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A Meshless RBF Method for Linear and Nonlinear Sobolev Equations

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Abstract

Radial Basis Functions are considered as important tools for scattered data interpolation. Collocation procedure is a powerful technique in meshless methods which is developed on the assumption of radial basis functions to solve partial differential equations in high dimensional domains having complex shapes. In this study, a numerical method, implementing the RBF collocation method and finite differences, is employed for solving not only 2-D linear, but also nonlinear Sobolev equations. First order finite differences and Crank-Nicolson method are applied to discretize the temporal part. Using the energy method, it is shown that the applied time-discrete approach is convergent in terms of time variable with order O (Δt). The spatial parts are approximated by implementation of two-dimensional MQ-RBF interpolation resulting in a linear system of algebraic equations. By solving the linear system, approximate solutions are determined. The proposed scheme is verified by solving different problems and error norms L_∞ and L_2 are computed. Computations accurately demonstrated the efficiency of the suggested method.

Keywords:
Radial basis functions (RBFs)
Finite differences
Crank-Nicolson method
Method of lines
Energy method
Sobolev equations

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INTRODUCTION

A wide spectrum of phenomena in different fields of engineering and sciences including physics, chemistry, and biology are simulated by partial differential equations (PDEs). Conducting investigations on solutions to such equations have always been of interest to scholars; however, most of them, especially nonlinear ones are too complex to be solved analytically. Thus, approximate solutions to them might be provided by numerical methods. In this study, linear and nonlinear 2-D Sobolev equations are numerically studied.

Consider the following two-dimensional PDE (Haq et al., 2019; Guo & Fang, 2012).

$$u_t - \Delta u_t - \alpha_1 \Delta u + (\alpha_2.\nabla)u + \alpha_3 \nabla.(u \nabla u) + \alpha_4 u^2 = \alpha_5 \nabla.(F(u)) + f(x,y,t), \quad (x,y,t) \in \Omega, t > 0, \tag{1}$$

with following conditions

$$\begin{cases} u(x,y,0) = p(x,y), & (x,y) \in \Omega, \\ u(x,y,t) = q(x,y,t) & t > 0, \ (x,y) \in \partial\Omega, \end{cases} \tag{2}$$

where u=u(x,y,t),F(u) , is the vector valued function, Ω R^2 , ,Δ, are gradient and Laplacian operators respectively, α_i (i=1,3,4,5) , are real constants and α_2 is a real constant vector.

If $\alpha_1=1$, and $\alpha_2=(0,0)$, $\alpha_3=\alpha_4=\alpha_5=0$ then Eq. 1 turns to the linear Sobolev equation as follows.

$$u_t - \Delta u_t - \Delta u = f(x,y,t), \qquad t > 0, \ (x,y) \in \Omega. \tag{3}$$

If $\alpha_4=\alpha_5=0$, α_2 (0,0), $\alpha_3=-1$ and If $\alpha_4=\pi^2$, then the following nonlinear Sobolev equation is obtained from Eq. 1.

$$u_t - \Delta u_t - \nabla.(u \nabla u) + \pi^2 u^2 = f(x,y,t), \quad t > 0, \ (x,y) \in \Omega. \tag{4}$$

Sobolev equations are derivative of evolution partial differential equations possessing an important physical background. They appear in the stream of fluids through cracked rock (Barenblatt et al., 1960; Cao & Pop, 2016), thermodynamics (Chen & Gurtin, 1968), the exchange in different media (Ting, 1974), the moisture penetration in soil (Shi, 1990) and many other usages in science and engineering. Various numerical methods suggested for the numerical solutions of Sobolev equations that mostly include finite element method. For instance, an expanded mixed finite element method for 2-D linear equations of Sobolev which was recently proposed by Li et al.(2019). In 2017, Gao et al. solved Sobolev equation by a weak Galerkin finite element method. Based on time discontinuous Galerkin space–time FEM method, a numerical method for non-linear Sobolev equation was proposed by Siriguleng et al.(2013). A Godunov-mixed finite element method used to change meshes was proposed and analyzed by Sun (2012) so as to solve the nonlinear Sobolev equations. The refence (shi et al.,2015), a low order nonconforming finite element method was combined with the method of characteristics so as to treat the nonlinear Sobolev equation with convection-dominated term.

There are many robust numerical methods including finite element method (FEM) and finite volume method (FVM) as well as finite difference method (FDM), etc. that have been successfully applied to various real-world problems (e.g.(Patil & Maniyeri, 2019; Jiang et al., 2020; Rossi et al.,2019; Gao & Keyes, 2019; Jose et al., 2017). In most of the aforementioned methods, it is needed to arrange the data in a simple domain, like a circle or rectangle. In such methods, grid generation is often needed which requires vast amount of time on computation. The accuracy of these methods decreases in non-smooth and non-regular domains because the solution of the problem is only provided on mesh points. Furthermore, in real applications, especially for multi-dimensional scattered data, it is not possible to construct a mesh. The Meshless methods were created to alleviate the aforementioned restrictions. In 1968, Ronald Hardy, an Iowa State geodesist, developed radial basis functions

(RBFs) for his desire to find an efficient way to interpolate the scattered data on a topographic surface. A set of N distinct points X={x_1^c , x_2^c ,…, x_N^C } in R^d which are called centers is used in RBF interpolation. There are no restrictions on the geometry of domains of the problem or on the position of the centers (Sarra, 2017). Hardy introduced the well-known radial basis function, multi-quadric (MQ) for the first time and described and named it in a paper appearing in 1971(Hardy, 1971). Another famous type of RBFs, thin-plate splines (TPS) were proposed by Duchon, in 1975 (Duchon, 1997). Although, Richard Franke, by performing many numerical experiments, showed that the MQ-RBFs scheme was the best scheme to interpolate scattered data among all known interpolation methods(Franke, 1979). In 1986, Charles Micchelli established that the system matrix of the MQ scheme was invertible implying that the problem of RBFs scattered data interpolation is well-posed(Micchelli, 1986). Edward Kansa was the pioneer of using MQs associated with the collocation method to solve PDEs in 1990 (Kansa, 1990a; Kansa, 1990b). After his worthy works, doing research in RBFs have actively been continued. During recent years, RBFs method has been considered as an effective tool for the purpose of solving various sort of problems including PDEs (Kazem et al., 2012; Vertnik & Šarler, 2013; Kadalbajoo et al., 2015; Casanova et al., 2019) integral equations (Dastjerdi & Ahmadabadi, 2017; Assari & Dehghan, 2018) and fractional equations (Chandhin et al., 2018; Piret & Hanert, 2013).

The goal of the present study, is to apply a numerical method which is based on the RBF collocation method and finite differences, in order to solve the 2-D linear and nonlinear Sobolev. In order to discretize the temporal part, finite differences are applied while two dimensional RBF interpolation is implemented for approximating the spatial parts.

The manuscript is organized as following. In section 2 a time discrete scheme is obtained and the convergence of the scheme is proved. The RBF interpolation method is explained by presenting basic concepts and definitions in Section 3. The suggested method is used in equations (3) and (4) in section 4. Then, the proposed method is applied for some test problems, and consequently the results are reported in Section 5. Finally, a conclusion is presented in Section 6.

TIME DISCRETE SCHEME

In this section, forward finite differences and also Crank-Nicolson scheme are used to discretize the time variable for the first-order time derivative. The convergence of this time-discrete scheme is analyzed using energy method subsequently.

Let $\Delta t = \frac{T}{N}$ and $t_k = k\,\Delta t$, $k = 0,\dots,N$.

Also, suppose that F(u) satisfies the Lipschitz condition

$$\left|F(u) - F(v)\right| \le M\,|u - v|,$$

where *M* is a Lipschitz constant.

Time discretization of the linear Sobolev equation

Consider Eq. 3 at tk. Then

$$\frac{u^{k+1}-u^k}{\Delta t} - \frac{\Delta u^{k+1}+\Delta u^k}{2} - \frac{\Delta u^{k+1}-\Delta u^k}{\Delta t} = f^k + R, \tag{5}$$

Where |R|<S Δt and is S a positive constant. Simplifying Eq. 5 gives the following equation.

$$u^{k+1} - u^k - \Delta u^{k+1}\left(1+\frac{\Delta t}{2}\right) + \Delta u^k\left(1-\frac{\Delta t}{2}\right) = \Delta t f^k + \Delta t R. \tag{6}$$

If the small term R is omitted then we have

$$U^{k+1} - \left(1+\frac{\Delta t}{2}\right)\Delta U^{k+1} + \left(1-\frac{\Delta t}{2}\right)\Delta U^k - U^k = \Delta t f^k. \tag{7}$$

The convergence analysis

Consider the functional spaces which are endowed with inner products and the standard norms as follows (Liu et al., 2011).

$$H^1(\Omega)=\left\{v\in L^2(\Omega),\frac{dv}{dx}\in L^2(\Omega)\right\},$$

$$H_0^1(\Omega)=\left\{v\in H^1(\Omega),v\,|_{\partial\Omega}=0\right\},$$

Where $L^2(\Omega)$ is the space of measurable functions. The definition of inner products of $H^1(\Omega)$ and $L^2(\Omega)$ respectively are defined as follows.

$$(u,v)=\int_\Omega uv\,d\Omega,\qquad (u,v)_1=(\nabla u,\nabla v)+(u,v).$$

The L^2 and H_1 norms are defined respectively as follows.

$$\|u\|=\sqrt{(u,u)},\quad \|u\|_1=\sqrt{(u,u)_1},\quad |u|_1=\sqrt{(\nabla u,\nabla u)}.$$

Here, the weighted H^1–norm (Liu et al., 2011) defined as follows is applied.

$$\|u\|_{*,1}=\sqrt{\|u\|^2+|u|_1^2}.$$

Lemma 1. *The following inequality is established for either functions; f (x) or g(x).*

$$(\nabla.|f|,g)\le|f|_1\|g\|.$$

Proof. (Dehghan et al., 2014).

Theorem 1. The proposed time discrete scheme is H1- convergent by convergence order O (Δt). When $u(x,t_k)=u^k$ and $U(x,t_k)=U^k\in H_0 1$ are the exact solution of Eq. 3 and the approximate solution respectively.

Proof. Subtracting (6) from (7) leads to

$$\lambda^{k+1}-\left(1+\frac{\Delta t}{2}\right)\Delta\lambda^{k+1}=\lambda^k-\left(1-\frac{\Delta t}{2}\right)\Delta\lambda^k+\Delta tR,\tag{8}$$

where

$$\lambda^k=U^k-u^k,\qquad \lambda^0=0.$$

Now, multiply both sides of (8) by λ^{k+1} then integrate on Ω.

$$\|\lambda^{k+1}\|^2-\left(1+\frac{\Delta t}{2}\right)(\Delta\lambda^{k+1},\lambda^{k+1})=(\lambda^k,\lambda^{k+1})-\left(1-\frac{\Delta t}{2}\right)(\Delta\lambda^k,\lambda^{k+1})+\Delta t(R,\lambda^{k+1}),$$

or

$$\|\lambda^{k+1}\|^2+\left(1+\frac{\Delta t}{2}\right)(\nabla\lambda^{k+1},\nabla\lambda^{k+1})=(\lambda^k,\lambda^{k+1})+\left(1-\frac{\Delta t}{2}\right)(\nabla\lambda^k,\nabla\lambda^{k+1})+\Delta t(R,\lambda^{k+1}),$$

Applying Lemma 1 and Schwarz inequality gives

$$\|\lambda^{k+1}\|^2+\left(1+\frac{\Delta t}{2}\right)|\lambda^{k+1}|_1^2\le\|\lambda^k\|\|\lambda^{k+1}\|+\left(1-\frac{\Delta t}{2}\right)|\lambda^k|_1|\lambda^{k+1}|_1+\Delta t|R|\|\lambda^{k+1}\|$$

$$\le\frac{1}{2}\left(\|\lambda^k\|^2+\|\lambda^{k+1}\|^2\right)+\left(\frac{1}{2}-\frac{\Delta t}{4}\right)\left(|\lambda^k|_1^2+|\lambda^{k+1}|_1^2\right)+\Delta t|R|\|\lambda^{k+1}\|.$$

Simplifying the above relation leads and multiplying both sides by 2 leads to

$$\|\lambda^{k+1}\|^2+\left(1+\frac{3\Delta t}{2}\right)|\lambda^{k+1}|_1^2\le\|\lambda^k\|^2+\left(1-\frac{\Delta t}{2}\right)|\lambda^k|_1^2+2\Delta t|R|\|\lambda^{k+1}\|.\tag{9}$$

According to Poincare inequality (Quarteroni & Valli, 2008):

$$\|V^{k+1}\|\le C|V^{k+1}|_1,\tag{10}$$

also the following inequality

$$ab\le\frac{1}{2\eta}a^2+\frac{\eta}{2}b^2,\qquad\forall\eta>0,\tag{11}$$

Eq. 9 is written as follows.

$$\left\|\lambda^{k+1}\right\|^2+\left|\lambda^{k+1}\right|_1^2(1+\frac{3\Delta t}{2})\le\left\|\lambda^k\right\|^2+\left|\lambda^k\right|_1^2\left(1-\frac{\Delta t}{2}\right)$$
$$+\frac{1}{\eta}\Delta t^2C^2R^2+\eta\left|\lambda^{k+1}\right|_1^2.$$

Now put η=Δt/2.

$$\left\|\lambda^{k+1}\right\|^2+(1+\Delta t)\left|\lambda^{k+1}\right|_1^2\le\left\|\lambda^k\right\|^2+$$
$$\left(1-\frac{\Delta t}{2}\right)\left|\lambda^k\right|_1^2+2\Delta tC^2R^2.$$

Since 1<1+Δt and 1-Δt/2<1 , the relation can be written as following.

$$\left\|\lambda^{k+1}\right\|^2+\left|\lambda^{k+1}\right|_1^2\le\left\|\lambda^k\right\|^2+\left|\lambda^k\right|_1^2+2\Delta tC^2R^2.$$

Using the weighted H^1 -norm results in

$$\left\|\lambda^{k+1}\right\|_{*,1}^2\le\left\|\lambda^k\right\|_{*,1}^2+2\Delta tC^2R^2$$
$$\le\left(\left\|\lambda^{k-1}\right\|_{*,1}^2+2\Delta tC^2R^2\right)+2\Delta tC^2R^2$$
$$\vdots$$
$$\le\left\|\lambda^0\right\|_{*,1}^2+2(k+1)\Delta tC^2R^2.$$

As $\lambda^0=0$

$$\left\|\lambda^{k+1}\right\|_{*,1}^2\le 2(k+1)\Delta tC^2R^2=2TC^2\left|R\right|^2.$$

Consequently

$$\left\|\lambda^{k+1}\right\|_{*,1}\le C\left|R\right|\sqrt{2T}\le CS\sqrt{2T}\,\Delta t,$$

which completes the proof.

Time discretization of the non-linear Sobolev equation

Considering Eq. 4 at t^k gives

$$\frac{u^{k+1}-u^k}{\Delta t}-\frac{\Delta u^{k+1}-\Delta u^k}{\Delta t}-\frac{\nabla F_1\left(u^{k+1}\right)+\nabla F_1\left(u^k\right)}{2}+$$
$$\pi^2\frac{F_2\left(u^{k+1}\right)+F_2\left(u^k\right)}{2}=f^k+R, \qquad (12)$$

Where F_1 (u)=u. u and F_1 (u)=u^2. Simplifying Eq.12 gives the following equation.

$$u^{k+1}-u^k-\Delta u^{k+1}+\Delta u^k-\frac{\Delta t}{2}\nabla F_1\left(u^{k+1}\right)-\frac{\Delta t}{2}\nabla F_1\left(u^k\right)$$
$$+\pi^2\frac{\Delta t}{2}F_2\left(u^{k+1}\right)+\pi^2\frac{\Delta t}{2}F_2\left(u^k\right)=\Delta tf^k+\Delta tR.$$

(13)

Omitting the small term R results in

$$U^{k+1}-U^k-\Delta U^{k+1}+\Delta U^k-\frac{\Delta t}{2}\nabla F_1\left(U^{k+1}\right)$$
$$-\frac{\Delta t}{2}\nabla F_1\left(U^k\right)$$
$$+\pi^2\frac{\Delta t}{2}F_2\left(U^{k+1}\right)+\pi^2\frac{\Delta t}{2}F_2\left(U^k\right)=\Delta tf^k.$$

(14)

The convergence analysis

Theorem 2. Assume u(x, t_k)=u^k is considered as the exact solution of Eq. 4 and U(x, t_k)=U^k ϵH_0^1 is the approximate solution, consequently the time discrete solution is H^1 - convergent and the convergence order is O(Δt).

Proof. When 13 is subtracted from 14, following relation is hold.

$$\lambda^{k+1}-\Delta\lambda^{k+1}-\frac{\Delta t}{2}\nabla.\left[F_1\left(U^{k+1}\right)-F_1\left(u^{k+1}\right)\right]+$$
$$\pi^2\frac{\Delta t}{2}\left[F_2\left(U^{k+1}\right)-F_2\left(u^{k+1}\right)\right]=$$
$$\lambda^k-\Delta\lambda^k+\frac{\Delta t}{2}\nabla.\left[F_1\left(U^k\right)-F_1\left(u^k\right)\right]-$$
$$\pi^2\frac{\Delta t}{2}\left[F_2\left(U^k\right)-F_2\left(u^k\right)\right]+\Delta tR.$$

(15)

If both sides of Eq. 15 are multiplied by λ^{k+1}, and also integrated on **Ω**, and finally using Lipschitz condition then

$$\left\|\lambda^{k+1}\right\|^2+\left|\lambda^{k+1}\right|_1^2+\frac{\Delta t}{2}M_1\left|\lambda^{k+1}\right|_1\left\|\lambda^{k+1}\right\|+\frac{\Delta t}{2}\pi^2M_2\left\|\lambda^{k+1}\right\|^2\le$$

$$\|\lambda^k\|\|\lambda^{k+1}\|+|\lambda^k|_1|\lambda^{k+1}|_1+\frac{\Delta t}{2}M_3|\lambda^k|_1|\lambda^{k+1}\|$$
$$-\frac{\Delta t}{2}\pi^2M_4\|\lambda^k\|\|\lambda^{k+1}\|+\Delta tR\|\lambda^{k+1}\|,$$

Where M_1, M_2, M_3, M_4 are Lipschitz constants.

Applying Schwarz inequality , Lemma 1, and (10) gives

$$\left(1+\frac{\Delta t}{2}\pi^2M\right)\|\lambda^{k+1}\|^2+\left(1+\frac{\Delta t}{2}MC\right)|\lambda^{k+1}|_1^2\leq$$
$$\frac{1}{2}\left(\|\lambda^k\|^2+\|\lambda^{k+1}\|^2\right)+\frac{1}{2}\left(|\lambda^k|_1^2+|\lambda^{k+1}|_1^2\right)+\frac{\Delta t}{4}M$$
$$\left(|\lambda^k|_1^2+\|\lambda^{k+1}\|^2\right)$$
$$-\frac{\Delta t}{4}\pi^2M\left(\|\lambda^k\|^2+\|\lambda^{k+1}\|^2\right)+\Delta tR\|\lambda^{k+1}\|,$$

where M is the maximum of $\{M_1, M_2, M_3, M_4\}$.

$$\left(1-\frac{\Delta t}{2}M+\frac{3\Delta t}{2}M\pi^2\right)\|\lambda^{k+1}\|^2+(1+\Delta tMC)|\lambda^{k+1}|_1^2\leq$$
$$\left(1-\frac{\Delta t}{2}\pi^2M\right)\|\lambda^k\|^2+\left(1+\frac{\Delta t}{2}M\right)|\lambda^k|_1^2+2\Delta tR\|\lambda^{k+1}\|.$$

Simplifying the above relation and multiplying both sides by 2 leads to

Because $\left(1-\frac{\Delta t}{2}\pi^2M\right)<\left(1+\frac{\Delta t}{2}M\right)<(1+\Delta tM)$

we can write

$$(1-\Delta tM)\|\lambda^{k+1}\|^2+(1+\Delta tMC)|\lambda^{k+1}|_1^2\leq$$
$$(1+\Delta tM)\|\lambda^k\|^2+(1+\Delta tM)|\lambda^k|_1^2+2\Delta tR\|\lambda^{k+1}\|.$$

According to relations (10) and (11), the above equation can be expressed in the following form:

$$(1-\Delta tM)\|\lambda^{k+1}\|^2+(1+\Delta tMC-\eta)|\lambda^{k+1}|_1^2\leq$$
$$(1+\Delta tM)\|\lambda^k\|^2+(1+\Delta tM)|\lambda^k|_1^2+\frac{1}{\eta}\Delta t^2C^2R^2.$$

Let η=ΔtMC. Therefore,

$$(1-\Delta tM)\left(\|\lambda^{k+1}\|^2+|\lambda^{k+1}|_1^2\right)\leq(1+\Delta tM)$$
$$\left(\|\lambda^k\|^2+|\lambda^k|_1^2+M'\Delta tCR^2\right).$$

Where $\bar{M}=\frac{1}{M}$. Applying the weighted H^1 -norm leads to

$$\|\lambda^{k+1}\|_{*,1}^2\leq\frac{1+\Delta tM}{1-\Delta tM}\left(\|\lambda^k\|_{*,1}^2+M'\Delta tCR^2\right)$$
$$\leq\left(\frac{1+\Delta tM}{1-\Delta tM}\right)\left(\left(\left(\frac{1+\Delta tM}{1-\Delta tM}\right)\left(\|\lambda^{k-1}\|_{*,1}^2+M'\Delta tCR^2\right)\right)+\right.$$
$$\left.M'\Delta tCR^2\right)$$
$$=\left(\frac{1+\Delta tM}{1-\Delta tM}\right)^2\|\lambda^{k-1}\|_{*,1}^2+M'\Delta tCR^2\left(\left(\frac{1+\Delta tM}{1-\Delta tM}\right)+\left(\frac{1+\Delta tM}{1-\Delta tM}\right)^2\right)$$
$$\vdots$$
$$\leq\left(\frac{1+\Delta tM}{1-\Delta tM}\right)^{k+1}\|\lambda^0\|_{*,1}^2+M'\Delta tCR^2\left(\left(\frac{1+\Delta tM}{1-\Delta tM}\right)\right.$$
$$\left.+\left(\frac{1+\Delta tM}{1-\Delta tM}\right)^2+\cdots+\left(\frac{1+\Delta tM}{1-\Delta tM}\right)^{k+1}\right).$$

As $\lambda^0=0$

$$\|\lambda^{k+1}\|_{*,1}^2\leq M'\Delta tCR^2\left(\left(\frac{1+\Delta tM}{1-\Delta tM}\right)+\left(\frac{1+\Delta tM}{1-\Delta tM}\right)^2\right.$$
$$\left.+\cdots+\left(\frac{1+\Delta tM}{1-\Delta tM}\right)^{k+1}\right)$$
$$\leq M'\Delta tCR^2\left(\left(\frac{1+\Delta tM}{1-\Delta tM}\right)^{k+1}+\left(\frac{1+\Delta tM}{1-\Delta tM}\right)^{k+1}+\cdots\right.$$
$$\left.+\left(\frac{1+\Delta tM}{1-\Delta tM}\right)^{k+1}\right)$$
$$=(k+1)M'\Delta tCR^2\left(\frac{1+\Delta tM}{1-\Delta tM}\right)^{k+1}$$
$$=TM'CR^2\left(\frac{1+\Delta tM}{1-\Delta tM}\right)^{k+1}.$$

On the other hand,

$$\lim_{k\to\infty}\left(\frac{1+\Delta t M}{1-\Delta t M}\right)^{k+1}=\lim_{k\to\infty}\left(\frac{1+\frac{TM}{k+1}}{1-\frac{TM}{k+1}}\right)^{k+1}=\frac{e^{TM}}{e^{-TM}}=e^{2TM}$$

Therefore,

$$\left\|\lambda^{k+1}\right\|_{*,1}\leq\sqrt{TM'Ce^{2TM}}R=Se^{TM}\sqrt{TM'C}\Delta t$$

That finishes the proof.

BASIC DEFINITIOMS

In this section, some basic concepts and definitions are expressed for the radial basis functions interpolation.

Definition 1. Let $\mathbb{R}^d$ be d-dimensional Euclidean space and $x^* \in \mathbb{R}^d$. A radial basis function is a function which is both continuous and multivariable like $\varphi:\mathbb{R}^d\to\mathbb{R}$ that its value at any point $x\in \mathbb{R}^d$, is dependent on the distance from a certain point $x^* \in \mathbb{R}^d$. This function could be written as $\varphi(r)=\varphi(\|x-x^*\|)$ where $r=\|x-x^*\|$ and $\|.\|$ is the Euclidean norm on R^d. The function φ is an univariable function in r and x^* is a center of RBF φ.

Definition 2. Given the data (x_i, f_i), with $i=1,\ldots,N$, $x_i \in \mathbb{R}^d$ and $f_i \in\mathbb{R}$, the scattered data interpolation problem is defined as finding a smooth function s such that $s(x_i)=f_i$, for $i=1,\ldots,N$. Function s is called an interpolant.

A radial basis function interpolant at centers $X=\{X_1^c , X_2^c ,\ldots, X_N^c \}$ $\mathbb{R}^d$ assumes the following form.

$$u(\mathbf{x})=\sum_{j=1}^{N}\alpha_j\varphi(r_j), \tag{16}$$

Where $r_j=\|X-X_j^c\|$, $\varphi(r)$ is a radial basis function, coefficients α_j ,$j=1,\ldots,N$ are constants to be determined by imposing the interpolation condition $u(X_i)=f_i$ at the set of N centers, X. This enforcement leads to the linear system

$$u=Ba \tag{17}$$

where $a=(\alpha_1,\ldots, \alpha_N)^T$,$u= (u_1,\ldots, u_N)^T$, and B is a N×N matrix called the interpolation matrix or the system matrix with entries

$$b_{i,j}=\phi\left(\left\|\mathbf{x}_i^c-\mathbf{x}_j^c\right\|_2\right), \qquad i,j=1,\ldots,N.$$

Definition 1 suggests that an RBF is independent of the spatial dimension. This property assists to easily transform a multivariable problem into a one-variable problem. This is the superiority of the RBF interpolation scheme to the other classical methods.

Generally, it is possible to fall RBFs into two separate major categories: Infinitely smooth and piecewise smooth; which are given in Table 1 and Table 2, respectively. Infinitely smooth RBFs include a free parameter which is called shape parameter, denoted by ε . Although this parameter can be chosen arbitrarily, a proper choice of its value is necessary. Because in an infinitely smooth RBF interpolation, the value of a shape parameter influences on the accuracy of the scheme (Rippa, 1999).

The MQ RBF is the focus of this paper because of its popularity in applications and its good approximation characteristic.

Table 1 : Infinitely smooth RBFs

Name of function	Definition
Multiquadric (MQ)	$\sqrt{1+\varepsilon^2 r^2}$
Inverse Multiquadric (IMQ)	$\frac{1}{\sqrt{1+\varepsilon^2 r^2}}$
Inverse Quadric (IQ)	$\frac{1}{(1+\varepsilon^2 r^2)}$
Gaussian (GA)	$e^{-\varepsilon^2 r^2}$

Table 2: Piecewise smooth RBFs

Name of function	Definition
Linear	r
Cubic	r^3
Thin Plate Spline (TPS)	$r^2\log(r)$

Piecewise smooth RBFs have algebraic convergence rates (Wendland, 2004). While infinitely smooth RBFs achieve spectral or exponential convergence rates (Jackson,1988; Platte, 2011).

IMPLEMENTATION OF THE METHOD

RBF collocation method is implemented on both linear and nonlinear Sobolev equations in this section.

Linear sobolev equation

In this section, Consider two-dimensional time-dependent linear PDE (3). for discretizing this equation in time, forward finite difference and Crank-Nicolson scheme are applied to temporal parts as follows.

$$\frac{u^{k+1}-u^k}{\Delta t}-\frac{\Delta u^{k+1}-\Delta u^k}{\Delta t}-\frac{\Delta u^{k+1}+\Delta u^k}{2}=f^k. \tag{18}$$

Simplifying this equation yields to

$$u^{k+1}-\left(1+\frac{\Delta t}{2}\right)\Delta u^{k+1}=u^k-\left(1-\frac{\Delta t}{2}\right)\Delta u^k+\Delta t f^k. \tag{19}$$

Discretizing Eq. 19 in space by RBF expansion (16) results in

$$\sum_{j=1}^{N}\alpha_j^{k+1}\varphi(r_j)-\left(1+\frac{\Delta t}{2}\right)\sum_{j=1}^{N}\alpha_j^{k+1}\Delta\varphi(r_j)=$$
$$\sum_{j=1}^{N}\alpha_j^{k}\varphi(r_j)-\left(1-\frac{\Delta t}{2}\right)\sum_{j=1}^{N}\alpha_j^{k}\Delta\varphi(r_j)+\Delta t f^k, \tag{20}$$

or

$$\left[B-\left(1+\frac{\Delta t}{2}\right)M\right]\mathbf{a}^{k+1}=\left[B-\left(1-\frac{\Delta t}{2}\right)M\right]\mathbf{a}^{k}+\Delta t\mathbf{f}^k, \tag{21}$$

Where a^k denotes the RBF expansion coefficients at time level t^k and $M=B_{XX}+B_{yy}$. B_{XX}. and B_{yy} are matrices of second derivative of the system matrix, B, respectively in x , and y.

Let

$$T_L=B-\left(1+\frac{\Delta t}{2}\right)M,$$

and

$$T_R=B-\left(1-\frac{\Delta t}{2}\right)M,$$

Assume that T_1 is non-singular. Then, a^{k+1} is given by

$$\mathbf{a}^{k+1}=T_L^{-1}T_R\mathbf{a}^k+\Delta t T_L^{-1}\mathbf{f}^k, \tag{22}$$

Recalling that $a^k=B^{-1}$ uk , the approximate PDE solution at t^{k+1} is obtained as follows.

$$u^{k+1}=Au^k+F \tag{23}$$

Where $A=BT_L^{-1}T_R B^{-1}$ and $F=\Delta t BT_L^{-1}f^k$.

Nonlinear sobolev equation

Consider the two dimensional time-dependent nonlinear PDE (4). The temporal parts of this equation are discretized within a time period using forward finite difference and Crank-Nicolson scheme as follows.

$$\frac{u^{k+1}-u^k}{\Delta t}-\frac{\Delta u^{k+1}-\Delta u^k}{\Delta t}-\frac{(u_x^2)^{k+1}+(u_x^2)^k}{2}$$
$$-\frac{(uu_{xx})^{k+1}+(uu_{xx})^k}{2}-\frac{(u_y^2)^{k+1}+(u_y^2)^k}{2}$$
$$-\frac{(uu_{yy})^{k+1}+(uu_{yy})^k}{2}+\pi^2\frac{(u^2)^{k+1}+(u^2)^k}{2}=f^k \tag{24}$$

The following formulas are used to approximate the non-linear terms.

$$\frac{(uu_{xx})^{k+1}+(uu_{xx})^k}{2}=\frac{u^{k+1}u_{xx}^k+u^ku_{xx}^{k+1}}{2},$$

$$\frac{(uu_{yy})^{k+1}+(uu_{yy})^k}{2}=\frac{u^{k+1}u_{yy}^k+u^ku_{yy}^{k+1}}{2},$$

$$\frac{(u_x^2)^{k+1}+(u_x^2)^k}{2}=u_x^ku_x^{k+1},$$

$$\frac{\left(u_y^2\right)^{k+1}+\left(u_y^2\right)^k}{2}=u_y^k u_y^{k+1}.$$

$$\frac{\left(u^2\right)^{k+1}+\left(u^2\right)^k}{2}=u^k u^{k+1} \tag{25}$$

when Eq. 25 is substituted in (24) and left and right sides of the equation are multiplied by Δt we have

$$u^{k+1}-u^k-\Delta u^{k+1}+\Delta u^k-\Delta t u_x^k u_x^{k+1}-\frac{\Delta t}{2}\left(u^{k+1}u_{xx}^k+u^k u_{xx}^{k+1}\right)$$
$$-\Delta t u_y^k u_y^{k+1}-\frac{\Delta t}{2}\left(u^{k+1}u_{yy}^k+u^k u_{yy}^{k+1}\right)+\Delta t\pi^2 u^k u^{k+1}=\Delta t f^k, \tag{26}$$

or

$$u^{k+1}-\left(1+\frac{\Delta t}{2}u^k\right)\Delta u^{k+1}-\frac{\Delta t}{2}\left(\Delta u^k-2\pi^2 u^k\right)u^{k+1}$$
$$-\Delta t\left(u_x^k u_x^{k+1}+u_y^k u_y^{k+1}\right)=u^k-\Delta u^k+\Delta t f^k, \tag{27}$$

Eq. 27 is discretized in space by RBF expansion (16) as follows.

$$\left(B-\left(I+\frac{\Delta t}{2}D^k\right)M-\frac{\Delta t}{2}\left(D_{xx}^k+D_{yy}^k-2\pi^2 D^k\right)\right.$$
$$\left.B-\Delta t\left(D_x^k B_x+D_y^k B_y\right)\right)\mathbf{a}^{k+1}=$$
$$\left(B-M\right)\mathbf{a}^k+\Delta t\mathbf{f}^k,$$

Where a^k denotes the RBF expansion coefficients at time level tk, D^k=diag(uk), D_x^k=diag(u_x^k) D_y^k=diag(u_y^k) and M=B_{xx} +B_{yy} .B_{xx} and B_{yy} are matrices of second derivative of the system matrix, B, respectively in x, and y. Bx and are matrices of first derivative of the system matrix, B, respectively in x, and y.

Let

$$T_L=B-\left(I+\frac{\Delta t}{2}D^k\right)M -$$
$$-\frac{\Delta t}{2}\left(D_{xx}^k+D_{yy}^k-2\pi^2 D^k\right)B-\Delta t\left(D_x^k B_x+D_y^k B_y\right),$$

and

T_R= B-M

By the assumption T_L is non-singular,a^{k+1} is given by (22). Finally, the approximate PDE solution at t^{k+1} namely u^{k+1} is obtained by (23).

NUMERICAL EXPERIMENTS

In this section, some problems as examples are numerically solved as instances for the purpose of verifying the ability of the proposed method with regards to Sobolev equations. Among all of the RBFs, MQ, the most popular RBF, is used in computations due to the rapid convergent rate. The domain Ω is chosen as the unit region, i.e. $\Omega=[0,1]^2$. In order to test the accuracy, two error norms, L_∞ and L_2 defined as follows are computed.

$$L_2=\sqrt{\sum_{i=1}^N\sum_{j=1}^N\left(\tilde{u}_{i,j}-u_{i,j}\right)^2},\quad L_\infty=\max_{1\le i,j\le N}\left|\tilde{u}_{i,j}-u_{i,j}\right|,$$

Where $\tilde{u}$ and u denote the approximate and exact solutions, respectively.

Example 1. Take following equation:

$$u_t-\Delta u_t-\Delta u=f(x,y,t),\quad t>0,\quad (x,y)\in\Omega.$$

with two cases of exact solutions:

Case I: $u(x,y,t)=\exp(-t)\sin(\pi x)\sin(\pi y),$

Case II: $u(x,y,t)=\exp(x-y-t)\sin(\pi x)\sin(\pi y).$

Initial and boundary conditions and the source term f(x,y,t) are taken from the exact solution.

The computed error norms t=1 are reported in Table 3. The results are weighed against those addressed in (Haq et al., 2019), which shows that the suggested method has higher degree of accuracy. Besides, they are compared with the work addressed in (Oruç, 2018). CPU time is also computed and given in this table indicating the efficiency of the method. It should be noted that, for every number of nodes, N×N, value of shape parameter varies. Here, the optimal value of shape parameter is determined by trial and error. Numerical and exact solutions and absolute error for cases I and II are depicted in Figs. 1 and 2, respectively. The figures show that exact and approximate solutions fit together.

Table 3: Error norms of Example 1 at t =1, Δt=0.01.

	N× N	L_∞	L_2	CPU time(s)	L_∞ (Haq et al., 2019)	L_∞ (Oruc., 2018: 40)	Shape parameter
case I	4×4	5.5840e-08	1.1168e-07	0.0916	7.4767e-03	4.2735e-04	9.1
	8×8	3.8312e-05	1.3227e-04	0.1118	2.1515e-03	1.4656e-04	10.5
	16×16	5.1334e-05	3.7155e-04	0.2320	4.9543e-04	1.2647e-05	9.6
	32×32	5.6155e-05	8.3930e-04	3.8208	5.9549e-05	3.7398e-06	9.8
case II	4×4	2.4997e-06	4.7833e-06	0.0953	6.27731e-03	3.6864e-03	30000
	8×8	2.9800e-06	1.1474e-05	0.1100	1.7252e-03	6.1921e-04	170000
	16×16	3.0726e-06	2.5068e-05	0.2341	3.8947e-04	7.7847e-05	800000
	32×32	3.1011e-06	5.1428e-05	3.3932	4.9551e-05	3.0337e-05	4000000

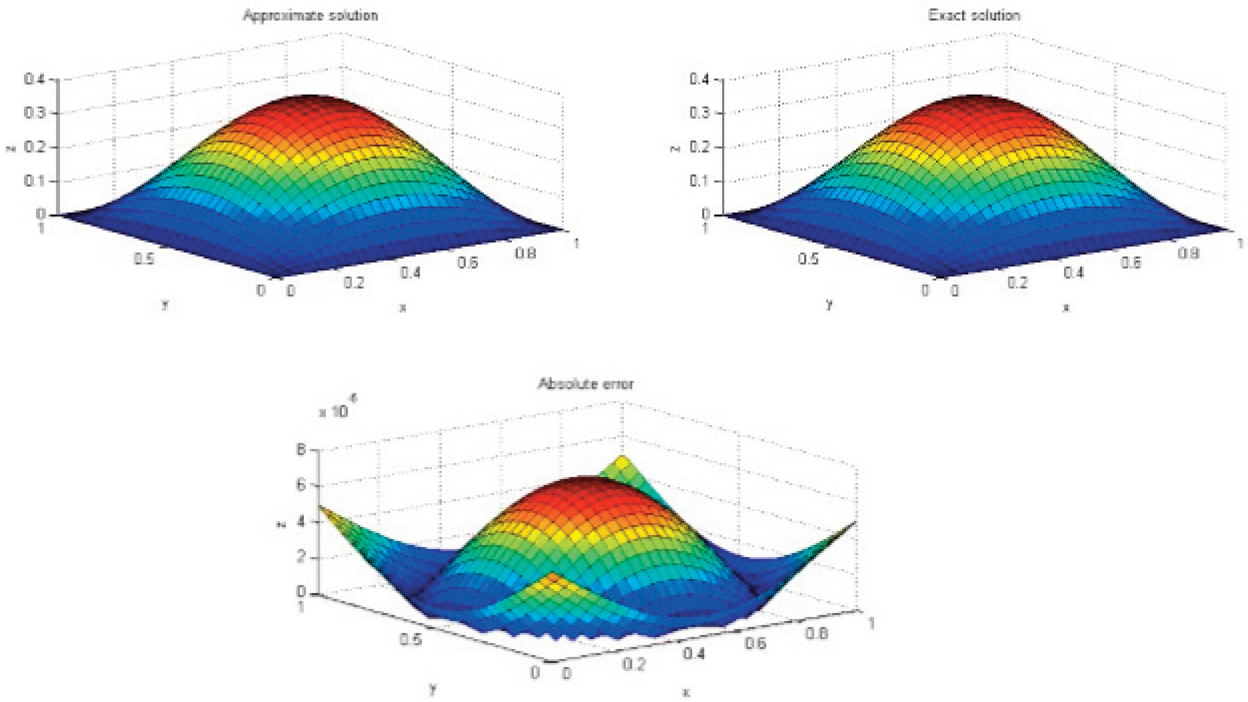

Fig .1. Approximate and exact solutions and absolute error graphs of Example 1 (case I) at t=1, N=32 and Δt=0.01

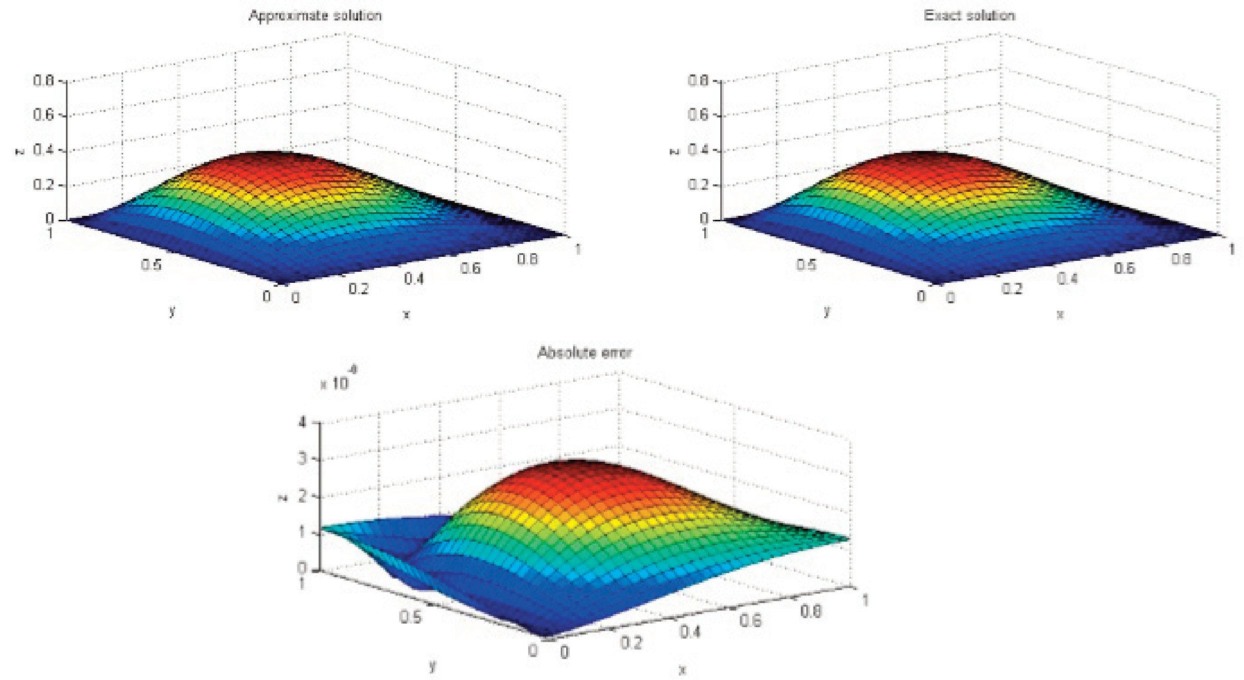

Fig. 2. Approximate and exact solutions and absolute error graphs of Example 1 (case II) at t=1, N=32 and Δt=0.01

Example 2. Consider the nonlinear Sobolev equation

$$u_t - \Delta u_t - \nabla.(u\nabla u) + \pi^2 u^2 = f(x,y,t),$$

$$t > 0,\ (x,y) \in \Omega.$$

with conditions

$$\begin{cases} u(x,y,0) = \sin(\pi x)\sin(\pi y), & (x,y)\in\Omega, \\ u(x,y,t) = \exp(t)\sin(\pi x)\sin(\pi y), \end{cases}$$

$$(x,y)\in\partial\Omega,\quad t>0.$$

and the source term

$$f(x,y,t) = \left(1+2\pi^2\right)\sin(\pi x)\sin(\pi y)\exp(t)$$
$$-\pi^2\exp(2t)\left(\cos^2(\pi x)\sin^2(\pi y)\right.$$
$$\left.-\cos^2(\pi y)\sin^2(\pi x) + 3\sin^2(\pi y)\sin^2(\pi x)\right)$$

with exact solution $u(x,y,t)=\sin(\pi x)\sin(\pi y)\exp(t)$.

The error norms at t=0.2 and t=1 are calculated and presented in Tables 4 and 5, respectively. The computed errors are compared with those in (Haq et al., 2019). Comparisons show that better approximations could be obtained by the proposed method, except for the case N=32 at t=1 where the infinite norm is larger than the corresponding value in (Haq et al., 2019). Fig. 3 illustrates the exact and approximate solutions and also absolute error. It is obviously observed that the approximate solutions are in good agreement with exact ones. This demonstrates that the method works effectively in non-linear problems.

CONCLUSION

In this study, an RBF meshless method has been suggested for solving linear Sobolev and non-linear Sobolev equations numerically. First, the finite difference formula and Crank Nicolson technique are implemented to discretized the time derivative. As a result, a time semi-discrete formula was obtained. The energy method was used to prove convergence of the time semi-discrete scheme. After that, a fully discrete formula was achieved by approximating the spatial terms by two-dimensional RBF interpolation. The numerical experiments suggest high degree of accuracy. By referring to the obtained results, it can be concluded that the results of proposed scheme are better than those formerly presented in the literature.

Table 4: Error norms of Example 2 at t =1, Δt=0.001

N× N	L_∞	L_2	L_∞ (Haq et al., 2019)	L_2 (Haq et al., 2019)	Shape parameter
4×4	5.2245e-04	1.5000e-03	2.4122e-02	5.9255e-02	0.6
8×8	1.4763e-04	5.8277e-04	6.6546e-03	2.9818e-02	1.4
16×16	1.6157e-04	1.2000e-03	1.6374e-03	1.4401e-02	3.7
32×32	1.7230e-04	2.5000e-03	3.4286e-04	6.0619e-03	15

Table 5: Error norms of Example 2 at t =1, Δt=0.001

N× N	L_∞	L_2	L_∞ (Haq et al., 2019)	L_2 (Haq et al., 2019)	Shape parameter
4×4	1.5000e-03	2.4000e-03	5.8662e-02	1.7284e-01	0.7
8×8	1.0000e-03	5.1000e-03	1.3147e-02	7.5881e-02	1.6
16×16	1.1000e-03	8.4000e-03	3.0030e-03	3.4724e-02	3.8
32×32	1.2000e-03	9.0000e-03	5.5208e-04	1.3063e-02	15.5

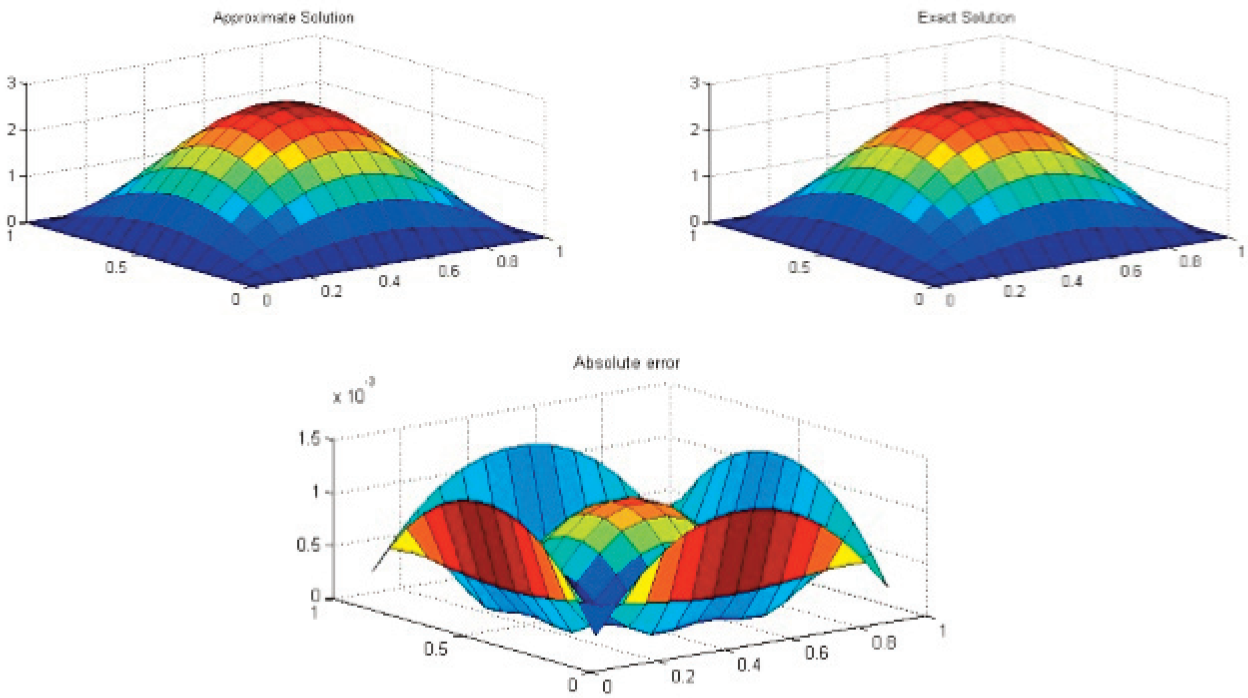

Fig. 3. Graphs of approximate and exact solutions and absolute error of Example 2 at t =1, Δt=0.001

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