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Numerical Simulation and Methodology Based on Improved Split Step Method for Studying Stochastic Models

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A B S T R A C T

In this paper, we improved the split step ϑ method to solve the stochastic differential equations. The strong convergence of this approximation for stochastic differential equations, whose drift and diffusion coefficients are globally Lipschitz continuous, are investigated. Furthermore, we analyse the stability in the mean square sense of our scheme by scalar stochastic differential equation with multi-dimensional Wiener processes. The study of stability shows the mean square stability of the method for $\vartheta \in [1/2,1]$. Finally, we present some numerical examples to describe the methodology and implementation of the split step ϑ method to solve linear and nonlinear one dimensional stochastic differential equations and the Lotka-Volterra stochastic system.

1. Introduction

Modelling is one of the most interesting topics of interest to researchers. Stochastic differential equations (SDEs) are one of the most important tools of this work, for example see [1, 2, 5, 9, 11, 12, 15]. Since most SDEs do not have explicit solutions, so we need to introduce the efficient numerical methods. Split step method is one of these methods presented in [7]. Basis this idea, many numerical methods presented for solving of SDEs. For instance, in [3] introduced split step θ (SSθ) method to solve nonlinear non autonomous SDEs, and proved that SSθ method converges strongly with the order one-half to the exact solution. Also, shown that this method is mean square stable if $\theta = 1$. In [6], constructed split step composite θ method for numerically solving SDEs of the Ito type. Recently, the authors of this paper present new method for solution of stiff SDEs by Rosenbrock ODE solver [14].

In this paper, for solution of Ito stochastic differential equations

$$dV(t) = A(t,V(t))dt + \sum_{n=1}^{m} B_n(t,V(t))Z_n(t), \quad V(0) = V_0 \in \mathbb{R}^d, \tag{1}$$

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We suggest improving split step ϑ (ISSϑ) method in following form

$$\begin{aligned} \bar{\nu}_k &= \nu_k + \vartheta h A(t_k + \vartheta h, \nu_k + \vartheta h A(t_k + \vartheta h, \bar{\nu}_k)), \quad \vartheta \in [0,1], \\ \nu_{k+1} &= \nu_k + h A(t_k + \vartheta h, \bar{\nu}_k) + \sum_{n=1}^{m} B_n(t_k + \vartheta h, \bar{\nu}_k)\Delta Z_{k,n}. \end{aligned} \tag{2}$$

where $A, B_n : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $h = (T_0 - t)/N$, $N = 1, 2, \ldots$ and $\nu_0 = V_0$. Note that in SDE (1), $Z_n(t)$ is the one dimensional Wiener process, whose increment $\Delta Z_{k,n} = Z_n(t_{k+1}) - Z_n(t_k)$ is a Gaussian random variable $N(0,h)$. By choosing $\vartheta = 0$, method (2) leads to the Euler-Maruyama method. For $0 < \vartheta \le 1$ the above scheme is an implicit equation in $\bar{\nu}_k$ that must be solved in order to obtain the intermediate approximation $\bar{\nu}_k$ [3, 6]. So, for solving nonlinear equations (2) we use classic Newton iterative method.

The rest of this paper is organized as follows. By using the fundamental Milstein theorem and some inequalities, the convergence analysis of the scheme will be investigated in Section 2. In Section 3, we analyse the mean square stability properties of the proposed method with respect to the linear test equation with multi-dimensional Wiener processes. We give several numerical simulations to illustrate our theoretical results in Section 4. Finally, in Section 5, some concluding remarks are given.

2. Mean-square convergence properties

According to [6, 10, 13, 14, 17], we obtain the strong convergence of the proposed method by following proposition and lemma, in this section.

Proposition 1. The functions A and B_n, $n = 1, \ldots, m$ in SDE (1) satisfy Lipschitz conditions

$$|A(\varsigma, l_1) - A(\varsigma, l_2)|^2 \vee \sum_{n=1}^{m} |B_n(\varsigma, l_1) - B_n(\varsigma, l_2)|^2 \le \kappa_1 |l_1 - l_2|^2, \tag{3}$$

linear growth bounds

$$|A(\varsigma, l_1)|^2 \vee \sum_{n=1}^{m} |B_n(\varsigma, l_1)|^2 \le \kappa_2 (1 + |l_1|^2), \tag{4}$$

and

$$|A(\varsigma_1, l_1) - A(\varsigma_2, l_1)|^2 \vee \sum_{n=1}^{m} |B_n(\varsigma_1, l_1) - B_n(\varsigma_2, l_1)|^2 \le \kappa_3 (1 + |l_1|^2) |\varsigma_1 - \varsigma_2|^2, \tag{5}$$

where positive constants $\kappa_1, \kappa_2, \kappa_3$ are independent of variables $l_1, l_2 \in \mathbb{R}^m$ and $\vee$ is a maximal operator.

Lemma 1. ([15]) Assume that for a one step discrete time approximation ν, the local mean error and mean square error for all $N = 1, 2, \ldots$ and $k = 0, 1, \ldots, N-1$ satisfy, respectively, the estimates

$$|E[(\nu_{k+1} - V(t_{k+1})) \,|\, \nu_k = V(t_k)]| \le \kappa (1 + |\nu_k|^2)^{1/2} h^{\upsilon_1}, \tag{6}$$

and

$$|E[|\nu_{k+1} - V(t_{k+1})|^2 \,|\, \nu_k = V(t_k)]|^{1/2} \le \kappa (1 + |\nu_k|^2)^{1/2} h^{\upsilon_2}, \tag{7}$$

where $\upsilon \ge 1/2$ *and* $\upsilon_1 \ge \upsilon_2 + 1/2$. Then

$$|E[|\nu_i - V(t_i)|^2 \,|\, \nu_0 = V(t_0)]|^{1/2} \le \kappa (1 + |\nu_0|^2)^{1/2} h^{\upsilon_2 - 1/2},$$

holds for each $i = 0, 1, \ldots, N$. Here, κ is independent of h but dependent on the length of the time interval $T - t_0$.

Now, we can prove mean square convergence of ISSϑ method (2) by above facts and following Euler-Maruyama approximation step

$$v_{k+1}^{EM} = v_k^{EM} + hA(t_k, v_k^{EM}) + \sum_{n=1}^{m} B_n(t_k, v_k^{EM})\Delta Z_{k,n}, \tag{8}$$

with the local mean and mean square errors, respectively:

$$\begin{aligned} &|E[(v_{k+1}^{EM} - V(t_{k+1})) \,|\, v_k^{EM} = V(t_k)]| = O(h^2),\\ &|E(|v_{k+1}^{EM} - V(t_{k+1})|^2 \,|\, v_k^{EM} = V(t_k))|^{1/2} = O(h), \end{aligned} \tag{9}$$

Theorem 1. Let v_n be the numerical approximation of $V(t_n)$ at time t_n after n steps with step size $h = (T - t_0)/N,\ N = 1,2,\dots$. For $h < \min(1, 1/(\vartheta\sqrt{2\kappa_1}))$, if applying the ISS ϑ method (2) to the SDE (1), for all $i = 0,1,\dots,N,$ one gets

$$|E[|v_i - V(t_i)|^2 \,|\, v_0 = V(t_0)]|^{1/2} = O(\sqrt{h}).$$

Proof. We first prove the local mean error by using (6) and first equation of (9),

$$\begin{aligned} H_1 &= |E[(v_{k+1} - V(t_{k+1})) \,|\, v_k = V(t_k)]|\\ &\le |E[(v_{k+1}^{EM} - V(t_{k+1})) \,|\, v_k^{EM} = V(t_k)]| + E[(v_{k+1} - v_{k+1}^{EM}) \,|\, v_k = v_k^{EM}]|\\ &\le O(h^2) + H_2. \end{aligned} \tag{10}$$

where, we have

$$\begin{aligned} H_2 &= |E[(v_{k+1} - v_{k+1}^{EM}) \,|\, v_k = v_k^{EM}]|\\ &= |E[v_k - v_k^{EM} + h(A(t_k + \vartheta h, \bar{v}_k) - A(t_k, v_k^{EM})) + \sum_{n=1}^{m} \Delta Z_{k,n}(B_n(t_k + \vartheta h, \bar{v}_k) - B_n(t_k, v_k^{EM})) \,|\, v_k^{EM} = V(t_k)]|\\ &\le h\sqrt{\kappa_1}\,|\bar{v}_k - v_k| + h|A(t_k + \vartheta h, v_k) - A(t_k, v_k)|\\ &\le h\sqrt{\kappa_1}\,|\bar{v}_k - v_k| + h\sqrt{\vartheta\kappa_3 h}(1 + |v_k|^2)^{1/2}, \end{aligned} \tag{11}$$

To obtain this inequality, we use the Lipschitz condition (3), inequality (5) and $E[\Delta Z_{k,n}] = 0$. By implicit equation (2) and proposition 1, we obtain

$$\begin{aligned} |\bar{v}_k - v_k| &\le \vartheta h(|A(t_k + \vartheta h, v_k + \vartheta h A(t_k + \vartheta h, \bar{v}_k)) - A(t_k + \vartheta h, v_k)| + |A(t_k + \vartheta h, v_k)|)\\ &\le \vartheta h(\vartheta h\sqrt{\kappa_1}\,|A(t_k + \vartheta h, \bar{v}_k)| + |A(t_k + \vartheta h, v_k) - A(t_k, v_k)| + |A(t_k, v_k)|)\\ &\le \vartheta h(\vartheta h\sqrt{\kappa_1}\,|\bar{v}_k - v_k| + (1 + \vartheta h\sqrt{\kappa_1})(|A(t_k + \vartheta h, v_k) - A(t_k, v_k)| + |A(t_k, v_k)|))\\ &\le \vartheta h \frac{\sqrt{\kappa_2} + \sqrt{\vartheta\kappa_3 h}}{1 - \vartheta h\sqrt{\kappa_1}}(1 + |v_k|^2)^{1/2}. \end{aligned} \tag{12}$$

Hence, from inequalities (10)-(12), we obtain $\upsilon_1 = 3/2$.

Using standard arguments, we show that inequality (7) with $\upsilon_2 = 1$ holds for the ISS ϑ method [6, 10, 13, 14]. Also, we can write

$$\begin{aligned} H_3 &= |E[|v_{k+1} - V(t_{k+1})|^2 \,|\, v_k = V(t_k)]|^{1/2}\\ &\le |E[|v_{k+1}^{EM} - V(t_{k+1})|^2 \,|\, v_k^{EM} = V(t_k)]|^{1/2} + |E[|v_{k+1} - v_{k+1}^{EM}|^2 \,|\, v_k = v_k^{EM}]|^{1/2}\\ &\le O(h) + \sqrt{H_4}. \end{aligned} \tag{13}$$

For computing $H4$, we can write

$$
\begin{aligned}
H_4 &= | E[| \nu_{k+1} - \nu_{k+1}^{EM} |^2 | \nu_k = \nu_k^{EM}] | \\
&= E[| \nu_k - \nu_k^{EM} + h(A(t_k + \vartheta h, \bar{\nu}_k) - A(t_k, \nu_k^{EM})) + \sum_{n=1}^{m} \Delta Z_{k,n} (B_n(t_k + \vartheta h, \bar{\nu}_k) - B_n(t_k, \nu_k^{EM})) |^2 | \nu_k = \nu_k^{EM}] | \\
&= h^2 | A(t_k + \vartheta h, \bar{\nu}_k) - A(t_k, \nu_k) |^2 + h \sum_{n=1}^{m} | B_n(t_k + \vartheta h, \bar{\nu}_k) - B_n(t_k, \nu_k^{EM}) |^2 \\
&\le 2h(1+h)(\kappa_1 | \bar{\nu}_k - \nu_k |^2 + \vartheta \kappa_3 h (1 + | \nu_k |^2)).
\end{aligned}
\tag{14}
$$

In above inequality, we use Lipschitz condition (3), inequality (5) and

$$
E[\Delta Z_{k,n_1} \Delta Z_{k,n_2}] = \begin{cases} h, n_1 = n_2, \\ 0, n_1 \ne n_2. \end{cases}
$$

Now, we estimate term $| \bar{\nu}_k - \nu_k |^2$ by implicit equation (2) and Proposition 1 as follows

$$
\begin{aligned}
| \bar{\nu}_k - \nu_k |^2 &\le 2(\vartheta h)^2 (| A(t_k + \vartheta h, \nu_k + \vartheta h A(t_k + \vartheta h, \bar{\nu}_k)) - A(t_k + \vartheta h, \nu_k) |^2 + | A(t_k + \vartheta h, \nu_k) |^2) \\
&\le 2(\vartheta h)^2 ((\vartheta h)^2 \kappa_1 | A(t_k + \vartheta h, \bar{\nu}_k) |^2 + 2 | A(t_k + \vartheta h, \nu_k) - A(t_k, \nu_k) |^2 + 2 | A(t_k, \nu_k) |^2) \\
&\le 4(\vartheta h)^2 ((\vartheta h \kappa_1)^2 | \bar{\nu}_k - \nu_k |^2 + (1 + 2(\vartheta h)^2 \kappa_1)(| A(t_k + \vartheta h, \nu_k) - A(t_k, \nu_k) |^2 + | A(t_k, \nu_k) |^2)) \\
&\le 4(\vartheta h)^2 \frac{\kappa_2 + \vartheta \kappa_3 h}{1 - 2(\vartheta h)^2 \kappa_1} (1 + | \nu_k |^2).
\end{aligned}
\tag{15}
$$

Thus, from inequalities (13)-(15), we obtain $\upsilon_2 = 1$. According to the above calculations, the Theorem 1 satisfies for $\upsilon_1 = 3/2$ and $\upsilon_2 = 1$. Thus, the proposed method strongly convergences to the exact solution with order 1/2.

3. Linear Mean square stability properties

In this section, we analyzed MS-stability of linear SDE with multi-dimensional Wiener processes

$$
V(t) = aV(t)dt + \sum_{n=1}^{m} b_n V(t) dZ_n(t), \tag{16}
$$

where the parameters $a, b_n \in \mathbb{C}, V_0 \ne 0$. The equilibrium position of test equation (16) is asymptotically MS-stable if and only if $\sum_{n=1}^{m} | b_n |^2 + 2\Re(a) < 0$, see [3, 6, 9, 13, 14, 16].

Definition 1. ([16]) The numerical scheme is said to be MS-stable if

$$
\overline{M}(a, \{b_n\}_{n=1}^m, h) = E[M^2(a, \{b_n\}_{n=1}^m, h, \{\xi_{k,n}\}_{n=1}^m)] < 1, \quad \xi_{k,n} \sim N(0,1),
$$

where $\overline{M}(a, \{b_n\}_{n=1}^m, h)$ is called MS-stability function and the $D_{MS} = \{(a, \{b_n\}_{n=1}^m) \in \mathbb{C} \times \mathbb{C} : \overline{M}(a, \{b_n\}_{n=1}^m, h) < 1\}$ set is called the MS-stability domain of the numerical method.

We apply the scheme (2) to solve the test equation (16) with step size $h > 0$. Since $A(t, Z(t)) = aZ(t)$ and $B_n(t, Z(t)) = b_n Z(t)$, we obtain

$$
\nu_{k+1} = M(a, \{b_n\}_{n=1}^m, \{\xi_{k,n}\}_{n=1}^m) \nu_k,
$$

where

$$M(a,\{b_n\}_{n=1}^{m},\vartheta,h,\{\xi_{k,n}\}_{n=1}^{m})=\frac{1+(1-\vartheta)ah+\sqrt{h}\sum_{n=1}^{m}b_n\xi_{k,n}}{1-\vartheta ah}.$$

Following the above definition, we can write

$$\begin{aligned}E[|M(a,\{b_n\}_{n=1}^{m},\vartheta,h,\{\xi_{k,n}\}_{n=1}^{m})|^2]&=E[M(a,\{b_n\}_{n=1}^{m},\vartheta,h,\{\xi_{k,n}\}_{n=1}^{m})\times\overline{M(a,\{b_n\}_{n=1}^{m},\vartheta,h,\{\xi_{k,n}\}_{n=1}^{m})}]\\&=E[\frac{1+(1-\vartheta)ah+\sqrt{h}\sum_{n=1}^{m}b_n\xi_{k,n}}{1-\vartheta ah}\times\frac{1+(1-\vartheta)\bar{a}h+\sqrt{h}\sum_{n=1}^{m}\bar{b}_n\xi_{k,n}}{1-\vartheta\bar{a}h}]\\&=\frac{1+2(1-\vartheta)\Re(a)h+(1-\vartheta)^2|a|^2h^2+h\sum_{n=1}^{m}|b_n|^2}{1-2\vartheta\Re(a)h+\vartheta^2|a|^2h^2}.\end{aligned}$$

According to Definition 1, the stability domain of the ISS ϑ scheme (2) applied with step size h is denoted by

$$E[|M(a,\{b_n\}_{n=1}^{m},\vartheta,h,\{\xi_{k,n}\}_{n=1}^{m})|^2]<1\Leftrightarrow 2\Re(a)h+(1-2\vartheta)|a|^2h^2+h\sum_{n=1}^{m}|b_n|^2<0.$$

Since $\sum_{n=1}^{m}|b_n|^2+2\Re(a)<0$, we conclude that ISS ϑ method MS-stable, when $1/2\le\vartheta\le1$. We take $x=h\Re(a), y_n^2=h|b_2|^2$ and drew the MS-stable regions of the method for $\vartheta\in[0,1]$ in Figures 1 and 2. Also, we can see that this figures supports of the obtained theoretical results.

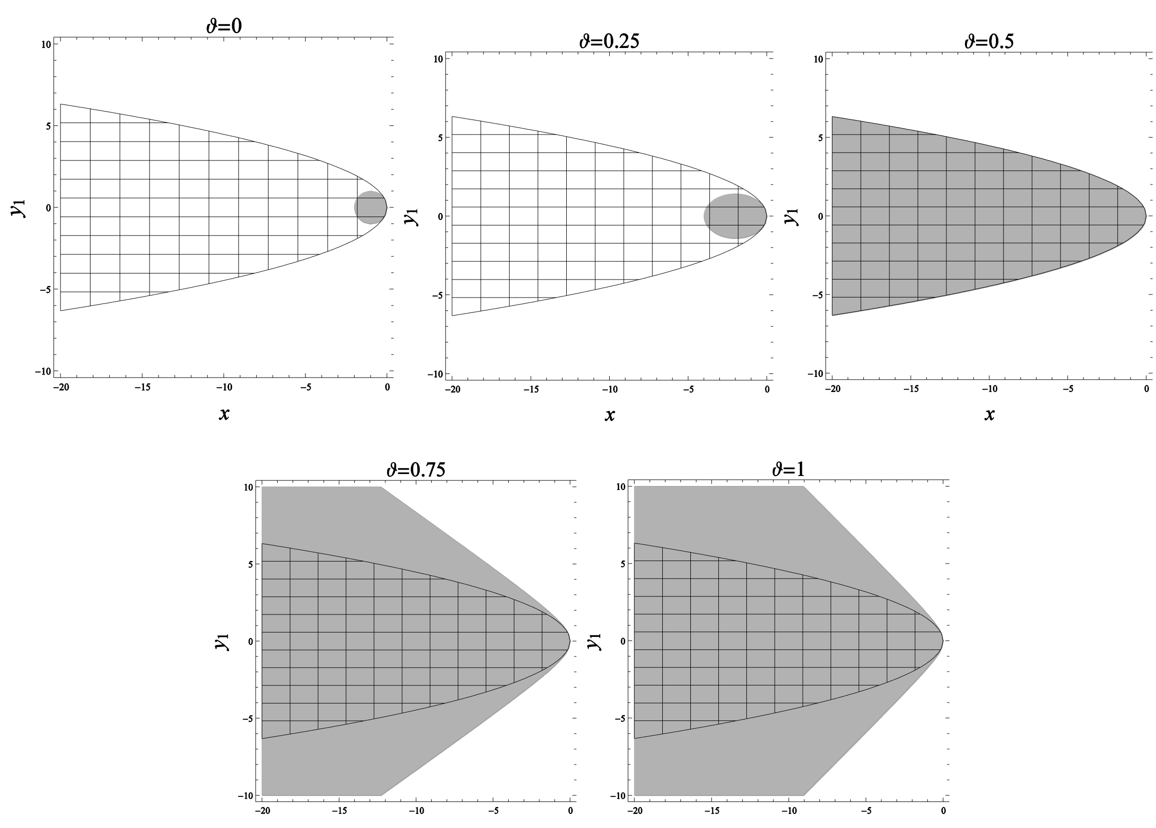

Figure 1: Gridded area: linear MS-stability of the test SDE (16). Gray area: linear MS-stability of ISS ϑ method (2).

4. Numerical results

In this section several numerical examples are given to illustrate our theoretical results in the previous sections. The numerical results obtained by applying MATLAB7.14.0.739 (R 2012a) on a PC with CPU Intel(R) Pentium(R) CPU G620 at 2.60 GHz, 2.00 GB of RAM, and the Windows 7 operating system.

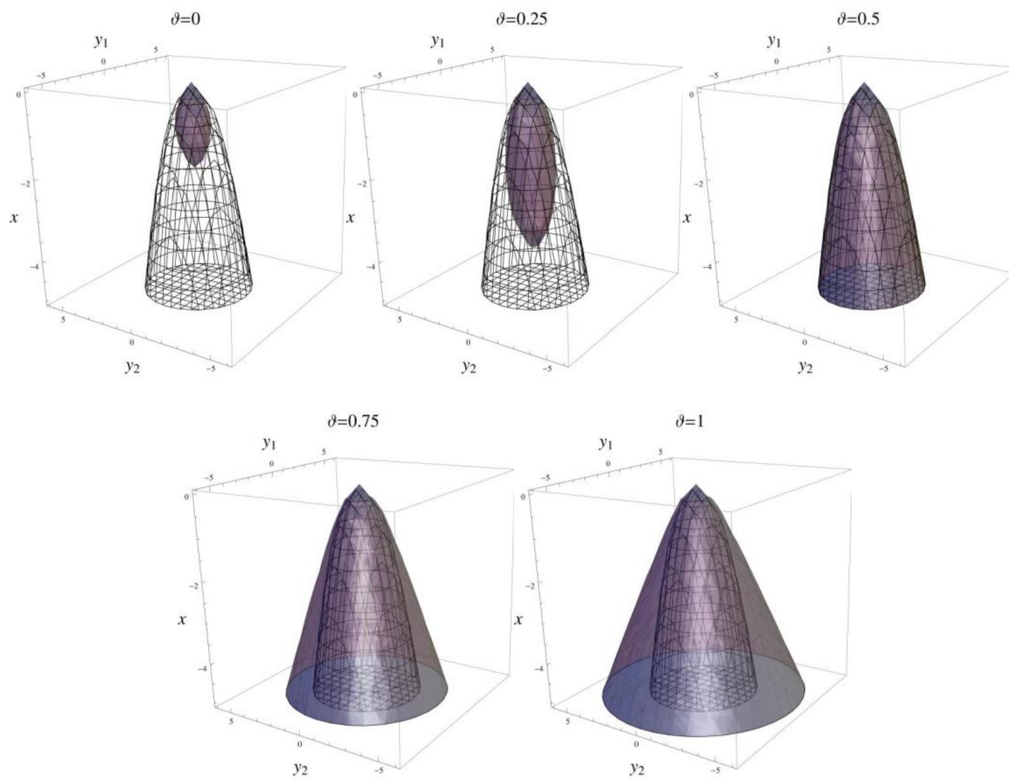

Figure 2: Gridded area: linear MS-stability of the test SDE (16). Gray area: linear MS-stability of ISS ϑ method (2).

Example 1. We consider the following one dimensional nonlinear SDE

$$dV(t) = (\frac{1}{2}\sin(2V(t)) + \frac{1}{8}\sin(4V(t)))dt + \frac{1}{2}\sin(2V(t))dZ(t), \quad V_0 = 1, \tag{17}$$

with exact solution $V(t) = \arctan(\tan(V_0)\exp(t + Z(t)))$.

Figure 3 shows a log-log plot of the MSEs (mean square errors), based on the 2000 sample paths with step sizes $h = 2^{-r}, r = 6, \ldots, 12$ for ISS ϑ method with $\vartheta = 0.1,$ at the terminal time $T = Nh = 1.$ A reference line of slope 1/2. We see that the results are consistent with the strong errors close to order 1/2.

Example 2. We consider the following one dimensional linear SDE

$$dV(t) = aV(t)dt + b_1V(t)dZ_1(t) + b_2V(t)dZ_2(t), \tag{18}$$

with the initial data $V_0 = 1.$ For parameters $a = -10, b_1 = \sqrt{9}, b_2 = \sqrt{10}$ and $T = 20,$ we illustrate the MS-stability of the our method for different step sizes in Figure 4. We take $h = 2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}$ and generate 50000 samples for each numerical scheme. We observe from Figure 4 that the equation (18) is MS-stable for any step size when $\vartheta \in [1/2, 1]$. Therefore, the theoretical results obtained in the previous section are confirmed.

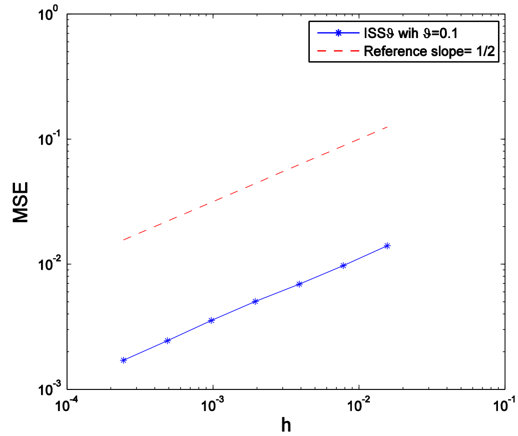

Figure 3: The convergence rate of the ISS ϑ method (2) for nonlinear stochastic equation (17).

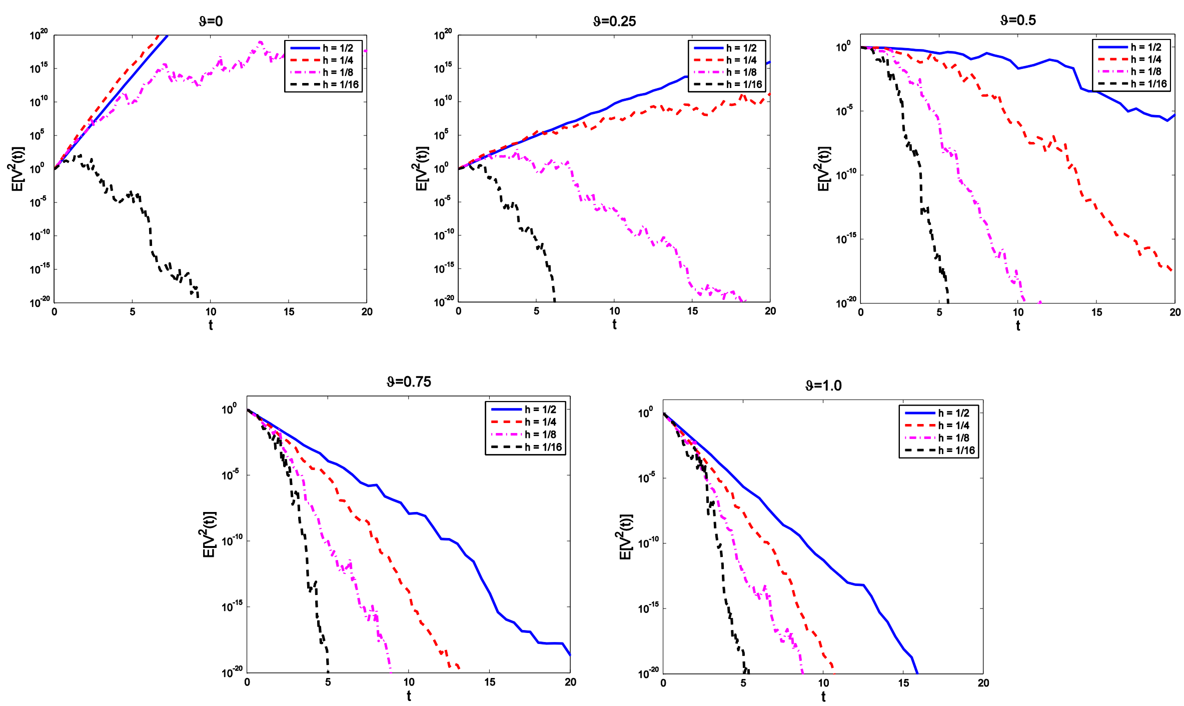

Figure 4: Simulation of $E[V^2(t)]$ by ISS ϑ method (2) with different step sizes applied to SDE (18).

Examples 3. Consider a Lotka-Volterra stochastic system [4, 8]

$$\begin{aligned} dV_1(t) &= V_1(t)(a_1 - a_2V_2(t))dt + \sigma_1 V_1(t)dZ_1(t), \\ dV_2(t) &= V_2(t)(b_1V_1(t) - b_2)dt + \sigma_2 V_2(t)dZ_2(t). \end{aligned} \tag{19}$$

For parameter values $a_1 = 2, a_2 = 0.25, b_1 = 0.2, b_2 = 3, \sigma_1 = 0.35, \sigma_2 = 0.25$ and initial data $V_1(0) = 2, V_2(0) = 1,$ we compare MSEs of ISS ϑ method (2) with $\vartheta = 0.1$ and EM method in Figure 4. To obtain the results of Figure 4, we compute 5000 sample paths to simulate the MSEs between the analytic solution with step size $h = 2^{-14}$ and numerical approximation with different step sizes $h = 2^{-r}, r = 5,\ldots,11.$

5. Concluding Remarks and future work

In the last two decades, various methods for solving stochastic differential equations have been proposed. Some of these numerical schemes have been developed using split step technique. By this strategy, improving split step ϑ method (2) was presented in this paper. We proved that the scheme with the strong convergence order 1/2 converges to the exact solution. Also, the mean square stability function of the method was obtained for linear test equation with multi-dimensional Wiener processes. In addition, it is proved that our method for any time step h is mean square stable if $1/2 \le \vartheta \le 1$. Finally, numerical examples were reported to verify the theoretical results and show efficiency of the constructed scheme. There are a number of potential directions in which the results achieved here can be extended. Future work will involve the development of a method (2) for other type of stochastic problems such as stochastic delay differential equations, stochastic difference equations and stochastic functional differential equation.

Conflict of interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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