Optimization
Iranian Journal of Optimization
Volume 13, Issue 4, 2021, 231-239
Research Paper

Online version is available on: www.ijo.rasht.iau.ir

Islamic Azad University
Rasht Branch
ISSN: 2588-5723
E-ISSN:2008-5427

Stability and Iterative Procedures for Quadrupled Fixed Point in Partially Ordered Metric Spaces

Aniki Samuel Adamariko*

Department of Mathematics, Faculty of Science, Confluence University of Science and Technology, Osara, Kogi State, Nigeria

Revise Date: 13 December 2021
Accept Date: 16 April 2022

Keywords:
Stability
Metric Space
Quadrupled Fixed Point
Mixed Monotone
Iteration Procedure

Abstract

The concept of fixed point is of great interest in mathematics as well as in many areas of applied sciences. Previously, the question of fixed points in partially ordered metric spaces had been studied, and quadrupled fixed point theorem is a continuation of the tripled fixed point theorem. This work presents the stability for quadrupled fixed point iterative procedure with the aid of some mathematical analysis properties, and establishes results for mixed monotone mappings which satisfy its contractive-type conditions. The results of this work extend some of the results in literature.

***Correspondence E-mail**: smlaniki@yahoo.com

INTRODUCTION

Metrical fixed point theory has significantly revolutionized the approaches of mathematics through the Banach contraction concept to sciences and its applications. This concept is a classical and powerful tool in nonlinear analysis because of its very useful structure. Coupled fixed point theorems have been given in different metric spaces. In the case of fixed points of an operator $T: X^2 \to X,$ its stability for fixed point iterative procedures was first studied by Ostrowski (1967) in the case of Banach contraction mappings and this subject was later developed for certain contractive definitions by several authors, Rhoades (1990, 1993), Osilike (1995, 1996), Jachymski (1997), Berinde (2003), Imoru and Olatinwo (2003), Owojori (2006), Olatinwo, Owojori and Imoru (2006).

Banach Principle was applied on partially ordered complete metric spaces and starting from the results, Bhaskar and Laksmikantham (2006) extend this theory to partially ordered metric spaces and introduce the concept of coupled fixed point for mixed-monotone operators of Picard type, obtaining results involving the existence and uniqueness of the coincidence points for mixed monotone operators $T: X^2 \to X$ in the presence of a contractive condition. This concept of coupled fixed points in partially ordered metric and cone metric spaces have been studied by several authors, including Ciric and Lakshmikantham (2009), Lakshmikantham and Ciric (2009), and Sabetghadam, Masiha and Sanatpour (2009), Karapinar (2010), Choudhury and Kundu (2010), Aniki and Rauf (2019).

Berinde and Borcut (2011) obtained extensions to the concept of tripled fixed points and tripled coincidence fixed points and also obtained tripled fixed points theorems and tripled coincidence theorems for mappings in partially ordered metric spaces. Work on tripled fixed point was advanced by Abbas, Aydi and Karapinar (2011), Amini-Harandi (2012) and Kishore (2011).

Recently, Rauf and Aniki, (2020) introduced quadrupled fixed point theorems for contractive type mappings in partially ordered cauchy spaces. Also, following the series, Aniki and Rauf, (2021) established the stability theorem and results for quadrupled fixed point of contractive type single valued operators. On the other hand, by adapting the stability concept of the iterative fixed point method, Olatinwo (2012) and Timis (2014) tested the stability of the related iterative fixed point method using several contractive conditions for which the existence of a unique coupled fixed point has been demonstrated in the literature.

MATERIALS AND METHODS

Firstly, we consider some notations that will be relevant in demonstrating our main findings.
If $(X, \leq)$ is a partially ordered set and d be a metric on X such that the pair (X,d) is a complete metric space. Then,X^4 is a product space with the following partial order

$$(p,q,r,s) \leq (u,v,w,x) \Leftrightarrow u \geq p,v \leq q,w \geq r,\, x \leq s \quad \forall (p,q,r,s),(u,v,w,x) \in X^4.$$

Definition 1 *(*Rauf & Aniki, 2021). Let $(X, \leq)$ be a partially ordered set and $T: X^4 \to X$ be a mapping. We say that T has the mixed monotone property if T(u,v,w,x) is monotone nondecreasing in u and w, and monotone nonincreasing in v and x, that is, for any $\boldsymbol{u,v,w,x \in X,}$

$$u_1,u_2 \in X, u_1 \leq u_2 \Rightarrow T(u_1,v,w,x) \leq T(u_2,v,w,x),$$

$$v_1,v_2 \in X, v_1 \leq v_2 \Rightarrow T(u,v_1,w,x) \geq T(u,v_2,w,x),$$

$$w_1,w_2 \in X, w_1 \leq w_2 \Rightarrow T(u,v,w_1,x) \leq T(u,v,w_2,x),$$

and

$$x_1,x_2 \in X, x_1 \leq x_2 \Rightarrow T(u,v,w,x_1) \geq T(u,v,w,x_2).$$

Definition 2 (Rauf & Aniki, 2021). An element $(u,v,w,x) \in X^4$ is called a quadrupled fixed point of the mapping $T: X^4 \to X,$ if $T(u,v,w,x) = u, T(v,u,v,x) = v, T(w,u,v,w) = w,$ and $T(x,w,v,u) = x.$

Definition 3 (Rauf & Aniki, 2021). The mapping $T: X^4 \to X$ is said to be $(\vartheta,\kappa,\lambda,\mu)$ −contraction if and only if there exists four constants $\vartheta \geq 0, \kappa \geq 0, \lambda \geq 0, \mu \geq 0, \vartheta + \kappa + \lambda + \mu < 1$, such that for all $u,v,w,x,p,q,r,s \in X$,

$$d[T(u,v,w,x),T(p,q,r,s)] \leq \vartheta d(u,p) + \kappa d(v,q) + \lambda d(w,r) + \mu d(x,s), \quad (1)$$

Let $A,B \in M_{(m,n)}(\mathbb{R})$ be two matrices. We write $A \leq B$;if $\alpha_{ij} \leq \beta_{ij}$ for all $i = \overline{1,m}$, $j = \overline{1,n}$.

In order to prove the main stability result in this research, the next are given;

Lemma 1. Let $\{\alpha_n\}$, $\{\beta_n\}$ be sequences of nonnegative numbers and h be a constant, such that $0 \leq h < 1$ and

$$\alpha_{n+1} \leq h\alpha_n + \beta_n, \quad n \geq 0,$$

If $\lim_{n\to\infty} \beta_n = 0$, then $\lim_{n\to\infty} \alpha_n = 0$.

Also, given in the next result is the extension of Lemma 1 to vector sequences where an inequality between vectors means inequalities on its elements.

Lemma 2. Let $\{p_n\},\{q_n\},\{r_n\},\{s_n\}$ be sequences of nonnegative real numbers. Consider a matrix $A \in M_{(4,4)}(\mathbb{R})$ with nonnegative elements, such that

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \\ r_{n+1} \\ s_{n+1} \end{pmatrix} \leq A.\begin{pmatrix} p_n \\ q_n \\ r_n \\ s_n \end{pmatrix} + \begin{pmatrix} \eta_n \\ \varepsilon_n \\ \delta_n \\ \gamma_n \end{pmatrix}, \quad n \geq 0, \quad (2)$$

With

i. $\lim_{n\to\infty} A^n = 0_4$,

ii. $\sum_{k=0}^{\infty} \eta_k < \infty$, $\sum_{k=0}^{\infty} \varepsilon_k < \infty$, $\sum_{k=0}^{\infty} \delta_k < \infty$, and $\sum_{k=0}^{\infty} \gamma_k < \infty$.

If $\lim_{n\to\infty} \begin{pmatrix} \eta_n \\ \varepsilon_n \\ \delta_n \\ \gamma_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, then $\lim_{n\to\infty} \begin{pmatrix} p_n \\ q_n \\ r_n \\ s_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

Proof. For $A = 0_4 \in M_{(4,4)}$, (2) is rewritten with $n = k$ and summing the inequalities obtained for $k = 0,1,2,\ldots,n$. Then, the following is obtained if $\eta, \varepsilon, \delta, \gamma$ are nonnegative.

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \\ r_{n+1} \\ s_{n+1} \end{pmatrix} \leq A^{n+1}.\begin{pmatrix} p_0 \\ q_0 \\ r_0 \\ s_0 \end{pmatrix} + \sum_{k=0}^{n} A^k \begin{pmatrix} \eta_{n-k} \\ \varepsilon_{n-k} \\ \delta_{n-k} \\ \gamma_{n-k} \end{pmatrix}, n \geq 0, \quad (3)$$

From condition (ii), it follows that the sequences of partial sums$\{\mathrm{H}_n\},\{\mathrm{E}_n\},\{\Delta_n\},\{\Gamma_n\}$ are given respectively by $\mathrm{H}_n = \eta_0 + \eta_1 + \cdots + \eta_n, \mathrm{E}_n = \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_n, \Delta_n = \delta_0 + \delta_1 + \cdots + \delta_n$, and $\Gamma_n = \gamma_0 + \gamma_1 + \cdots + \gamma_n$, for $n \geq 0$, converge respectively to some $\mathrm{H}_n \geq 0, \mathrm{E}_n \geq 0, \Delta_n \geq 0$, and $\Gamma_n \geq 0$ and hence, they are bounded.

Let $M > 0$ be such that

$$\begin{pmatrix} \mathrm{H}_n \\ \mathrm{E}_n \\ \Delta_n \\ \Gamma_n \end{pmatrix} \leq M.\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \forall \ n \geq 0.$$

By condition (ii), then $\forall e > 0$, there exists $N = N(e)$ such that $A^n \leq \frac{e}{2M}.I_4, \forall n \geq N, \ M > 0$.

Write

$$\sum_{k=0}^{n} A^k \begin{pmatrix} \eta_{n-k} \\ \varepsilon_{n-k} \\ \delta_{n-k} \\ \gamma_{n-k} \end{pmatrix}$$

$$= A^n \begin{pmatrix} \eta_0 \\ \varepsilon_0 \\ \delta_0 \\ \gamma_0 \end{pmatrix} + \cdots + A^N \begin{pmatrix} \eta_{n-N} \\ \varepsilon_{n-N} \\ \delta_{n-N} \\ \gamma_{n-N} \end{pmatrix}$$

$$+ A^{N-1} \begin{pmatrix} \eta_{n-N+1} \\ \varepsilon_{n-N+1} \\ \delta_{n-N+1} \\ \gamma_{n-N+1} \end{pmatrix} + \cdots$$

$$+ I_4 \begin{pmatrix} \eta_n \\ \varepsilon_n \\ \delta_n \\ \gamma_n \end{pmatrix}$$

But

$$A^n \begin{pmatrix} \eta_0 \\ \varepsilon_0 \\ \delta_0 \\ \gamma_0 \end{pmatrix} + \cdots + A^N \begin{pmatrix} \eta_{n-N} \\ \varepsilon_{n-N} \\ \delta_{n-N} \\ \gamma_{n-N} \end{pmatrix}$$

$$\leq \frac{e}{2M} . I_4 \left[\begin{pmatrix} \eta_0 \\ \varepsilon_0 \\ \delta_0 \\ \gamma_0 \end{pmatrix} + \cdots \right.$$

$$\left. + \begin{pmatrix} \eta_{n-N} \\ \varepsilon_{n-N} \\ \delta_{n-N} \\ \gamma_{n-N} \end{pmatrix} \right]$$

$$= \frac{e}{2M} . I_4 \begin{pmatrix} \mathrm{H}_{n-N} \\ \mathrm{E}_{n-N} \\ \Delta_{n-N} \\ \Gamma_{n-N} \end{pmatrix} \leq \frac{e}{2M} . I_4 . M \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{e}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \forall\, n \geq N.$$

On the other hand, let $S = max\{I_4, A, \ldots, A^{N-1}\}$, the following is obtained

$$A^{N-1} \begin{pmatrix} \eta_{n-N+1} \\ \varepsilon_{n-N+1} \\ \delta_{n-N+1} \\ \gamma_{n-N+1} \end{pmatrix} + \cdots + I_4 \begin{pmatrix} \eta_n \\ \varepsilon_n \\ \delta_n \\ \gamma_n \end{pmatrix}$$

$$\leq S \left[\begin{pmatrix} \eta_{n-N+1} \\ \varepsilon_{n-N+1} \\ \delta_{n-N+1} \\ \gamma_{n-N+1} \end{pmatrix} + \cdots + \begin{pmatrix} \eta_n \\ \varepsilon_n \\ \delta_n \\ \gamma_n \end{pmatrix} \right]$$

$$= S \begin{pmatrix} \mathrm{H}_n - \mathrm{H}_{n-N} \\ \mathrm{E}_n - \mathrm{E}_{n-N} \\ \Delta_n - \Delta_{n-N} \\ \Gamma_n - \Gamma_{n-N} \end{pmatrix}.$$

Since N is fixed, then $\lim_{n\to\infty} \mathrm{H}_n = \lim_{n\to\infty} \mathrm{H}_{n-N} = \mathrm{H}$, $\lim_{n\to\infty} \mathrm{E}_n = \lim_{n\to\infty} \mathrm{E}_{n-N} = \mathrm{E}$, $\lim_{n\to\infty} \Delta_n = \lim_{n\to\infty} \Delta_{n-N} = \Delta$, $\lim_{n\to\infty} \Gamma_n = \lim_{n\to\infty} \Gamma_{n-N} = \Gamma$, which shows that there exists a positive integer k such that

$$S \begin{pmatrix} \mathrm{H}_n - \mathrm{H}_{n-N} \\ \mathrm{E}_n - \mathrm{E}_{n-N} \\ \Delta_n - \Delta_{n-N} \\ \Gamma_n - \Gamma_{n-N} \end{pmatrix} < \frac{e}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \forall\, n \geq k.$$

Now, for $m = max\{k,N\}$, the following is gotten

$$A^n \begin{pmatrix} \eta_0 \\ \varepsilon_0 \\ \delta_0 \\ \gamma_0 \end{pmatrix} + \cdots + I_4 \begin{pmatrix} \eta_n \\ \varepsilon_n \\ \delta_n \\ \gamma_n \end{pmatrix} < e \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \forall\, n \geq m,$$

and therefore, $\lim_{n\to\infty} \sum_{k=0}^{n} A^k \begin{pmatrix} \eta_{n-k} \\ \varepsilon_{n-k} \\ \delta_{n-k} \\ \gamma_{n-k} \end{pmatrix} = 0.$

Now, letting limit in (3), as $\lim_{n\to\infty} A^n = 0_4$, then

$$\lim_{n\to\infty} \begin{pmatrix} p_n \\ q_n \\ r_n \\ s_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

as required.

MAIN RESUTS

Definition 4. Let (X,d) be a metric space and $T: X^4 \to X$ be a mapping. For $(u_0,v_0,w_0,x_0) \in X^4$ the sequence $\{(u_n,v_n,w_n,x_n)\} \subset X^4$ defined by

$$u_{n+1} = T(u_n,v_n,w_n,x_n),\ v_{n+1} = T(v_n,u_n,v_n,x_n),\ w_{n+1} = T(w_n,u_n,v_n,w_n), x_{n+1} = T(x_n, w_n, v_n, u_n) \quad (4)$$

with $n = 0,1,2,\dots$, is the quadrupled fixed point iterative procedure.

Definition 5. Let (X,d) be a complete metric space and

$$Fix_t(T) = \{(u^*,v^*,w^*,x^*) \in X^4 / T(u^*,v^*,w^*,x^*) = u^*, T(v^*,u^*,v^*,x^*) = v^*, T(w^*,u^*,v^*,w^*) = w^*, T(x^*,w^*,v^*,u^*) = x^*\}$$

is the set of quadrupled fixed point of T.

Definition 6. Let $\{(p_n,q_n,r_n,s_n)\} \subset X^4$ be an arbitrary sequence. For all $n = 0,1,2,\dots$ setting $\eta_n = d\big(p_{n+1},T(p_n,q_n,r_n,s_n)\big)$, $\varepsilon_n = d\big(q_{n+1}, = ,T(q_n,p_n,q_n,s_n)\big)$, $\delta_n = d\big(r_{n+1},T(r_n,p_n,q_n,r_n)\big)$, $\gamma_n = d\big(s_{n+1},T(s_n,r_n,q_n,p_n)\big)$.

Then, the quadrupled fixed point iterative procedure defined by (4) is $T-$stable or stable with respect to T, if and only if $\lim_{n\to\infty}(\eta_n, \varepsilon_n,\delta_n,\gamma_n) = 0_{\mathbb{R}^4}$ implies that $\lim_{n\to\infty}(p_n,q_n,r_n,s_n) = (u^*,v^*,w^*,x^*)$.

Theorem 1. Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that the pair (X,d) is a complete metric space. Let $T: X^4 \to X$ be a continuous mapping having the mixed monotone property on X and satisfying (1). If there exist $u_0,v_0,w_0,x_0 \in X$ such that $u_0 \leq T(u_0,v_0,w_0,x_0), v_0 \geq T(v_0,u_0,v_0,x_0), w_0 \leq T(w_0,u_0,v_0,w_0)$, and $x_0 \geq T(x_0,w_0,v_0,u_0)$, then there exist$u^*,v^*,w^*,x^* \in X$ such that$u^* = T(u^*,v^*,w^*,x^*), v^* = T(v^*,u^*,v^*,x^*), w^* = T(w^*,u^*,v^*,w^*)$, and $x^* = T(x^*,w^*,v^*,u^*)$.

Assuming that for every $(u,v,w,x), (u_1,v_1,w_1,x_1) \in X^4$, there exist$(p,q,r,s) \in X^4$ that is comparable to (u,v,w,x) and (u_1,v_1,w_1,x_1). For $(u_0,v_0,w_0,x_0) \in X^4$, let $\{(u_n,v_n,w_n,x_n)\} \subset X^4$ be the quadrupled fixed point iterative procedure defined by (4). Then, the quadrupled fixed point iterative procedure is stable with respect to T.

Proof.

Let $\{(u_n,v_n,w_n,x_n)\} \subset X^4$, $\eta_n = d\big(p_{n+1},T(p_n,q_n,r_n,s_n)\big)$, $\varepsilon_n = d\big(q_{n+1},T(q_n,p_n,q_n,s_n)\big)$, $\delta_n = d\big(r_{n+1},T(r_n,p_n,q_n,r_n)\big)$, $and\ \gamma_n = d\big(s_{n+1},T(s_n,r_n,q_n,p_n)\big)$.

Assume also that $\lim_{n\to\infty}\eta_n = \lim_{n\to\infty}\varepsilon_n = \lim_{n\to\infty}\delta_n = \lim_{n\to\infty}\gamma_n = 0$,

in order to establish that $\lim_{n\to\infty}p_n = u^*, \lim_{n\to\infty}q_n = v^*, \lim_{n\to\infty}r_n = w^*$, and $\lim_{n\to\infty}s_n = x^*$.

Therefore, using the $(\vartheta,\kappa,\lambda,\mu)-$contraction condition (1), the following is obtained

$$\begin{aligned} d(p_{n+1},u^*) &\leq d\big(p_{n+1},T(p_n,q_n,r_n,s_n)\big) + d(T(p_n,q_n,r_n,s_n),u^*) \\ &= d\big(T(p_n,q_n,r_n,s_n),T(u^*,v^*,w^*,x^*)\big) + \eta_n \\ &\leq \vartheta d(p_n,u^*) + \kappa d(q_n,v^*) + \lambda d(r_n,w^*) + \mu d(s_n,x^*) + \eta_n, \end{aligned} \quad (5)$$

$$\begin{aligned} d(q_{n+1},v^*) &\leq d\big(q_{n+1},T(q_n,p_n,q_n,s_n)\big) + d(T(q_n,p_n,q_n,s_n),v^*) \\ &= d\big(T(q_n,p_n,q_n,s_n),T(v^*,u^*,v^*,x^*)\big) + \eta_n \\ &\leq \vartheta d(p_n,u^*) + \kappa d(q_n,v^*) + \lambda d(r_n,w^*) + \mu d(s_n,x^*) + \varepsilon_n \\ &= (\vartheta + \lambda) d(q_n,v^*) + \kappa d(p_n,u^*) + \mu d(s_n,x^*) + \varepsilon_n \\ &= \kappa d(p_n,u^*) + (\vartheta + \lambda) d(q_n,v^*) + \mu d(s_n,x^*) + \varepsilon_n \end{aligned} \quad (6)$$

$$\begin{aligned} d(r_{n+1},w^*) &\leq d\big(r_{n+1},T(r_n,p_n,q_n,r_n)\big) + d(T(r_n,p_n,q_n,r_n),w^*) \\ &= d\big(T(r_n,p_n,q_n,r_n),T(w^*,u^*,v^*,w^*)\big) + \delta_n \\ &\leq \vartheta d(r_n,w^*) + \kappa d(p_n,u^*) + \lambda d(q_n,v^*) + \mu d(r_n,w^*) + \delta_n \\ &= \kappa d(p_n,u^*) + \lambda d(q_n,v^*) + (\vartheta + \mu) d(r_n,w^*) + \delta_n \end{aligned} \quad (7)$$

$$\begin{aligned} d(s_{n+1},x^*) &\leq d\big(s_{n+1},T(s_n,r_n,q_n,p_n)\big) + d(T(s_n,r_n,q_n,p_n),x^*) \\ &= d\big(T(s_n,r_n,q_n,p_n),T(x^*,w^*,v^*,u^*)\big) + \gamma_n \\ &\leq \vartheta d(s_n,x^*) + \kappa d(r_n,w^*) + \lambda d(q_n,v^*) + \mu d(p_n,u^*) + \gamma_n \\ &= \mu d(p_n,u^*) + \lambda d(q_n,v^*) + \kappa d(r_n,w^*) + \vartheta d(s_n,x^*) + \gamma_n \end{aligned} \quad (8)$$

From (5)-(8), the following is obtain

$$\begin{pmatrix} d(p_{n+1},u^*) \\ d(q_{n+1},v^*) \\ d(r_{n+1},w^*) \\ d(s_{n+1},x^*) \end{pmatrix} \leq \begin{pmatrix} \vartheta & \kappa & \lambda & \mu \\ \kappa & \vartheta+\lambda & 0 & \mu \\ \kappa & \lambda & \vartheta+\mu & 0 \\ \mu & \lambda & \kappa & \vartheta \end{pmatrix} . \begin{pmatrix} d(p_n,u^*) \\ d(q_n,v^*) \\ d(r_n,w^*) \\ d(s_n,x^*) \end{pmatrix} + \begin{pmatrix} \eta_n \\ \varepsilon_n \\ \delta_n \\ \gamma_n \end{pmatrix}.$$

Denote $A = \begin{pmatrix} \vartheta & \kappa & \lambda & \mu \\ \kappa & \vartheta+\lambda & 0 & \mu \\ \kappa & \lambda & \vartheta+\mu & 0 \\ \mu & \lambda & \kappa & \vartheta \end{pmatrix}$, where $0 \leq \vartheta + \kappa + \lambda + \mu < 1$ as in (1).

In order to apply Lemma 2, we need that $A^n \to 0$ as $n \to \infty$.

As a way of simplification, denote

$$A = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ e_1 & f_1 & g_1 & h_1 \\ i_1 & j_1 & k_1 & l_1 \\ m_1 & n_1 & p_1 & q_1 \end{pmatrix},$$

where

$$a_1 + b_1 + c_1 + d_1 = e_1 + f_1 + g_1 + h_1 = i_1 + j_1 + k_1 + l_1 = m_1 + n_1 + p_1 + q_1 = \vartheta + \kappa + \lambda + \mu < 1, \qquad (9)$$

Then

$$A^2 = \begin{pmatrix} \vartheta^2+\kappa^2+\kappa\lambda+\mu^2 & 2\vartheta\kappa+\lambda(\kappa+\lambda+\mu) & 2\vartheta\lambda+\mu(\kappa+\lambda) & \mu(2\vartheta+\kappa) \\ \kappa(\vartheta+\lambda)+\vartheta\kappa+\mu^2 & (\vartheta+\lambda)^2+\kappa^2+\lambda\mu & \kappa\lambda+\kappa\mu & \mu(\vartheta+\lambda)+\kappa\mu+\vartheta\mu \\ \kappa(\vartheta+\mu)+\vartheta\kappa+\kappa\lambda & \lambda(\vartheta+\mu)+\kappa^2+\lambda(\vartheta+\lambda) & (\vartheta+\mu)^2+\kappa\lambda & \kappa\mu+\lambda\mu \\ \vartheta\mu+\kappa^2+\kappa\lambda+\vartheta\mu & \vartheta\lambda+\kappa\lambda+\lambda(\vartheta+\lambda)+\kappa\mu & \vartheta\kappa+\kappa(\vartheta+\mu)+\lambda\mu & \vartheta^2+\lambda\mu+\mu^2 \end{pmatrix}$$

$$= \begin{pmatrix} a_2 & b_2 & c_2 & d_2 \\ e_2 & f_2 & g_2 & h_2 \\ i_2 & j_2 & k_2 & l_2 \\ m_2 & n_2 & p_2 & q_2 \end{pmatrix},$$

where

$$\begin{aligned} a_2 + b_2 + c_2 + d_2 &= e_2 + f_2 + g_2 + h_2 \\ &= i_2 + j_2 + k_2 + l_2 \\ &= m_2 + n_2 + p_2 + q_2 \\ &= \vartheta^2 + \kappa^2 + \lambda^2 + \mu^2 + 2\vartheta\kappa \\ &\quad + 2\vartheta\lambda + 2\vartheta\mu + 2\kappa\lambda + 2\lambda\mu \\ &\quad + 2\kappa\mu \\ &= (\vartheta + \kappa + \lambda + \mu)^2 < \vartheta + \kappa + \lambda + \mu \\ &< 1 \end{aligned} \qquad (10)$$

Now, proving by induction that

$$A^n = \begin{pmatrix} a_n & b_n & c_n & d_n \\ e_n & f_n & g_n & h_n \\ i_n & j_n & k_n & l_n \\ m_n & n_n & p_n & q_n \end{pmatrix},$$

where

$$\begin{aligned} a_n + b_n + c_n + d_n &= e_n + f_n + g_n + h_n \\ &= i_n + j_n + k_n + l_n \\ &= m_n + n_n + p_n + q_n \\ &= (\vartheta + \kappa + \lambda + \mu)^n < \vartheta + \kappa + \lambda + \mu \\ &< 1 \end{aligned} \qquad (11)$$

Assume that (11) is true, then

$$A^{n+1} = A^n . A = \begin{pmatrix} a_n & b_n & c_n & d_n \\ e_n & f_n & g_n & h_n \\ i_n & j_n & k_n & l_n \\ m_n & n_n & p_n & q_n \end{pmatrix} . \begin{pmatrix} \vartheta & \kappa & \lambda & \mu \\ \kappa & \vartheta+\lambda & 0 & \mu \\ \kappa & \lambda & \vartheta+\mu & 0 \\ \mu & \lambda & \kappa & \vartheta \end{pmatrix}$$

Then

$$\begin{aligned} a_{n+1} + b_{n+1} + c_{n+1} + d_{n+1} &= a_n\vartheta + b_n\kappa + c_n\kappa + d_n\mu \\ &\quad + a_n\kappa + b_n(\vartheta+\lambda) + c_n\lambda + d_n\lambda \\ &\quad + a_n\lambda + c_n(\vartheta+\mu) + d_n\kappa + a_n\mu \\ &\quad + b_n\mu + d_n\vartheta \\ &= a_n(\vartheta+\kappa+\lambda+\mu) + b_n(\vartheta+\kappa+\lambda+\mu) \\ &\quad + c_n(\vartheta+\kappa+\lambda+\mu) \\ &\quad + d_n(\vartheta+\kappa+\lambda+\mu) \\ &= (a_n + b_n + c_n + d_n)(\vartheta+\kappa+\lambda+\mu) \\ &= (\vartheta+\kappa+\lambda+\mu)^n(\vartheta+\kappa+\lambda+\mu) \\ &= (\vartheta+\kappa+\lambda+\mu)^{n+1} \\ &< \vartheta+\kappa+\lambda+\mu < 1, \end{aligned}$$

similarly,

$$\begin{aligned} e_{n+1} + f_{n+1} + g_{n+1} + h_{n+1} &= i_{n+1} + j_{n+1} + k_{n+1} + l_{n+1} \\ &= m_{n+1} + n_{n+1} + p_{n+1} + q_{n+1} \\ &= (\vartheta+\kappa+\lambda+\mu)^{n+1} \\ &< \vartheta+\kappa+\lambda+\mu < 1. \end{aligned}$$

Therefore, $\lim_{n\to\infty} A^n = 0_4$, and having satisfying the conditions of the hypothesis of Lemma 2, on applying we get

$$\lim_{n\to\infty}\begin{pmatrix}p_n\\ q_n\\ r_n\\ s_n\end{pmatrix}=\begin{pmatrix}u^*\\ v^*\\ w^*\\ x^*\end{pmatrix},$$

So the quadrupled fixed point iteration procedure defined by (4) is T −stable or stable with respect to its operator.

Corollary 1. Let $(X,\leq)$ be a partially ordered set and suppose there is a metric d on X such that the pair (X,d) is a complete metric space. Let $T\colon X^4\to X$ be a continuous mapping having the mixed monotone property on X.
There exist $h\in[0,1)$ such that T satisfies the following contraction condition

$$\begin{aligned}&d\big(T(u,v,w,x),T(p,q,r,s)\big)\\ &\qquad\leq\frac{h}{4}\big(d(u,p)+d(v,q)+d(w,r)\\ &\qquad+d(x,s)\big),\qquad (12)\end{aligned}$$

for each $u,v,w,x,p,q,r,s\in X$, with $u\geq p, v\leq q, w\geq r, x\leq s$.If there exists $u_0,v_0,w_0,x_0\in X$ such that $u_0\leq T(u_0,v_0,w_0,x_0), v_0\geq T(v_0,u_0,v_0,x_0), w_0\leq T(w_0,u_0,v_0,w_0)$, and $x_0\geq T(x_0,w_0,v_0,u_0)$, then there exists $u^*,v^*,w^*,x^*\in X$ such that $u^*=T(u^*,v^*,w^*,x^*), v^*=T(v^*,u^*,v^*,x^*), w^*=T(w^*,u^*,v^*,w^*)$, and $x^*=T(x^*,w^*,v^*,u^*)$. Assuming that for every $(u,v,w,x),(u_1,v_1,w_1,x_1)\in X^4$, there exists $(p,q,r,s)\in X^4$ that is comparable to (u,v,w,x) and (u_1,v_1,w_1,x_1). For $(u_0,v_0,w_0,x_0)\in X^4$, let $\{(u_n,v_n,w_n,x_n)\}\subset X^4$ be the quadrupled fixed point iterative procedure defined by (4). Then, the quadrupled fixed point iterative procedure is stable with respect to T.

Proof. Applying Theorem 1 for $\vartheta=\kappa=\lambda=\mu=\frac{h}{4}$

The following is obtained on using the contraction condition (1),

$$\begin{aligned}d(p_{n+1},u^*)&\leq\frac{h}{4}d(p_n,u^*)+\frac{h}{4}d(q_n,v^*)\\ &\quad+\frac{h}{4}d(r_n,w^*)+\frac{h}{4}d(s_n,x^*)\\ &\quad+\eta_n,\qquad (13)\end{aligned}$$

$$\begin{aligned}&d(q_{n+1},v^*)\\ &\leq\frac{h}{4}d(p_n,u^*)+\frac{h}{2}d(q_n,v^*)+\frac{h}{4}d(s_n,x^*)\\ &+\varepsilon_n\qquad (14)\end{aligned}$$

$$\begin{aligned}&d(r_{n+1},w^*)\\ &\leq\frac{h}{4}d(p_n,u^*)+\frac{h}{4}d(q_n,v^*)+\frac{h}{2}d(r_n,w^*)\\ &+\delta_n\qquad (15)\end{aligned}$$

$$\begin{aligned}d(s_{n+1},x^*)&\leq\frac{h}{4}d(p_n,u^*)+\frac{h}{4}d(q_n,v^*)\\ &\quad+\frac{h}{4}d(r_n,w^*)+\frac{h}{4}d(s_n,x^*)\\ &\quad+\gamma_n\qquad (16)\end{aligned}$$

From (13)-(16), the following is obtain

$$\begin{pmatrix}d(p_{n+1},u^*)\\ d(q_{n+1},v^*)\\ d(r_{n+1},w^*)\\ d(s_{n+1},x^*)\end{pmatrix}\leq\begin{pmatrix}\frac{h}{4}&\frac{h}{4}&\frac{h}{4}&\frac{h}{4}\\ \frac{h}{4}&\frac{h}{2}&0&\frac{h}{4}\\ \frac{h}{4}&\frac{h}{4}&\frac{h}{2}&0\\ \frac{h}{4}&\frac{h}{4}&\frac{h}{4}&\frac{h}{4}\end{pmatrix}\cdot\begin{pmatrix}d(p_n,u^*)\\ d(q_n,v^*)\\ d(r_n,w^*)\\ d(s_n,x^*)\end{pmatrix}+\begin{pmatrix}\eta_n\\ \varepsilon_n\\ \delta_n\\ \gamma_n\end{pmatrix}.$$

Denote $A=\begin{pmatrix}\frac{h}{4}&\frac{h}{4}&\frac{h}{4}&\frac{h}{4}\\ \frac{h}{4}&\frac{h}{2}&0&\frac{h}{4}\\ \frac{h}{4}&\frac{h}{4}&\frac{h}{2}&0\\ \frac{h}{4}&\frac{h}{4}&\frac{h}{4}&\frac{h}{4}\end{pmatrix}$, where $0\leq h<1$ as in (1).

Applying Lemma 2, need that $A^n\to 0$ as $n\to\infty$.
As a way of simplification, denote

$$A=\begin{pmatrix}a_1&b_1&c_1&d_1\\ e_1&f_1&g_1&h_1\\ i_1&j_1&k_1&l_1\\ m_1&n_1&p_1&q_1\end{pmatrix},$$

where

$$\begin{aligned}a_1+b_1+c_1+d_1&=e_1+f_1+g_1+h_1\\ &=i_1+j_1+k_1+l_1\\ &=m_1+n_1+p_1+q_1=h<1,\end{aligned}$$

then

$$A^2 = \begin{pmatrix} \frac{h^2}{4} & \frac{5h^2}{16} & \frac{h^2}{4} & \frac{3h^2}{16} \\ \frac{h^2}{4} & \frac{3h^2}{8} & \frac{h^2}{8} & \frac{h^2}{4} \\ \frac{h^2}{4} & \frac{5h^2}{16} & \frac{5h^2}{16} & \frac{h^2}{8} \\ \frac{h^2}{4} & \frac{5h^2}{16} & \frac{h^2}{4} & \frac{3h^2}{16} \end{pmatrix} = \begin{pmatrix} a_2 & b_2 & c_2 & d_2 \\ e_2 & f_2 & g_2 & h_2 \\ i_2 & j_2 & k_2 & l_2 \\ m_2 & n_2 & p_2 & q_2 \end{pmatrix},$$

where

$$a_2 + b_2 + c_2 + d_2 = e_2 + f_2 + g_2 + h_2 = i_2 + j_2 + k_2 + l_2 = m_2 + n_2 + p_2 + q_2 = h^2 < h < 1$$

Now, proving by induction that

$$A^n = \begin{pmatrix} a_n & b_n & c_n & d_n \\ e_n & f_n & g_n & h_n \\ i_n & j_n & k_n & l_n \\ m_n & n_n & p_n & q_n \end{pmatrix},$$

where

$$a_n + b_n + c_n + d_n = e_n + f_n + g_n + h_n = i_n + j_n + k_n + l_n = m_n + n_n + p_n + q_n = h^n < h < 1 \qquad (17)$$

Assuming that (17) is true for n, then

$$A^{n+1} = \begin{pmatrix} a_n & b_n & c_n & d_n \\ e_n & f_n & g_n & h_n \\ i_n & j_n & k_n & l_n \\ m_n & n_n & p_n & q_n \end{pmatrix} \cdot \begin{pmatrix} \frac{h}{4} & \frac{h}{4} & \frac{h}{4} & \frac{h}{4} \\ \frac{h}{4} & \frac{h}{2} & 0 & \frac{h}{4} \\ \frac{h}{4} & \frac{h}{4} & \frac{h}{2} & 0 \\ \frac{h}{4} & \frac{h}{4} & \frac{h}{4} & \frac{h}{4} \end{pmatrix},$$

we have

$$a_{n+1} + b_{n+1} + c_{n+1} + d_{n+1} = e_{n+1} + f_{n+1} + g_{n+1} + h_{n+1} = i_{n+1} + j_{n+1} + k_{n+1} + l_{n+1} = m_{n+1} + n_{n+1} + p_{n+1} + q_{n+1}$$

$$= \frac{h}{4}(a_n + b_n + c_n + d_n) + \frac{h}{4}(e_n + f_n + g_n + h_n) + \frac{h}{4}(i_n + j_n + k_n + l_n) + \frac{h}{4}(m_n + n_n + p_n + q_n) = \frac{h}{4}(h^n + h^n + h^n + h^n) = h^{n+1} < h < 1.$$

Therefore, $\lim_{n\to\infty} A^n = 0_4$ and now having satisfied the conditions of Lemma 2, then

$$\lim_{n\to\infty} \begin{pmatrix} p_n \\ q_n \\ r_n \\ s_n \end{pmatrix} = \begin{pmatrix} u^* \\ v^* \\ w^* \\ x^* \end{pmatrix},$$

which shows that the quadrupled fixed point iteration procedure defined by (4) is T −stable.

CONCLUSION

This study shows that the quadrupled iterative fixed point method for contractile-type mapping in a partially ordered metric space with mixed monotonic properties is stable. This result is a continuation of the results of Timis (2014), from triple fixed point stability to quadrupled fixed point satisfying various contractive conditions.

ACKNOWLEDGEMENTS

I remain grateful to all whose contributions had enriched this manuscript.

REFERENCES

Amini-Harandi, A. (2012). Coupled and tripled fixed point theory in partially ordered metric spaces with application to initial value problem. *Mathematical and Computer Modeling*, 57, 2343-2348.

Aniki, S.A. & Rauf, K. (2019). Stability results on coupled fixed point iterative procedures in complete metric spaces. *Islamic University Multidisciplinary Journal*, 6(3), 175-186. ISSN 2617-6513.

Aniki, S.A. & Rauf, K. (2020). Stability theorem and results for quadrupled fixed point of contractive type single valued operators. *Iranian Journal of Optimization,* 12(2).

Abbas, M., Aydi H. & Karapinar E. (2011). Tripled fixed point of multivalued nonlinear contraction mappings in partially ordered metric spaces. *Abstract and Applied Analysis,* (2), 1-12.

Bhaskar, T.G. & Lakshmikantham, V. (2006). Fixed point theorems in partially ordered metric spaces and applications. *Nonlinear Analysis,* 65(7), 1379-1393.

Berinde V. (2003). Summable almost stability of fixed point iteration procedures. *Carpathian Journal of Mathematics,* 19(2), 81–88.

Berinde, V. & Borcut, M. (2011). Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces. *Nonlinear Analysis*, 74(15), 4889-4897.

Ciric L.B. & Lakshmikantham V. (2009). Coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces. *Stochastic Analysis and Applications,* 27(6), 1246–1259.

Choudhury, B.S. & Kundu, A.A. (2010) Coupled coincidence point result in partially ordered metric spaces for compatible mappings. *Nonlinear Analysis*, 73(8), 2524-2531.

Imoru, C.O. & Olatinwo M.O. (2003). On the stability of Picard andMann iteration processes. *Carpathian Journal of Mathematics,* 19(2), 155–160.

Jachymski J.R. (1997). An extension of A. Ostrowski's theorem on the round-off stability of iterations. *Aequationes Mathematicae*, 53(3), 242–253.

Karapinar, E. (2010). Coupled foxed point theorems for nonlinear contractions in cone metric spaces. *Computer & Mathematics with Applications,* 59(12), 3656-3668.

Lakshmikantham, V. & Ciric, L.B. (2009). Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. *Nonlinear Analysis,* 70(12), 4341-4349.

Olatinwo, M.O., Owojori, O.O. & Imoru, C.O. (2006). Some stability results for fixed point iteration processes. *Australian Journal of Mathematical Analysis and Applications,* 2(1), 1–7.

Osilike, M.O. (1996). A stable iteration procedure for quasi-contractive maps. *Indian Journal of Pure and Applied Mathematics,* 27(1), 25–34.

Osilike, M.O. (1995). Stability results for fixed point iteration procedure. *Journal of the Nigerian Mathematical Society,* 14, 17–29.

Ostrowski, A.M. (1967). The round-off stability of iterations. *Journal of Applied Mathematics and Mechanics,* 47(1), 77–81.

Rauf, K. & Aniki, S.A. (2021). Quadrupled fixed point theorems for contractive type mappings in partially ordered cauchy spaces. *Confluence Journal of Pure and Applied Sciences,* 4(1), 18-30.

Rao, K.P.R. & Kishore, G.N.V. (2011). A Unique Common tripled fixed point theorem in partially ordered cone metric spaces. *Bulletin of Mathematical Analysis and Applications,* 3(4), 213-222.

Rhoades, B.E. (1990). Fixed point theorems and stability results for fixed point iteration procedures. *Indian Journal of Pure and Applied Mathematics,* 21(1), 1–9.

Rhoades, B.E. (1993) Fixed point theorems and stability results for fixed point iteration procedures II. *Indian Journal of Pure and Applied Mathematics,* 24(11), 691–703.

Sabetghadam, F., Masiha, H.P. & Sanatpour, A.H. (2009). Some coupled fixed point theorems in cone metric spaces. *Fixed Point Theory and Applications,* Article ID 125426. doi:10.1155/2009/125426

Timis, I. (2014). Stability of tripled fixed point iteration procedures for mixed monotone mappings. *Carpathian Mathematical Publications,* 6(2), 377-388.