of the DEA, attempts are made to link DEA models to game theory. Cooperative games have a significant share in these combined studies. Nash bargaining and Shapley's value has the largest share in the methods obtaining cooperative game solutions. Wu et al. [29] used the Nash bargaining game to find a common weight vector and performance score. Li and Liang [15] ranked the importance of input and output variables using the Shapley value as a solution of a cooperative game. Wu et al.[31] also introduced a way to choose the best competitor using Shapley value. Wu et al. [13] used the Nucleolus solution and the Shapley value [28] in the cooperative game to determine the final weights at cross efficiency performance. Nakabayashi and Tone [19] applied the solution of Shapley and Nucleus to distribute a prize fairly between the players of a cooperative game. Lee et al. [16] and Hinujosa et al. [10] utilizing cooperative game, ranked efficient units in a common framework. An et al. [2] approach fixed-cost allocation of two-stage systems by considering cooperation among DMUs. Do et al. [8] for evaluating the performance of two-stage networks and Zhou et al. [34] for decomposing the performance in a centralized two-stage model applied the bargaining game. Mahmoudi et al. [18] proposed a DEA-Game model to evaluate the performance of network structure in which initially divided into several subnets. In this model, input variables are categorized to measure the efficiency of subnets in each input group. And then the network efficiency is calculated by collecting the efficiency scores of the subnets in each group. A marginal probability transformation method based on Shapley value proposed to rectify the problems in transformation of basic probability assignment into a probability distribution function [12].

Since the introduction of entropy as a tool to measure, by Claude Shannon [22], Shannon's concept of entropy has been used in different disciplines. It is used as a measure of dispersal of trips, the amount of disorder of a system, the degree of randomness in the event, a measure of fuzziness, a weighting tool. In DEA studies, Shannon entropy is used as an acceptable weighting tool based on the degree of diversification for aggregation. By examining the combined studies of DEA and entropy, we arrive at the application of Shannon entropy in the fields of ranking, cross-evaluation, common weights, discrimination, namely. Soleimani-Damaneh and Zarepisheh [24] utilized the Shannon's entropy for combining the efficiency results of different types of DEA models. On the other hand, Yang et al. [33] used bootstrap method to measure influential DMU in the data envelopment analysis. To prevent making unrealistic assumptions about the true distribution they estimate the basic distribution for efficiency scores. Moreover, Wu et al. [30] applied the Shannon entropy instead of averaging to determine the weights for ultimate cross efficiency scores. Hsiao et al. [11] performed evaluation in DEA by introducing entropy in the Russell and slack-based measure. Qi, and Guo [20] combined the DEA common weights with Shannon's entropy. Xie et al.[32] used Shannon's entropy to improve the discriminatory power of DEA. Wang et al. [26] presented an entropy cross-efficiency model for decision making units with interval data. Storto [17] With the help of DEA cross-efficiency and Shannon's entropy method, provided ecological efficiency based ranking of cities. Later Ghosh et al.[9] quantify the relative performance of aerosols on photovoltaic cells by combining both Shannon's entropy and DEA. Rotela et al. [21] proposed a new approach for portfolio optimization by using Entropic data envelopment analysis. Çakır[5] proposed the imprecise Shannon's entropy method and the acceptability index for resource allocation, an interval inverse DEA model is performed. Lee [14] combined cross efficiency scores obtained from different evaluation models, using Shannon's entropy. Si and Ma [23] expressed DEA cross-efficiency ranking method based on grey correlation degree and relative entropy. Su and Lu [25] defined Cross-Efficiency based on entropy with the technology of variable returns to scale.

Although DEA and game theory, as well as DEA and entropy, have been used to rank units before, a combined approach of all three DEA, game theory, and entropy has not yet been used for ranking. Also, Shannon entropy has not been used to weigh the Shapley value in non-DEA studies. The initial defined Shapley value [7] is based on the aggregation with the same weight of all the marginal effects of a player in different coalitions, or based on its share in various entrance to the permutations in grand coalition. In this paper, it is suggested that Shannon entropy is used as the required weight in the definition of Shapley value. In fact, instead of using the usual average to aggregate marginal shares in Shapley value, we use the entropy of marginal shares of each player for this purpose. This study intends to offer a new approach for ranking efficient DMUs in DEA, based on the solution concept of Shapley value in cooperative game weighted by Shannon's entropy. This research includes the following: First efficient DMUs are identified as players in a cooperative game. A subset of these players is defined as coalition S. Then the sum of the efficiency of inefficient DMUs with respect to the production possibility set (PPS) which is made of inefficient DMUs and the member of coalition S, is defined as the characteristic function of the coalition S, which is used to determine the marginal effect of efficient DMUs in various coalitions. Ultimately, the Shannon's entropy and Shapley Value is utilized to determine the solution of the cooperative game, which in turn is used to rank efficient DMUs.

The rest of this research includes: Section 2 encompassing a brief introduction of DEA, game theory and Shannon entropy. Section 3, represents the proposed method. Section 4 explains the proposed model by giving a real data example. Section 5 presents the conclusions of the research and suggestions.

PRELIMINARIES

Data envelopment analysis

Assume that there are n independent Decision Making Units (DMUs). Each DMU_j ($j \in \{1,\dots,n\}$) uses m inputs $x_{ij}, i \in \{1,\dots,m\}$ to produce s output y_{rj}، $r \in \{1,\dots,s\}$.Production Possibility Set (PPS) with the assumption of constant return to scale (CCR) is a set of (x,y) in which the input of x produce the output of y.

$$T_c = \{(X,Y) \mid X \ge \sum_{j=1}^{n} \lambda_j X_j, Y \le \sum_{j=1}^{n} \lambda_j Y_j, \lambda_j \ge 0, j = 0,\dots,n\} \qquad (1)$$

By adding the constraint $\sum_{j=1}^{n} \lambda_j = 1$ on the PPS, we arrive at PPS with variable return to scale situation (T_v). The efficiency of DMUs can be estimated by using the CCR model as follows:

$$\theta_o = \min \theta - \varepsilon \left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \right)$$
$$s.t \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{io} \quad , i = 1,\dots,m$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro} \quad , r = 1,\dots,s$$
$$\lambda_j \geq 0 \quad , j = 1,\dots,n \qquad (2)$$
$$s_i^- \geq 0 \quad , i = 1,\dots,m$$
$$s_r^+ \geq 0 \quad , r = 1,\dots,s$$
$$\theta \text{ free.}$$

θ_o is the efficiency of DMU_o when the PPS is made by $DMU_1, \dots, DMU_n$. If $\theta_o = 1 s_r^+ = 0, s_i^- = 0$, then DMU_o is CCR efficient and which is also at the frontier of T_c .If $\theta_o < 1$, Thus DMU_o is CCR inefficient and inside of T_c. Now by adding the constraint $\sum_{j=1}^{n} \lambda_j = 1$ to the CCR model, The BCC model (input orientation) is obtained. From the considerations above, the efficiency of all efficient DMUs is one, so there is no distinction among efficient DMUs.

Cooperative games and Shapley value

Consider a competitive situation in which there is a set of players. The players of the game can be competed in two ways:

1) Non-cooperative game: the players play individually and payoff is personal. In this game, we are interested in knowing what strategy each player chooses to maximize his payoff.

Cooperative game: in this game, it is predicted that players create a coalition to maximize their payoff. The cooperative games are identified by players and a characteristic function as *<N,C(S)>*. Suppose that coalition S is a subset of the players, the value of characteristic function C(S) as the payoff of players, is an achievement that the members of coalition S are sure will get, if they cooperate. It is clear that whatever the players of coalition achieve, should be fairly divided among them.

Let's assume imputation vector $X=\{x_1, x_2, \dots, x_n\}$ be the prize of players e.g., $x_i$is the prize of the player i. The imputation vector should be a part of two conditions:

1. C(N)=$\sum_{i=1}^{n} x_i$Group rationality
2. $x_i \geq C(\{i\})$Individual rationality

There are different solutions such as Kernel, Stable set, Core, Nucleolus and Shapley value to find a value of the imputation vector. Among the mentioned solutions Shapley Value is more understandable and easy to interpret.

This method has significant application in dividing the prize of the cooperative game.

In the Shapley Value solution, the prize of the player i is computed as follows:

$$x_i = \sum_{\forall S \subseteq N \text{which } i \notin S} \frac{(s-1)!(n-s)!}{n!} \left(C(S) - C(S \cup \{i\}) \right) \qquad (3)$$

Shannon entropy

Entropy is a well-known method to obtain the weights for amultiple attribute decision making (MADM) problem (Table 1), especially when obtaining a suitable weight based on the preferences and the experiments of the DMU is not possible.

Table 1: Structure of the Alternative Performance

	Criterion 1	Criterion 2	…	Criterion n
Alternative 1	X_{11}	X_{12}	…	X_{1n}
Alternative 2	X_{21}	X_{22}	…	X_{2n}
⋮	⋮	⋮	⋮	⋮
Alternative m	X_{m1}	X_{m2}	…	X_{mn}
	W_1	W_2	…	W_n

The procedure of Shannon's entropy can be expressed in the series of the following steps:

S1: Normalize the decision matrix.

Set

$$n_{ij} = \frac{x_{ij}}{\sum_{j=1}^{m} x_{ij}}, \quad j=1,...,m, \quad i=1,...,n \qquad (4)$$

The row data are normalized to eliminate anomalies with different measurement units and scales. This process transforms different scales and units among various criteria into common measurable units to allow for comparisons of different criteria.

S2: Compute entropy e_i as

$$e_i=-e_0\sum_{j=1}^{m}n_{ij}.\ln n_{ij},\ i=1,...,n \qquad (5)$$

,where e_0 is the entropy constant which is equal to $(\ln n)^{-1}$, also $n_{ij}.\ln n_{ij}$ is equals as 0 if $n_{ij}=0$.

S3: set

$$d_i=1-e_i,\ i=1,...,n \qquad (6)$$

as the degree of diversification.

S4: Set

$$w_i=\frac{d_i}{\sum_{s=1}^{n}d_s},\ i=1,...,n \qquad (7)$$

as the degree of importance of attribute i.

We should describe some basic definitions, terminology as well as a brief description of literature.

USING SHANO'S ENTROPY AND SHAPLEY VALUE FOR RANKING EFFICIENT UNITS IN DEA

In this section, we describe the combined approach of game theory and entropy for ranking efficient units in DEA. The proposed method entails distinguishing the efficient and inefficient DMUs by utilizing an appropriate model (CCR or BCC). The efficiency is computed and DMUs are divided to two classes, efficient and inefficient.

$$E=\{DMU_j|\theta_j=1\} \qquad (8)$$
$$N=\{DMU_j\,|\,\theta_j<1\}.$$

The current method defines the characteristic function of a coalition as follows:

If S is a subset of efficient DMUs, then the characteristics function of the coalition S, in which its value is the payoff of members of coalition S, is defined as follows:

$$C(S)=\sum_{t\in N}\theta_t^s. \qquad (9)$$

C(S) is the sum of efficiency of inefficient DMUs, where PPS is constructed of all inefficient DMUs and efficient DMUs of coalition S and θ_t^s is defined as follow:

$$\begin{aligned}
&\theta_t^s=\min\theta\\
&s.t.\ \sum_{j\in(N\cup S)}\lambda_j x_{ij}+s_i^-=\theta x_{it}, && i=1,\dots,m\\
&\sum_{j\in(N\cup S)}\lambda_j y_{rj}-s_r^+=y_{rt}, && r=1,\dots,s\\
&\lambda_j\ge 0, && j=1,\dots,n \qquad (10)\\
&s_i^-\ge 0, && i=1,\dots,m\\
&s_r^+\ge 0, && r=1,\dots,s\\
&\theta\ free.
\end{aligned}$$

N is a set of inefficient DMUs and S is the subset of efficient DMUs. Moreover, the payoff of the efficient members of the coalition S alongside the efficient unit of k where $k\notin S$ is as follows:

$$C(S\cup\{k\})=\sum_{t\in N}\theta_t^{s\cup\{k\}}. \qquad (11)$$

$\theta_t^{s\cup\{k\}}$ is the efficiency of the inefficient unit of t where PPS is made by inefficient DMUs, the whole number of efficient DMUs acquired the coalition S and the efficient unit of k, as follows:

$$\begin{aligned}
&\theta_t^{s\cup\{k\}}=\min\theta\\
&s.t.\ \sum_{j\in(N\cup S\cup\{k\})}\lambda_j x_{ij}+s_i^-=\theta x_{it}, && i=1,\dots,m\\
&\sum_{j\in(N\cup S\cup\{k\})}\lambda_j y_{rj}-s_r^+=y_{rt}, && r=1,\dots,s\\
&\lambda_j\ge 0, && j=1,\dots,n \qquad (12)\\
&s_i^-\ge 0, && i=1,\dots,m\\
&s_r^+\ge 0, && r=1,\dots,s\\
&\theta\ free.
\end{aligned}$$

If S is a grand coalition, then the C(S) is the sum of efficiency of inefficient units when PPS is formed by all the DMU's. If S is empty coalition, then C(S) is the sum of efficiency of inefficient units where PPS is made of the inefficient DMUs.

In the next step, the marginal effect of an efficient DMU on the efficiency of the inefficient DMUs is determined. Marginal effect of the efficient unit of k in changing the sum of efficiency of inefficient DMUs is defined as follows:

To arrive at a marginal effect of a DMU in coalition S (S is a subset of efficient DMUs): First, the sum of efficiency of inefficient units in PPS which is made of inefficient units and the members of coalition S ($C(S)$) and the sum of efficiency of inefficient units in PPS that is made of inefficient units and the members of coalition S and the $DMU_k C(S \cup \{k\})$is measured. The difference between $C(S)$ and $C(S \cup \{k\})$ is defined as

$$EP^S(k) = C(S) - C(S \cup \{k\}) \qquad (13)$$

By adding a new DMU, the PPS expands or stays unchanged so, the efficiency of inefficient units becomes smaller or remains unchanged, from which it follows that the defined marginal effect is always greater than or equal to zero. As much as the marginal effect of an efficient unit in a coalition is greater, the DMU has an important role in the coalition.

It is reasonable to evaluate DMU_k contribution in the entire game as the weighted aggregation of marginal contribution to coalitions which include k. From the above considerations, the proportional Shapley value which is a aggregation of marginal effects weighted by Shannon's entropy can be considered.

In this paper we compute all orderings (permutations) of players and evaluate their marginal contribution to the coalition. We also illustrate the importance of each permutations by using the weights based on the entropy. For this purpose, we should consider each permutation as a criterion in an MADM problem. Thus, we have the proportional Shapley value of each player as the entropy based aggregation of the marginal contributions. Finally the Shannon's based Shapley value $\varphi_k$as a solution of cooperative game for ranking efficient DMUs is applied.

$$\varphi_k = \sum_{\forall i} ((\text{Entropy weight of per i}) * (\text{Marginal effect of } DMU_k \text{ in per i})) \qquad (14)$$

The greater the value of φ_k , the higher the rank of DMU_k.

The ranking procedure based on Shapley value and Shannon's entropy consists of following steps:

Step 1: Taking multiple inputs and multiple outputs. estimate the efficiency of DMUs by using a DEA model (CCR model (2), BCC or another DEA model).

Step 2: Classification DMUs in efficient set E and inefficient set N.

Step 3: Defining the characteristic function of coalition S by (9)

Step 4: Finding the marginal effect of efficient DMU in coalition S using (9)-(13)

Step 5: Constituting all permutations.

Step 6: Normalizing the permutation table by (4)

Step 7: Computation the entropy for each permutation by (5)

Step 8: Computing the degree of diversification (6)

Step 9: Obtaining the importance of each permutation by (7)

Step 10: Calculating the weighted Shapley value for each efficient DMU by (14).

Step 11: Ranking Efficient DMUs by the values of step 10. The greater the value, the higher the rank.

To illustrate the proposed method, consider the following numerical example. Suppose that there is a collection of seven DMUs with one input and one output (Table 2). According to BCC model, the DMUs of 1, 2, 3 and 4 are efficient.

Table2: Input, output and the efficiency of DMUs

	Input	Output	θ_{BCC}
DMU_1	1	1	1
DMU_2	2	2	1
DMU_3	4	4	1
DMU_4	6	5	1
DMU_5	4	2	0.50
DMU_6	7	3	0.43
DMU_7	6	0.5	0.17

Considering this four DMUs as players of a cooperative game, to rank the efficient DMUs, the proposed method in this paper, which is based on Shannon entropy and Shapley value, is used.

Thus, the Shapley value of each DMU has been computed by utilizing its marginal effect. In the first column of Table 4 all possible coalitions of efficient units were collected. In the columns 2-5, the marginal effect of DMUs in different coalitions is given. Also, in Table 5 using PPS of inefficient DMUs and efficient DMUs of coalition S, the efficiency of inefficient units are shown, clearly the sum of foregoing is C(S). Similarly, the efficiency of inefficient units was computed with PPS of inefficient DMUs and efficient DMUs of coalition S and DMU_j, the sum of foregoing is C(SU{j}). It is defined C(x) - c(SU{j}) as the marginal effect of efficient, if the difference between C(S) and C(SU{j}) added to the coalition S.

If DMU_3 is added to S={} (Fourth column, second row, table 3), then in the obtained PPS which is made by inefficient units and S={}, both DMU_5 and DMU_6 are efficient and the DMU_7 with the efficiency $\theta_7 = 0.6667$ is inefficient.

By adding DMU_3 to the coalition S={}, the PPS which is also made by inefficient units and S={}, the DMU_3 and DMU_5 are remained efficient, the DMU_6 with the amount of efficiency $\theta_6 = 0.5714$ becomes inefficient and the amount of the efficiency of DMU_7 changes to $\theta_7 = 0.6666$. Therefore, the marginal effect of adding DMU_3 to the coalition S={} is as follows:

$$EP^{\{\}}(3)$$
$$= (1+1+0.6667) - (1+0.5714+0.6666)$$
$$= 0.4287$$

Now, in the Table 3 if DMU_3 is added to S={1}, (Fourth column, third row) in the resulting PPS which is made by inefficient units and S={1}, both DMU_5 and DMU_6 are efficient and DMU_7 with the efficiency of $\theta_7 = 0.1667$ is inefficient.

On the hand, in the obtained PPS making by inefficient units, S={1} and DMU_3, it is shown that DMU_5 with the efficiency value of $\theta_5 = 0.5000$ and DMU_6 with amount $\theta_6 = 0.4286$ are inefficient, but the efficiency of DMU_7 with the value $\theta_7 = 0.1667$ is remained. Hence, the marginal effect of adding DMU_3 to the coalition S={1} is given as follows:

$$EP^{\{1\}}(3)$$
$$= (1+1+0.1667) - (0.5000+0.4286+0.1667)$$
$$= 1.0714$$

In the Table 3, Fourth column, seventh row, if DMU_3 is added to S={1,2}, then in the resulting PPS which is made by the inefficient units and S={1,2}, it is seen that the DMU_6 is efficient, DMU_5 and DMU_7 are inefficient with the efficiency values of $\theta_5 = 0.5$ and $\theta_7 = 0.1667$, respectively.

Now, by adding the DMU_3 to the coalition S={1,2}, the resulting PPS made by inefficient units, S={1,2} and DMU_3, the efficiency of DMU_5 remains at $\theta_5 = 0.5$, the efficiency of DMU_7 remains at $\theta_7 = 0.1667$ and finally the DMU_6 with the amount value of efficiency $\theta_6 = 0.4286$ becomes inefficient. Therefore, the marginal effect of adding DMU_3 to the coalition S={1,2} is computed as follows:

$$EP^{\{1,2\}}(3)$$
$$= (0.5+1+0.1667) - (0.5000+0.4286+0.1667)$$
$$= 0.5714$$

Table3: The marginal effect of efficient DMUs in all possible coalitions

Coalition	DMU_1	DMU_2	DMU_3	DMU_4
{}	0.5	0.8334	0.4287	0.3333
{1}	0	0.5	1.0174	0.9375
{2}	0.1666	0	0.5719	0.5238
{3}	1.1427	0.9761	0	0
{4}	1.1042	1.0239	0.0954	0
{1,2}	0	0	0.5714	0.5238
{1,3}	0	0	0	0

{1,4}	0	0.0863	0.1339	0
{2,3}	0.1666	0	0	0
{2,4}	0.1666	0	0.0476	0
{3,4}	1.1427	0.9761	0	0
{1,2,3}	0	0	0	0
{1,2,4}	0	0	0.0476	0
{1,3,4}	0	0	0	0
{2,3,4}	0.1666	0	0	0

Now we consider all the orderings including all the efficient players. These data are shown in Table 4. Each efficient unit in the columns 3 to 6, represents the marginal contribution of that efficient unit to its entry the coalition. To aggregate the values of each column as the Shapley value, we first obtain an entropy weight for any permutation in each row. The values of entropy, the degree of diversification and degree of importance of permutation i are all listed as the weights for aggregation in the columns 7, 8 and 9 in the Table 4. Using Equation (14), the Shapley value is obtained for each of the efficient units. The rank corresponding to each unit is also given in the last line.

Table 4: Permutations, marginal effects, entropy, weighted Shapley value, and ranking

	Permutation	A	B	C	D	Entropy	Degree of diversification	Weights
1	A←B←C←D	0.5	0.5	0.57	0	0.791067	0.208933	0.023561
2	A←B←D←C	0.5	0.5	0.05	0.52	0.868909	0.131091	0.014783
3	A←C←B←D	0.5	0	1.07	0	0.451356	0.548644	0.061869
4	A←C←D←B	0.5	0	1.07	0	0.451356	0.548644	0.061869
5	A←D←B←C	0.5	0.08	0.05	0.94	0.672999	0.327001	0.036875
6	A←D←C←B	0.5	0	0.13	0.94	0.633202	0.366798	0.041363
7	B←A←C←D	0.16	0.84	0.57	0	0.674605	0.325395	0.036694
8	B←A←D←C	0.16	0.84	0.05	0.52	0.752447	0.247553	0.027916
9	B←C←A←D	0.16	0.84	0.57	0	0.674605	0.325395	0.036694
10	B←C←D←A	0.16	0.84	0.57	0	0.674605	0.325395	0.036694
11	B←D←A←C	0.16	0.84	0.05	0.52	0.752447	0.247553	0.027916
12	B←D←C←A	0.16	0.84	0.05	0.52	0.752447	0.247553	0.027916
13	C←A←B←D	1.14	0	0.43	0	0.423492	0.576508	0.065011
14	C←A←D←B	1.14	0	0.43	0	0.423492	0.576508	0.065011
15	C←B←A←D	0.16	0.98	0.43	0	0.635938	0.364062	0.041054
16	C←B←D←A	0.16	0.98	0.43	0	0.635938	0.364062	0.041054
17	C←D←A←B	1.14	0	0.43	0	0.423492	0.576508	0.065011
18	C←D←B←A	0.16	0.98	0.43	0	0.635938	0.364062	0.041054
19	D←A←B←C	1.11	0.08	0.05	0.33	0.601913	0.398087	0.044891
20	D←A←C←B	1.11	0	0.13	0.33	0.562116	0.437884	0.049379
21	D←B←A←C	0.16	1.03	0.05	0.33	0.683029	0.316971	0.035744
22	D←B←C←A	0.16	1.03	0.05	0.33	0.683029	0.316971	0.035744
23	D←C←A←B	1.14	0	0.1	0.33	0.530642	0.469358	0.052928
24	D←C←B←A	0.16	0.98	0.1	0.33	0.743088	0.256912	0.028971
Weighted Shapley value		0.574	0.411	0.377	0.206	-	-	-
Rank		***1***	***2***	***3***	***4***	-	-	-

REAL DATA EXAMPLE

In this section, we use the proposed method for ranking 18 DMUs. The data are extracted from the reference [27] are shown in Table 5.

Table5: Coalitions and the marginal effects

Coalitions	DMU 2	DMU 6	DMU 10
{2}	0.62	0	0
{6}	0	1.14	0
{10}	0	0	0.84
{2,6}	2.23	2.75	0
{2,10}	1.46	0	1.68
{6,10}	0	0.55	0.25
{2,6,10}	2.41	1.5	0.43

The efficiency values obtained from the implementation of the CCR model are listed in column 7. Accordingly, units 2, 6, and 10 are efficient units and the rest are inefficient. To rank the efficient units, according to the proposed method, we find the marginal shares of the units in 6 permutations including these efficient units in Table 6. By forming all the permutations, we find the weighted result of the entropy method or the weighted Shapley value. The results show that units 2, 6, and 10 have ranks of 1, 2 and 3, respectively.

Table 6: Permutations, marginal effects, entropy, weighted Shapley value, and ranking for real data example

DMU	Coalition	DMU 2	DMU 6	DMU 10	Entropy	Degree of diversification	Weights
1	2←6←10	0.62	2.75	0.43	0.560068	0.439932	0.217976
2	2←10←6	0.62	1.5	1.68	0.738359	0.261641	0.129637
3	6←2←10	2.23	1.14	0.43	0.664034	0.335966	0.166464
4	6←10←2	2.41	1.14	0.25	0.598017	0.401983	0.199173
5	10←2←6	1.46	1.5	0.84	0.770465	0.229535	0.113729
6	10←6←2	2.41	0.55	0.84	0.650802	0.349198	0.17302
Weighted Shapley		1.64976	1.47647	0.67376			
Rank		1	2	3			

CONCLUSION

In this research, a method based on the concept of game theory for ranking was introduced. In the mentioned method, different PPSs including all inefficient units and different coalitions of efficient units were constructed. Development of the proposed method for ranking groups including units, use of entropy for weighting of other cooperative game methods, definition of marginal effect of efficient units based on theirs impact on the entropy of efficiencies of inefficient units in different PPS, use of interval and fuzzy entropy for ranking efficient units in the presence of imprecise data can be considered for future studies.

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