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Optimization of Solution Stiff Differential Equations Using MHAM and RSK Methods

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Abstract

In this paper, a nonlinear stiff differential equation is solved by using the Rosenbrock iterative method, modified homotpy analysis method and power series method. The approximate solution of this equation is calculated in the form of series which its components are computed by applying a recursive relation. Some numerical examples are studied to demonstrate the accuracy of the presented methods.

Keywords:
optimization
Stiff differential equations
Power series method
MHAM method
RSK method

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INTRODUCTION

The goal of this paper is to implement the power seies method, MHAM method and RSK method the stiff differential equations, which are often encounter in physical and electrical circuits problems. This equation models long wave in a nonlinear dispersive system. So some numerical techniques prefer to consider overcoming this type of problem. In recent years, much research has been focused on numerical solution of the system of differential equations by using technique of Adomian's decomposition methods. In the literature, the Adomian's decomposition is used to find approximate numerical and analytic solutions of a wide class of linear or nonlinear differential equations (Adomian 1994,1998; Kaya, 2004; Repaci, 1990). The basic purpose of this paper is to illustrate advantages of using Rosenbrock method over the other methods namely Padé approximation method and Adomian's decomposition method in terms of numerical comparisons. However, it shall be considered the Rosenbrock method (Wanner & Hairer, 1996; Rosenbrock 1960; Schmitt & Weiner 2004; Zhao et al., 2005), Padé approximation method (Baker and Morris 1996), and the Adomian's decomposition method in order to solve the system of first order differential equations. A lot of works have been done in order to find numerical solution of this equation. For example (Adomian, 1994; Behiry et al.,2007; Perturbation, 2003; Corliss & Chang, 1982; Wanner & Hairer, 1996; Wahlbin, 1974; Kaya, 2004; Repaci, 1990).

ANALYSIS OF THE NUMERICAL METHODS

Power series method

Suppose k is a positive integer and $f_1, f_2, \ldots, f_k$ are k real continuous functions defined on some domain . To obtain k differentiable functions $y_1, y_2, \ldots, y_k$ defined on the interval I such that $(t, y_1(t), y_2(t), \ldots, y_k(t)) \in G$ for $t \in I$. Let us consider the problems in the following system of ordinary differential equations:

$$\frac{dy_i(t)}{dt} = f_i(t, y_1(t), y_2(t), \ldots, y_k(t)) \qquad (1)$$

$$y_i(t)\big|_{t=0} = \beta_i$$

where β_i is a specified constant vector, $y_i(t)$ is the solution vector for i=$1,2,\ldots,k$. In the power series method, (1) is approximated by the operators in the form: $Lyi(t)=fi(t, y1(t), y2(t), \ldots, yk(t))$ where L is the first order operator defined by $l=d/dt$ and $i=1,2,\ldots,k$. Assuming the inverse operator of L is $L-1$ which is invertible and denoted by $L-1(0)=\int t0t(0)dt$, then applying $L-1$ to $lyi(t)$ yields

$$L^{-1}Ly_i(t) = L^{-1}f_i(t, y_1(t), y_2(t), \ldots, y_k(t))$$

where i=1,2,…,k .. Thus $y_i(t)=y_i(t_0)+L^{-1} f_i(t, y_1(t), y_2(t), \ldots, y_k(t))$. Hence the power series method consists of representing $y_i(t)$ in the series form given by

$$y_i(t) = \sum_{n=0}^{\infty} f_{i,n}(t, y_1(t), y_2(t), \ldots, y_k(t))$$

where the components $y_{i,n}$,$n \geq 1$ and $i=1,2,\ldots,k$ can be computed readily in a recursive manner. Then the series solution is obtained as

$$y_i(t) = y_{i,0}(t) + \sum_{n=1}^{\infty}\{L^{-1}f_{i,n}(t, y_1(t), y_2(t), \ldots, y_k(t)),$$

MHAM method

The MHAM method is a zeroth order approximating search algorithm that does not require any derivatives of the desired function, just only simple evaluations of the objective function are used. This method is particularly well suited when the objective function does not require a great deal of computing power. In such a case, it is useless to use very complicated optimization algorithms which are needed to loose more spare time in the optimization calculations, instead of making a little bit more evaluations of the objective function that will lead, at the end, to a shorter calculation time. The numerical solution of systems of ordinary differential equation of the form, $y'(x)=f(y(x)), y(x_0)=y_0$. The MHAM method searches to look for the solution of the form $y_{n+1}=y_n+h\sum_{i=1}^{s} c_i k_i$ where the corrections k_i are found by solving linear equations that generalize the structure in

$$(1-\gamma hf')k_i = hf\left(y_0 + \sum_{j=1}^{i-1} \alpha_{ij}k_j\right) + hf' \sum_{j=1}^{k-1} \gamma_{ij}k_j \qquad (2)$$

$$i = 1, \ldots, s$$

Here Jacobi matrix is denoted by $f' = (\frac{\partial f}{\partial y})(y_n)$ The coefficients y, c_i, α_{ij} and y_{ij} are fixed constants independent of the problem.

RKS Method

Suppose that $f(x)$ has a Maclaurin expansion about the zero such as $f(x)=\sum_{i=0}^{\infty} c_i x^i$ where $c_i=0,1,2,\ldots$ .The RKS is a rational function (Zhao et al., 2005).

$$P_f\,[L/M]=\frac{\alpha_0+\alpha_1 x+\alpha_2 x^2+\cdots+\alpha_L x^L}{\beta_0+\beta_1 x+\beta_2 x^2+\cdots+\beta_M x^M} \tag{3}$$

which has a Maclaurin expansion which agrees with power series as far as possible. Notice that P_f has a quotient of polynomials, which has $L+1$ terms numerator coefficients and $M+1$ denominator coefficients. There is a more or less irrelevant common factor between them, and for definiteness we take $\beta_0=1$. This choice turns out to be an essential part of the precise definition and P_f is here conventional notation with this choice for β_0. So there are $L+1$. independent numerator coefficients and M independent denominator coefficients which makes $L+M+1$. unknown coefficients in all. This number suggests that normally the $[[L/M]]$ must to fit the power series through the orders $1, x, x^2, x^3, \ldots, x(^{L+M})$ in the notion of formal power series

$$\sum_{i=0}^{\infty} c_i x^i=\frac{\alpha_0+\alpha_1 x+\cdots+\alpha_L x^L}{\beta_0+\beta_1 x+\cdots+\beta_M x^M}+O(x^{L+M+1}) \tag{4}$$

Substituting (4) in (3), then

$$(\beta_0+\beta_1 x+\cdots+\beta_M x^M)(c_0+c_1 x+\cdots)=\alpha_0+\alpha_1 x+\alpha_2 x^2+\cdots+\alpha_L x^L+O(x^{L+M+1}) \tag{5}$$

Equating the coefficients of $x^{L-1}, x^{L-2}, \ldots, x^{L-M}$ in (5), then

$$\begin{aligned}
&\beta_M c_{L-M+1}+\beta_{M-1} c_{L-M+2}+\cdots+\beta_0 c_{L+1}=\\
&\beta_M c_{L-M+2}+\beta_{M-1} c_{L-M+3}+\cdots+\beta_0 c_{L+2}=0\\
&\qquad\qquad\ldots\\
&\beta_M c_L+\beta_{M-1} c_{L+1}+\cdots+\beta_0 c_{L+M}=0
\end{aligned} \tag{6}$$

are obtained. If $J<0$, then $c_j=0$ is defined for consistency. Since $\beta_0=1$, the system which is given by (6) become a set of M linear equations for unknown denominator coefficients as follows:

$$\begin{bmatrix} c_{L-M+1} & c_{L-M+2} & c_{L-M+3} & \cdots & c_L \\ c_{L-M+2} & c_{L-M+3} & c_{L-M+4} & \cdots & c_{L+1} \\ c_{L-M+3} & c_{L-M+4} & c_{L-M+5} & \cdots & c_{L+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{L-M+1} & c_{L-M+1} & c_{L-M+1} & \cdots & c_{L-M+1} \end{bmatrix}\begin{bmatrix} \beta_M \\ \beta_{M-1} \\ \beta_{M-2} \\ \\ \beta_1 \end{bmatrix}=-\begin{bmatrix} c_{L+1} \\ c_{L+2} \\ c_{L+3} \\ \\ c_{L+M} \end{bmatrix}$$

Thus β_I can be calculated easily. The numerator coefficients $\alpha_0, \alpha_1, \ldots, \alpha_L$ follow immediately from (5) by equating the coefficients $1, x,'' x^2$ x^3 ,, …, x^L, then $\alpha_0 = c_0$, , $\alpha_1= c_1+\beta_1 c_0$,$\alpha_2=c_2+\beta_1 c_1+\beta_2 c_0$,…, , $\alpha_L=c_L+\sum_{i=1}^{\min(L,M)} \beta_i c_{L-i}$ are obtained.

NUMERICAL EXAMPLES

Example 1. Let us to consider the following differential equation as follows (Rosenbrock, 1960):

$$\begin{aligned} y_1' &= -10y_1+6y_2 \\ y_2' &= 13.5y_1-10y_2 \end{aligned} \tag{7}$$

with initial conditions

$y_1(0)=4e/3 \quad y_2(0)=0 \quad 0\le x\le 1,$

The exact solution of this problem is

$$y_1(x)=\tfrac{2}{3}e(e^{-x}+e^{-19x}),\ y_2(x)=e(e^{-x}-e^{-19x}).$$

where initial conditions

$$\begin{aligned} &y_1(0)=4e/3,\ y_2(0)=0,\ y_{1,0}=y_1(0),\ y_{2,0}=y_2(0). \\ &y_1(x)=y_{0,1}+\varepsilon_1 x=4e/3+\varepsilon_1 x \\ &y_2(x)=y_{0,2}+\varepsilon_2 x=\varepsilon_2 x \end{aligned} \tag{8}$$

Re-

$$\begin{aligned} y_1(x)&=\frac{4e}{3}-\frac{40e}{3}x+\frac{362e}{3}x^2-\frac{6860e}{9}x^3+\frac{65161e}{18}x^4-\frac{123805e}{9}x^6-\frac{44693587e}{378}x^7 \\ &+\frac{8491781521e}{30240}x^8-\frac{2304912127e}{3888}x^9+\frac{3065533128901e}{2721600}x^{10}+O(x^{11}) \end{aligned}$$

$$\begin{aligned} y_2(x)&=18ex-180ex^2+1143ex^3-5430ex^4+\frac{412683e}{20}x^5-\frac{130683e}{2}x^6 \\ &+\frac{49659541e}{280}x^7-\frac{11794141e}{28}x^8+\frac{1991899369e}{2240}x^9-\frac{3406147921e}{2016}x^{10}+O(x^{11}) \end{aligned}$$

$$\bar{y}_1[\tfrac{5}{5}](x)=\frac{3.624375770-2.057914264x+129.6082474x^2-73.74538199x^3+348.9539654x^4-203.1856237x^5}{1+9.432201738x+39.58217461x^2+93.52709832x^3+126.3975849x^4+79.78502234x^5}$$

$$\bar{y}_2[\tfrac{5}{5}](x)=\frac{48.92907290x-20.92483208x^2+413.5655807x^3-169.3803363x^4+136.1017021x^5}{1+9.572343585x+40.67578447x^2+97.11894158x^3+132.3740297x^4+84.12285991x^5}$$

$$y_1=\frac{4e}{3}/-\frac{40e}{3}x+\frac{362e}{3}x^2-\frac{6890e}{9}x^3+\frac{65161e}{18}x^4-\frac{123805e}{9}x^5+\frac{23522941e}{540}x^6-\cdots$$

$$y_2=18ex-180ex^2+1143ex^3-5430ex^4+\frac{412683}{20}ex^5-\frac{130683}{2}ex^6+\cdots$$

garding to these approximate solutions, it is illustrated with Tables in which the errors can be seen as follows:

X	Exact	MHAM	RKS	Abs. Err. MHAM	RKS. Err. Abs
0.0	0	-9.35E-08	0.0011905	0	$\le 01^{\,2-}$
0.1	0.1617584	0.1617584	0.1619634	$\le 01^{\,6-}$	$\le 01^{\,3-}$
0.2	0.3512394	0.3512394	0.3512605	$\le 01^{\,6-}$	$\le 01^{\,4-}$
0.3	0.5630688	0.5630689	0.5630697	$\le 01^{\,6-}$	$\le 01^{\,5-}$
0.4	0.7859815	0.7859816	0.7859815	$\le 01^{\,6-}$	$\le 01^{\,8-}$
0.5	1	1.0000001	1	$\le 01^{\,7-}$	0
0.6	1.1725466	1.1725469	1.1725467	$\le 01^{\,5-}$	$\le 01^{\,7-}$
0.7	1.2531327	1.253133	1.2531337	$\le 01^{\,5-}$	$\le 01^{\,5-}$
0.8	1.166156	1.1661561	1.1661811	$\le 01^{\,6-}$	$\le 01^{\,4-}$
0.9	0.8011947	0.8011947	0.8014527	$\le 01^{\,6-}$	$\le 01^{\,3-}$
1	0	7.203E-07	0.0015873	$\le 01^{\,5-}$	$\le 01^{\,2-}$
E_∞				$\le 01^{\,5-}$	$\le 01^{\,2-}$

Example 2. We consider the system as follows:

$$y_1' = 32y_1 + 66y_2 + 2x/3 + 2/3$$
$$y_2' = -66y_1 - 133y_2 - x/3 - 1/3 \qquad (9)$$

with initial conditions

$$y_1(0) = 1/3 \quad y_2(0) = 1/3$$

The exact solution of this problem is

$$y_1(x) = \tfrac{2}{3}x + \tfrac{2}{3}e^{-x} - \tfrac{1}{3}e^{-100x}$$
$$y_2(x) = -\tfrac{1}{3}x - \tfrac{1}{3}e^{-x} + \tfrac{2}{3}e^{-100x}$$

$$\bar{y}_1[5/5](x) = \frac{0.3333333333 + 49.92473248x + 359.7697630x^2 + 13875.94938x^3 - 30787.42799x^4 + 340369.8101x^5}{1 + 49.77419745x + 1100.889544x^2 + 13693.77341x^3 + 97315.37512x^4 + 322673.3475x^5}$$

$$\bar{y}_2[5/5](x) = \frac{0.3333333333 - 50.01912345x + 373.1626781x^2 - 13931.77445x^3 + 33749.15668x^4 - 337628.9958x^5}{1 + 49.94262964x + 1108.513962x^2 + 13839.31067x^3 + 98725.83778x^4 + 328651.2020x^5}$$

$$y_1 = \frac{1}{3} + \frac{100}{3}x - \frac{4999}{3}x^2 + \frac{499999}{9}x^3 - \frac{49999999}{36}x^4 + \frac{4999999999}{180}x^5 - \frac{499999999999}{1080}x^6 + \cdots$$

$$y_2 = \frac{1}{3} - \frac{200}{3}x + \frac{1999}{6}x^2 - \frac{199999}{18}x^3 + \frac{19999999}{72}x^4 - \frac{1999999999}{360}x^5 + \frac{199999999999}{2160}x^6 - \cdots$$

X	**Exact Sol**	**MHAM**	**RKS**	**Abs. Err. MHAM**	**RKS. Err. Abs**
0.0	0	-1E-08	-0.0001984	:::; 10^{-8}	:::; 10^{-3}
0.1	0.2582482	0.2582482	0.2582214	:::; 10^{-7}	:::; 10^{-4}
0.2	0.3613712	0.3613712	0.3613692	:::; 10^{-7}	:::; 10^{-4}
0.3	0.3271114	0.3271114	0.3271114	:::; 10^{-6}	:::; 10^{-6}
0.4	0.1907226	0.1907226	0.1907226	:::; 10^{-7}	:::; 10^{-9}
0.5	0	2E-10	0	:::; 10^{-9}	0
0.6	-0.1907226	-0.1907226	-0.1907226	:::; 10^{-7}	:::; 10^{-9}
0.7	-0.3271114	-0.3271114	-0.3271114	:::; 10^{-6}	$< 10^{-6}$
0.8	-0.3613712	-0.3613712	-0.3613692	:::; 10^{-7}	:::; 10^{-4}
0.9	-0.2582482	-0.2582482	-0.2582214	:::; 10^{-7}	:::; 10^{-4}
1	0	1E-08	0.0001984	:::; 10^{-8}	:::; 10^{-3}
E_∞				$< 10^{-6}$	:::; 10^{-3}

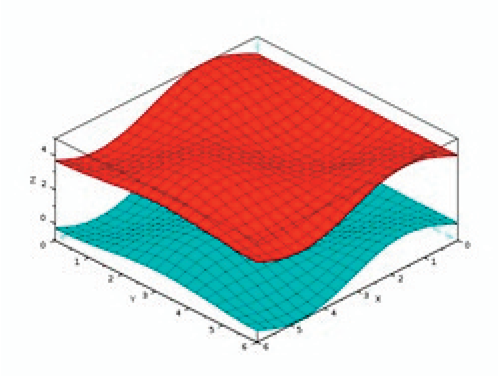

Fig. 1

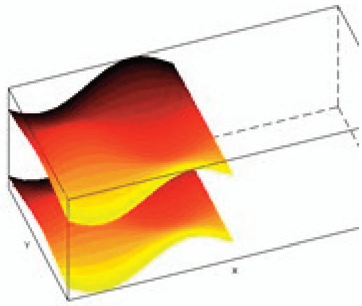

Fig. 2

CONCLUSION

The RKS and MHAM methods have been shown to solve effectively, easily and accurately a large class of nonlinear stiff problems with the approximations which convergent are rapidly to exact solutions. In this work, the RKS and MHAM methods have been successfully employed to obtain the approximate analytical solution of the stiff differential equations. The solution is rapidly convergent by using this methods.

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