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Exact Solutions of Fuzzy Linear System Equations

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A B S T R A C T

Systems of simulations linear equations play major role in various areas such as mathematics, statistics, and social sciences. Since in many applications, at least some of the system's parameters and measurements are represented by fuzzy rather than crisp numbers, therefore, it is important to develop mathematical models and numerical procedures that would appropriately treat general fuzzy linear systems and solve them. In this paper, a new method based on fuzzy operations approach for solving Fuzzy Linear System (FLS) is introduced. The related theorems are proved in details. Finally, the proposed method is illustrated by solving two numerical examples.

1. Introduction

There are many linear equation systems in many areas of science and engineering. In the linear equation systems, exact numerical data might be unrealistic, but there could be considered uncertain data as more aspects of a real word problem.

Recently, fuzzy systems are used to study a variety of problems ranging from fuzzy topological spaces to control chaotic systems [5, 14, 17], fuzzy metric spaces [7], fuzzy differential equations [1, 2], fuzzy linear systems [3, 4, 20] and particle physics [9, 11- 14]. The concept of fuzzy numbers and arithmetic operations with this numbers were first introduced and investigated by Zadeh [6] and others. All of which observed the fuzzy number as a collection of α−levels, $0 \leq \alpha \leq 1$, [19]. Additional related material can be found in [8, 15, 16].

Any linear system representing real-world situations involves a lot of parameters whose values are assigned by experts, and in the conventional approach, they are required to fix an exact value to the aforementioned parameters. However, both experts and the decision-maker (DM) frequently do not precisely know the value of those parameters. If exact values are suggested these are only statistical inference from past data and their stability is doubtful, so the parameters of the problem are usually defined by the DM in a un- certain way or by means of language statement parameters. Therefore, it is useful to consider the knowledge of experts about the parameters as fuzzy data. This paper considers FLS problems $A\tilde{X} = \tilde{b}$ where $\tilde{b}$ and then $\tilde{X}$ are fuzzy

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numbers. The aim of this paper is to introduce a strategy for solving FLS based on ranking method, and fuzziness index function, [18]. In section 2, some notation and basic definitions related to subjects are explained. In Section 3, we show how we use our method to solve FLS in order to define the fuzziness index function. Finally in Section 4, we solve a numerical example. Finally, conclusion is drawn in Section 5.

2. Notation and Basic Definitions

A fuzzy set $\tilde{A}$ is a pair $(A^l(\alpha), A^r(\alpha))$; $0 \le \alpha \le 1$ which $A^l(\alpha)$ and $A^r(\alpha)$ satisfy the following requirements:

a: $A^l(\alpha)$ is a bounded monotonic increasing left continuous function;

b: $A^r(\alpha)$ is a bounded monotonic decreasing left continuous function;

c: $A^l(\alpha) \le A^r(\alpha)$; $0 \le \alpha \le 1$.

A fuzzy number $\tilde{A}$ is nonnegative if $A^l(\alpha) \ge 0$.

For arbitrary $\tilde{A} = (A^l(\alpha), A^r(\alpha))$ and B˜ = (Bl(α), Br(α)) and we define addition $\tilde{A}+\tilde{B}$ and scalar multiplication by $k \in \mathbf{R}$ as addition and multiplication as defined by Eqs. (1) and (2) is denoted by E1 and is a convex cone.

$$\tilde{A}+\tilde{B} = (A^l(\alpha)+B^l(\alpha), A^r(\alpha)+B^r(\alpha)) \tag{1}$$

$$k\tilde{A} = \begin{cases} (kA^l(\alpha), kA^r(\alpha)), & k \ge 0 \\ (kA^r(\alpha), kA^l(\alpha)), & k < 0 \end{cases} \tag{2}$$

The collection of all the fuzzy numbers with addition and multiplication defined by Eqs. (1) and (2) is denoted by E^1 and it is a convex cone.

Definition 1 [18]. For arbitrary $\tilde{A} = (A^l(\alpha), A^r(\alpha))$ the number

$$A^0 = \frac{1}{2}(A^l(1) + A^r(1)) \tag{3}$$

is said to be a location index number of $\tilde{A}$ and two nonincreasing left continuous functions

$$A^{l*}(\alpha) = A^0 - A^l(\alpha), \qquad A^{r*}(\alpha) = A^r(\alpha) - A^0 \tag{4}$$

are called the left fuzziness index function and the right fuzziness index function, respectively.

According to Definition 1, every fuzzy number $\tilde{A}$ can be represented by $(A^0, A^{l*}(\alpha), A^{r*}(\alpha))$.

Theorem 1. [1], Define

$$j : E^1 \to R \times L \times L;\ \tilde{A} \mapsto (A^0, A^{l*}(\alpha), A^{r*}(\alpha)). \tag{5}$$

Then, j is a bijection, i.e., j is a *one-to-one* mapping, where $R \times L \times L$ is the Cartesian product and $L = \{h \mid h : [0,1] \to [0,\infty)$ is nondecreasing and left continuous.

Definition 2. Suppose $\tilde{A} = (A^l(\alpha), A^r(\alpha))$ and $\tilde{B} = (B^l(\alpha), B^r(\alpha))$ then

$$\tilde{A} = \tilde{B} \Leftrightarrow \{A^0 = B^0, A^{l*}(\alpha) = B^{l*}(\alpha), A^{r*}(\alpha) = B^{r*}(\alpha),\ 0 \le \alpha \le 1\}. \tag{6}$$

Proposition 1. For arbitrary fuzzy number $\tilde{A} = (A^0, A^{l*}(\alpha), A^{r*}(\alpha))$ and $\tilde{B} = (B^0, B^{l*}(\alpha), B^{r*}(\alpha))$ and $k \in \mathbf{R}$ from definition 1,

$$\tilde{A}+\tilde{B} = (A^0 + B^0, A^{l*}(\alpha) + B^{l*}(\alpha), A^{r*}(\alpha) + B^{r*}(\alpha)) \tag{7}$$

$$k\tilde{A} = \begin{cases} (kA^0, kA^{l*}(\alpha), kA^{r*}(\alpha)), & k \ge 0 \\ (kA^0, -kA^{r*}(\alpha), -kA^{l*}(\alpha)), & k < 0 \end{cases} \tag{8}$$

Proof. Let, $\tilde{A} = (A^l(\alpha), A^r(\alpha))$ and $\tilde{B} = (B^l(\alpha), B^r(\alpha))$ then

$$(A+B)^0=\frac{1}{2}((A+B)^l(1)+(A+B)^r(1))=\frac{1}{2}(A^l(1)+B^l(1)+A^r(1)+B^r(1))= \\ \frac{1}{2}(A^l(1)+A^r(1))+\frac{1}{2}(B^l(1)+B^r(1))=A^0+B^0 \quad (9)$$

$$(A+B)^{l*}=(A+B)^0-(A+B)^l(\alpha)=A^0+B^0-(A^l(\alpha)+B^l(\alpha))=A^0-A^l(\alpha)+B^0-B^l(\alpha)= \\ A^{l*}(\alpha)+B^{l*}(\alpha). \quad (10)$$

The proof of $(A+B)^{r*}=A^{r*}(\alpha)+B^{r*}(\alpha)$ is similar.

If $k\geq 0$

$$(kA)^0=\frac{1}{2}((kA)^l(1)+(kA)^r(1))=\frac{1}{2}(kA^l(1)+kA^r(1))=k\frac{1}{2}(A^l(1)+A^r(1))=kA^0 \quad (11)$$

$$(kA)^{l*}=(kA)^0-(kA)^l(\alpha)=kA^0-kA^l(\alpha)=k(A^0-A^l(\alpha))=kA^{l*}(\alpha) \quad (12)$$

In case $k<0$, the proof is similar. Also it is clear that $(kA)^{r*}=kA^{r*}(\alpha)$.

3. Fuzzy Linear System

Consider the following fuzzy linear system

$$\sum_{j=1}^{n}a_{ij}\tilde{x}_j=\tilde{b}_i \quad ;i=1,\ldots,m \quad (13)$$

where $a_{ij}\in\mathbb{R}$ and $\tilde{b}_i=(b_i^0,b_i^{l*}(\alpha),b_i^{r*}(\alpha))$ and $\tilde{x}_j=(x_j^0,x_j^{l*}(\alpha),x_j^{r*}(\alpha))$ are fuzzy numbers for $i=1,\ldots,m$ *and* $j=1,\ldots,n$.

Model (13) is rewriten as follows:

$$\sum_{a_{ij}<0}a_{ij}\tilde{x}_j+\sum_{a_{ij}>0}a_{ij}\tilde{x}_j=\tilde{b}_i \quad ;i=1,\ldots,m. \quad (14)$$

Regarding Proposition 1 and Definition 2, the model (14) is given as follows:

$$(\sum_{a_{ij}<0}a_{ij}x_j^0+\sum_{a_{ij}>0}a_{ij}x_j^0,\sum_{a_{ij}>0}a_{ij}x_j^{l*}(\alpha)-\sum_{a_{ij}<0}a_{ij}x_j^{r*}(\alpha),\ \sum_{a_{ij}>0}a_{ij}x_j^{r*}(\alpha)-\sum_{a_{ij}<0}a_{ij}x_j^{l*}(\alpha))=(b_i^0,b_i^{l*}(\alpha),b_i^{r*}(\alpha)) \quad (15)$$

for $i=1,\ldots,m$.

For solving model (15), first we solve the following location model.

$$location\,model:\ \sum_{a_{ij}<0}a_{ij}x_j^0+\sum_{a_{ij}>0}a_{ij}x_j^0=b_i^0 \quad i=1,\ldots,m. \quad (16)$$

If the location model is infeasible, then the model (17) will be infeasible, otherwise we solve the following fuzziness model:

$$\begin{aligned} max \quad & \sum_{j=1}^{n}x_j^{l*}(\alpha)+\sum_{j=1}^{n}x_j^{r*}(\alpha) \\ s.t \quad & \sum_{j=1}^{n}\beta_{ij}x_j^{l*}(\alpha)-\sum_{j=1}^{n}\gamma_{ij}x_j^{r*}(\alpha)=b_i^{l*}(\alpha) \quad i=1,\ldots,m \\ & \sum_{j=1}^{n}\beta_{ij}x_j^{r*}(\alpha)-\sum_{j=1}^{n}\gamma_{ij}x_j^{l*}(\alpha)=b_i^{r*}(\alpha) \quad i=1,\ldots,m \\ & x_j^{l*}(\alpha)\geq 0,x_j^{r*}(\alpha)\geq 0 \end{aligned} \quad (17)$$

where

$$\beta_{ij}=\begin{cases} a_{ij} & a_{ij}>0 \\ 0 & otherewise \end{cases} \quad and \quad \gamma_{ij}=\begin{cases} 0 & a_{ij}\geq 0 \\ a_{ij} & otherewise \end{cases} \tag{18}$$

We say if fuzziness model (17) is feasible then $\tilde{x}_j=(x_j^0, x_j^{l*}(\alpha), x_j^{r*}(\alpha));\ j=1,...n$ is an solutions of the model (13) for any $\alpha\in[0,1]$.

Theorem 2. The $\tilde{x}_j,\ j=1,...n$ are fuzzy numbers.

Proof. Regarding to definition 1, it is sufficient to show $x_j^{l*}(\alpha)$ and $x_j^{r*}(\alpha)$ are nondecreasing left continuous functions on $\alpha\in[0,1]$, which from the models (17) is clear.

4. Numerical Examples

To illustrate our method, we consider following examples.

Example 1. Consider the following linear system.

$$\begin{cases} -2\tilde{x}_1 & +\tilde{x}_2 & =\tilde{b}_1 \\ 2\tilde{x}_1 & -3\tilde{x}_2 & =\tilde{b}_2 \end{cases} \tag{19}$$

where $\tilde{b}_1=(-3,1,2,7)$ and $\tilde{b}_2=(-17,-10,-7,1)$ are trapezoidal fuzzy numbers.
First, we solve the following location model

$$\begin{cases} -2x_1^0 & +x_2^0 & =1.5 \\ 2x_1^0 & -3x_2^0 & =-8.5 \end{cases} \tag{20}$$

which the solutions are $x_1^0=1$ *and* $x_2^0=3.5$. Now, we solve the following fuzziness model

$$\begin{array}{llllll} max & x_1^{l*}(\alpha)+ & x_2^{l*}(\alpha)+ & x_1^{r*}(\alpha)+ & x_2^{r*}(\alpha) & \\ s.t & & x_2^{l*}(\alpha)+ & 2x_1^{r*}(\alpha) & & =4.5-4\alpha \\ & 2x_1^{l*}(\alpha)+ & & & x_2^{r*}(\alpha) & 5.5-5\alpha \\ & 2x_1^{l*}(\alpha)+ & & & 3x_2^{r*}(\alpha) & =8.5-7\alpha \\ & & 3x_2^{l*}(\alpha)+ & 2x_1^{r*}(\alpha) & & 9.5-8\alpha \\ & x_1^{l*}(\alpha)\geq 0 & x_2^{l*}(\alpha)\geq 0 & x_1^{r*}(\alpha)\geq 0 & x_2^{r*}(\alpha)\geq 0 & \end{array} \tag{21}$$

The results are in Table 1 for $\alpha=0,.1,.2,...,.9,1$.

Table 1. Location index number and two fuzziness indexes for optimal solutions

$\boxed{\alpha}$	$\boxed{(x_1^0, x_1^{l*}(\alpha), x_1^{r*}(\alpha))}$	$\boxed{(x_2^0, x_2^{l*}(\alpha), x_2^{r*}(\alpha))}$
0	(1, 2, 1)	(3.5, 2.5, 1.5)
.1	(1, 1.8, .9)	(3.5, 2.3, 1.4)
.2	(1, 1.6, .8)	(3.5, 2.1, 1.3)
.3	(1, 1.4, .7)	(3.5, 1.9, 1.2)
.4	(1, 1.2, .6)	(3.5, 1.7, 1.1)
.5	(1, 1, .5)	(3.5, 1.5, 1)
.6	(1, .8, .4)	(3.5, 1.3, .9)
.7	(1, .6, .3)	(3.5, 1.1, .8)
.8	(1, .4, .2)	(3.5, .9, .7)
.9	(1, .2, .1)	(3.5, .7, .6)
1	(1, 0, 0)	(3.5, .5, .5)

Note that $\tilde{x}_1 = (-1, 1, 2)$ is triangular and $\tilde{x}_2 = (1, 3, 4, 5)$ is trapezoidal fuzzy numbers where satisfy on FLS (19).

Example 2. A manufacturing company makes three types of computers A, B, and C. Computer A takes about 19 hours for assembling (the ingredients), 2 hours for testing (the hardware), 2 hours for installing (the software). Computer B takes about 12 hours for assembling, 4 hours for testing, 2 hours for installing. Computer C takes about 6 hours for assembling, 1 hours for testing, 4 hours for installing.

The company has a factory for which it works about $\tilde{b}_1 = (1469.3, 1897, 2433.2)$ labor-hours each month for assembling, $\tilde{b}_2 = (358.3, 434.5, 543.8)$ labor-hours for testing, and $\tilde{b}_3 = (447.3, 535.5, 667.4)$ hours for installing. How many computers of each kind can the factory make in a month? We firstly utter the linear equations that can describe this situation. Let x_1, x_2 and x_3 show the number of computers of type A, B and C, respectively. Together, these equations form a fuzzy linear system

$$\begin{cases} 19\tilde{x}_1 & +12\tilde{x}_2 & +6\tilde{x}_3 & =\tilde{b}_1 & \textit{for assembling}; \\ 2\tilde{x}_1 & +4\tilde{x}_2 & +\tilde{x}_3 & =\tilde{b}_2 & \textit{for testing}; \\ 2\tilde{x}_1 & +2\tilde{x}_2 & +4\tilde{x}_3 & =\tilde{b}_2 & \textit{for installing}; \end{cases} \tag{22}$$

where $\tilde{b}_1 = (1469.3, 1897, 2433.2)$, $\tilde{b}_2 = (358.3, 434.5, 543.8)$ and $\tilde{b}_3 = (447.3, 535.5, 667.4)$ are triangular fuzzy numbers.

First, we solve the following location model

$$\begin{cases} 19x_1^0 & +12x_2^0 & +6x_3^0 & =1897 \\ 2x_1^0 & +4x_2^0 & +x_3^0 & =434.5 \\ 2x_1^0 & +2x_2^0 & +4x_3^0 & =535.5 \end{cases} \tag{23}$$

which the solutions are $x_1^0 = 26.41176471$, $x_2^0 = 74.57352942$ *and* $x_3^0 = 83.38235292$. Now, with solving the fuzziness model results are in Table 2 for $\alpha = 0, .1, .2, \ldots, .9, 1$.

Table 2. Location index number and two fuzziness indexes for optimal solutions

α	$(x_1^0, x_1^{l*}(\alpha), x_1^{r*}(\alpha))$	$(x_2^0, x_2^{l*}(\alpha), x_2^{r*}(\alpha))$	$(x_3^0, x_3^{l*}(\alpha), x_3^{r*}(\alpha))$
0	(26.41176471,12.86,11.7012)	(74.57352942,9.96,16.7924)	(83.38235292,10.64,18.7282)
0.1	(26.41176471,11.574,10.5311)	(74.57352942,8.964,15.1131)	(83.38235292,9.576,16.8554)
0.2	(26.41176471,10.288,9.3609)	(74.57352942,7.968,13.4339)	(83.38235292,8.512,14.9826)
0.3	(26.41176471,9.002,8.1908)	(74.57352942,6.972,11.7547)	(83.38235292,7.448,13.1098)
0.4	(26.41176471,7.716,7.0207)	(74.57352942,5.976,10.0754)	(83.38235292,6.384,11.2369)
0.5	(26.41176471,6.43,5.8500)	(74.57352942,4.98,8.3962)	(83.38235292,5.32,9.3641)
0.6	(26.41176471,5.144,4.6805)	(74.57352942,3.984,6.7169)	(83.38235292,4.256,7.4913)
0.7	(26.41176471,3.858,3.5104)	(74.57352942,2.988,5.0377)	(83.38235292,3.193,5.6185)
0.8	(26.41176471,2.572,2.3402)	(74.57352942,1.992,3.3585)	(83.38235292,2.128,3.7456)
0.9	(26.41176471,1.286,1.1701)	(74.57352942,.996,1.6792)	(83.38235292,1.064,1.8728)
1.0	(26.41176471,0,0)	(74.57352942,0,0)	(83.38235292,0,0)

Note that $\tilde{x}_1 = (13.55176471, 26.41176471, 38.11296471)$, $\tilde{x}_2 = (64.61352942, 74.57352942, 91.36592942)$ and $\tilde{x}_3 = (72.74235292, 83.38235292, 102.1105529)$ are triangular fuzzy numbers where satisfy on FLS (22).

5. Conclusion

In this paper, a new method based on fuzzy operations approach, [18], for solving Fuzzy Linear system is introduced. First we solved the location model and then decided about feasibility or infeasibility of FLS. The decision maker can intervene in all the steps of the decision process which makes our approach very useful to be applied in a lot of real-world problems where the information is uncertain or incomplete, like environmental management, project investment, marketing and quantum physics [2, 3].

Conflict of interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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