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Efficiency of Extended Two-Stage Systems in Presence of Triangular Fuzzy Number

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A B S T R A C T

Data envelopment analysis (DEA) is an excellent method to evaluate the efficiency of homogeneous Decision Making Units (DMUs). This procedure considers the internal structure of DMUs as black box systems. In the real world, many systems are a combination of two stages that are connected with intermediate measures. In fact, intermediate measures are considered as the outputs of the first stage and the inputs of the second stage. To evaluate the efficiency of these systems, network DEA (NDEA) models are presented. In some cases, these systems may have shared inputs, and also part of the intermediate measures will be allocated as inputs of second-stage. We also frequently deal uncertain information such as stochastic data, fuzzy data and so on. Therefore, in practice, it is not easy to obtain the exact values of these inputs and assign them to each of the stages. Also, it is not possible to easily determine the use of the second stage of intermediate measures. Therefore, in this paper, we shall combine fuzzy DEA and NDEA models which introduce a model based on the multiplicative approach. To this end, we use the non-compensatory property of the multiplication operator and the $\alpha-$cut procedure. These models calculate the $\alpha-$cut interval of overall efficiency and efficiency of stages in the presence of triangular fuzzy numbers (TFNs). Also, specify the optimal portion of stages in the use of shared inputs and the portion of the second stage of intermediate measures is specified. Furthermore, the product of the upper (the lower) bound of the $\alpha-$cut of the stages efficiency is considered as the upper (the lower) bound overall efficiency. Finally, we will illustrate the proposed models by using a numerical example extracted from the extant literature.

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1. Introduction

DEA was first introduced by Charnes et al. [2] which is a non-parametric tool for evaluating the efficiency of decision-making units (DMUs) (Also, see [1]). Afterward, numerous studies have been introduced that evaluate the efficiency of systems in a variety of applications. Recently, in order to stock evaluation and portfolio selection under data ambiguity a new optimistic and pessimistic fuzzy DEA procedure is resented by Peykani et al. [28]. Also, a new fuzzy stochastic DEA model is proposed by Golami [7] that measure the efficiency of DMUs in presence of undesirable outputs [23, 30, 32].

In the real-life problems, DMUs may have network structures. Hence, to assess the efficiency of such systems, conventional DEA models do not work well. Actually, these models treat DMUs as a black box system and ignore their internal structure. Therefore, the decision maker (DM) does not have comprehensive information about the internal factors that cause inefficiency. Hence, NDEA models were introduced to solve this problem. Two-stage systems as a special case of network system have very important in many applications such as banks, hospitals, airports, and others. For example, each bank branch can be viewed as a two-stage system which stage1 as the "attract resources" stage and the second stage as the "resources allocation" stage. In two-stage systems, the produced outputs of stage1 become the inputs for stage 2 which are named by intermediate measures.

Recently, several models are presented in order to measure the efficiency of two-stage systems. Firstly, Seiford and Zhu [31], suggested models for measuring the efficiency of two-stage systems, independently. Based on the proposed model, the whole system may be efficient but stages1, 2 are not efficient that is a weakness. Hence, in order to overcome this weakness, Kao and Hwang [13] consider the relations of stages and introduced a model that measures the efficiency of system and stages simultaneously (under the constant returns to scale (CRS)). And the overall efficiency is the product of the efficiency of stages. Their model cannot measure the efficiency of two-stage systems under the variable returns to scale (VRS). Therefore, Chen et al. [3] presented the models that can evaluate the efficiency of two-stage systems under CRS and VRS. Also, a weighted average of the efficiencies of stages considered as the overall efficiency.

After that, numerous studies focus on the extended structures of two-stage systems and suggested the models for evaluating these systems. In the subsequent years, based on multiplicative and additive approaches of NDEA, various researches have been conducted to evaluate the performance of extended two-stage systems. In many of these articles, using the DM's point of view in choosing a cooperative and non-cooperative perspective, the efficiency of the system and its stages have been obtained. For example, Zha and Liang [41], using a cooperative approach, calculated the efficiency of two-stage systems in the presence of shared inputs. Also, Li et al. [20] in order to measure the efficiency of two-stage systems with additional input in the second stage, applied both cooperative and non-cooperative perspectives. They also used a heuristic algorithm to solve the derived (obtained) model from the cooperative approach. The efficiency of two-stage systems with shared inputs and free intermediate measures was evaluated by Jianfeng [11] using additive efficiency decomposition method. Li et al. [19] also calculated the efficiency of two-stage systems in the presence of shared inputs and shared outputs by using the additive decomposition approach.

Toloo et al. [36] suggested a novel DEA model for measuring the efficiency of two-stage systems with shared inputs. Izadikhah et al. [10] measured the efficiency of systems with freely distributed initial inputs and shared intermediate outputs. In order to evaluate the efficiency of multi-period two-stage systems, a model based on a slacks-based measure (SBM) was introduced by Esfidani et al. [5], in which the efficiency of 20 branches of Mellat Bank was also evaluated. Nemati et al. [26], Considering the minor effects between inputs and outputs, evaluated the efficiency of systems that have several production lines with a two-stage network structure and each production line uses the inputs according to its needs.

All the mentioned studies are limited to the use of crisp inputs and outputs and intermediate measures. However, one of the major challenges in application problems with two-stage structure is to obtain quantities values for inputs, intermediate measures and outputs (Such as the quality of life, the quality of service, … or for example, we cannot precisely measure the quality of life.). In other words, the data may be imprecise (uncertain) (for example stochastic data, fuzzy data, interval data and so on). Therefore, in the field of application, it is especially important to obtain the performance of these systems in the presence of uncertain data. In this regard, different formulations of conventional DEA models and NDEA models are suggested that measure the efficiency of systems in the presence of uncertain data. For example, Jiang et al. [12] used uncertainty theory and introduced new two-stage network models in which inputs, intermediate measures, and outputs are considered uncertain variables. Also, Esfidani et al. [6] evaluated the efficiency of two-stage systems with stochastic data.

One way to deal with uncertain data is to use the fuzzy concept in DEA models. There are many studies that have discussed the efficiency of systems in the fuzzy environment. Firstly, Sengupta [31] suggested a fuzzy approach in order to solve fuzzy DEA models. Also, Hatami-Marbini [8] calculated the fuzzy efficiency of DMUs using an interactive method. Then, Hatami-Marbini et al. [9] presented classification schemes of fuzzy approaches. Among these approaches have widely used $\alpha -$ cut technique to measure efficiency [4, 16, 17, 26]. Also, Kao and Liu [15] used the extension principle and suggested FNDEA models that calculate the $\alpha -$ cut efficiency of two-stage systems. Furthermore, a new procedure was presented by Lozano [22] in which, firstly, $\alpha -$ cut efficiency of two-stage systems are calculated. Then, the $\alpha -$ cut efficiency of each stage is measured while the overall efficiency is unchanged. Liu [21] proposed a method for ranking the fuzzy efficiency of two-stage systems. Soltani et al. [35] proposed two-stage fuzzy DEA model based on fuzzy arithmetic. Ostovan et al. [27] suggested models to measure the average efficiency of two-stage networks using DEA and DEA-R with fuzzy data. In practice, fuzzy numbers (FN) have special computational efficiency. Among FNs, TFNs are widely used by researchers due to their simplicity in calculations. Existing researches focused on measuring the fuzzy efficiency of simple two-stage systems.

Our motivation in this article is twofold. (1) We have considered the developed structure of two-stage systems in which part of the inputs of the second stage is freely determined by the DM and also part of the intermediate measures produced in the first stage are considered as final outputs of whole system. And a group of inputs is divided between the stages and the portion of stages of these inputs is determined by solving the proposed model. (2) In the real world, there are many cases where observations are very difficult to measure, such as qualitative data. In this case, the values used for this data are ambiguous. None of the presented papers evaluated the performance of two-stage systems developed in the presence of fuzzy data. Therefore, by combining these two modes, in this paper, a model based on the multiplicative approach is introduced to measure the efficiency of these systems in the presence of TFNs. In fact, this article uses triangular fuzzy numbers for simplicity in calculations and notation. The proposed approach can also be implemented on all fuzzy data. And also, for solving the proposed model, we will use the concept of $\alpha -$cut efficiency and non-cooperative procedure. Actually, in this procedure, it is assumed that one of the stages is more important from the DM's point of view and is selected as the leader stage and the other stage is considered as the follower stage. Then the efficiency of the first stage will be calculated, separately. Then, the efficiency of the second stage is also calculated while the efficiency of the leader stage is unchanged. Based on our suggested models, the lower (the upper) bound of $\alpha -$cut of the overall efficiency is equal to product of the lower (the upper) bounds of $\alpha -$cut of the efficiency of the stages. Therefore, it can be concluded that the whole system is efficient at the upper bound of $\alpha -$cut if and only if it is efficient in the upper bound $\alpha -$cut of the efficiency of the stages. Finally, the proposed models will be illustrated using a numerical example.

The organization of this paper is as follows: In Section 2, firstly, we reviewed TFNs and then, two-stage DEA model is briefly reviewed that evaluates the efficiency of two-stage systems. In Section 3, firstly, the structure of an extended two-stage system is presented. Then, we suggested the models for measuring the efficiencies of this systems. And also, decompositions of the efficiencies based on the intervals of $\alpha -$cuts are presented. Finally, in Section 4, a numerical illustration of proposed fuzzy DEA models is exhibited. Conclusions of study, are inserted in Section 5.

2. Preliminaries

In this section, firstly, the definitions of fuzzy set, fuzzy number and triangular fuzzy number are reviewed. Also, the two-stage DEA model of Kao and Hwang [13] is presented.

1.1 Triangular Fuzzy Number (TFN)

Suppose X is a universe set. A fuzzy set $\tilde{N}$ is defined as $\tilde{N}=\{(x,\mu_{\tilde{N}}(x))\,|\,x\in X\}$ where $0<\mu_{\tilde{N}}(x)\le 1$ is the degree of membership of element $x\in X$ to the set $\tilde{N}\subset X$ (Zimmermann [40]). And also, let $S(\tilde{N})$ is as $S(\tilde{N})=\{x\in X\,|\,\mu_{\tilde{N}}(x)>0\}$ that denote the support of $\tilde{N}$. The $\alpha-$ cut set of $\tilde{N}$ is defined as $\tilde{N}_\alpha=\{x\in S(\tilde{N})\,|\,\mu_{\tilde{N}}(x)>\alpha\}$.

Definition 1. Suppose $\tilde{N}\subset R$ be a fuzzy set. $\tilde{N}$ is called FN, if the following conditions are hold:

- $\tilde{N}$ is fuzzy set convex set if the membership function is fuzzy convex set: $\forall x_1,x_2\in R$, $\forall\lambda\in[0,1]$: $\mu_{\tilde{N}}(\lambda x_1+(1-\lambda)x_2)\ge\min\{\mu_{\tilde{N}}(x_1),\mu_{\tilde{N}}(x_2)\}$
- There is at least one $x'\in R$ such that $\mu_{\tilde{N}}(x')=1$.
- The membership $\mu_{\tilde{N}}(x)$ function is semi-continuous.

Definition 2. A FN $\tilde{N}\subset R$ is a TFN with membership function $\mu_{\tilde{N}}(x)$ of the following form:

$$\mu_{\tilde{N}}(x)=\begin{cases}\dfrac{x - x^m+x^l}{l_\mu}, & x^m-x^l\le x\le x^m\\[2ex] \dfrac{x^m+x^r-x}{x^r}, & x^m\le x\le x^m+x^r\end{cases}$$

Here, x^m is called mean value and x^r,x^l called the right and the left spreads of membership function, respectively. We denote the TFN by $\tilde{N}=(x^l,x^m,x^r)$.

Moreover, $\alpha-$cut set of TFN is defined as follows that:

$$\tilde{N}_\alpha=[\tilde{N}^L_\alpha,\tilde{N}^U_\alpha]=[(x^m-x^l)+\alpha x^l,(x^m+x^r)-\alpha x^r].$$

2.2 Two-stage DEA Model

Suppose there are n $DMUs$ with two-stage structure. Each $DMU_j\,(j=1,\dots,n)$, in stage 1 consumes inputs $x_{ij}\,(i=1,\dots,m)$ to produce intermediate measures $z_{dj}\,(d=1,\dots,D)$. Then, in stage2, these intermediate measures, are used to produce final outputs $y_{rj}\,(r=1,\dots,s)$. The structure of this system is shown in Figure1.

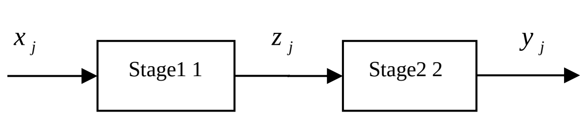

Figure 1. Two-stage production system

In order to measure the efficiency of DMU_o (DMU under evaluation), Kao and Hwang [13], proposed the following model:

$$\begin{aligned}
E_o^* = \max \quad & \sum_{r=1}^{s} u_r y_{ro} \\
s.t \quad & \sum_{i=1}^{m} v_i x_{io} = 1 \\
& \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} \le 0 \\
& u_r, v_i, w_d \ge 0 \quad r=1,\dots,s \;\; i=1,\dots,m \;\; d=1,\dots,D
\end{aligned} \tag{1}$$

Suppose (u_r^*, v_i^*, w_d^*) is an optimal solution of model (1). The overall efficiency (E_o^*) and efficiency of stages ($E_o^{\mathrm{I}*}, E_o^{\mathrm{II}*}$) are as follows:

$$E_o^* = \frac{\sum_{r=1}^{s} u_r^* y_{ro}}{\sum_{i=1}^{m} v_i^* x_{io}}, \quad E_o^{\mathrm{I}*} = \frac{\sum_{d=1}^{D} w_d^* z_{do}}{\sum_{i=1}^{m} v_i^* x_{io}}, \quad E_o^{\mathrm{II}*} = \frac{\sum_{r=1}^{s} u_r^* y_{ro}}{\sum_{d=1}^{D} w_d^* z_{do}}$$

Theorem 1. DMU_o is overall efficient ($E_o^* = 1$) if and only if $E_o^{\mathrm{I}*} = E_o^{\mathrm{II}*} = 1$.

3. Proposed fuzzy two-stage models with TFNs

In this section, firstly we indicate the structure of extended two-stage systems that were presented by Jianfeng [11]. Then, based on the formulation of the multiplicative efficiency decomposition model of (Kao and Hwang [13]), we will present the model to measure the efficiency of extended two-stage systems in presence of TFNs.

Suppose we have n $DMUs$ with an extended two-stage structure where some inputs are shared between two-stages. In stage1, each DMU_j produces intermediate measures $z_{dj}(d=1,\dots,D)$ by consuming input $x_{ij}(i=1,\dots,m)$ and shared input $x'_{hj}(h=1,\dots,H)$. And also, stage2 consumes inputs $x''_{fj}(f=1,\dots,F)$ (that are associated with stage2 directly), shared inputs $x'_{hj}(h=1,\dots,H)$ and the part of intermediate measures $z_{dj}(d=1,\dots,D)$ to generate outputs $y_{rj}(r=1,\dots,s)$. The structure of an extended two-stage system is depicted in Figure 2.

In this system, the contribution of each stage in the use of shared inputs is not known. Also, the portion of the second stage for using intermediate measures (as input) is not specified. Therefore, we use parameters α'_{hj}, β_{dj} to identify the contribution of each stage. In this regard, we show the portion of stage1 of the i-th shared input with $\alpha'_{hj} x'_{hj}$ and the portion of the stage 2 with $(1-\alpha'_{hj})x'_{hj}$. And the portion of the second stage of d-th intermediate product is also displayed with $\beta_{dj} z_{dj}$. Note that, $(1-\beta_{dj})z_{dj}$ is considered as the final output in the whole system. In order to better interpret of the results, it is assumed that $\alpha'_{hj}, \beta_{dj} \in [0,1]$.

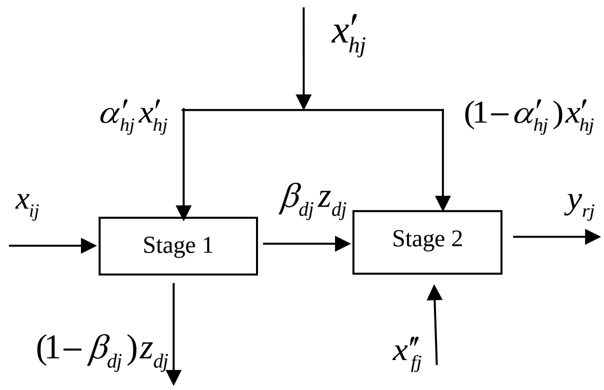

Figure 2. Extended two-stage system

Jianfeng [11] presented the additive efficiency decomposition models for evaluating the overall efficiency of these systems and stages. Note that, in many practical problems, there are systems with imprecise information such as interval data, stochastic data, fuzzy number and so on. In many two-stage systems, the simultaneous presence of both stages in the final production is required. And the shortcoming of one stage is not compensated by another stage. In such circumstances (in such cases), it is of particular important for the DM to evaluate the performance of these systems. Therefore, considering these mentioned cases, we will use the non-compensatory property of the Multiplicative operator and suggest a multiplicative model that calculates the efficiency of these systems in the presence of triangular fuzzy numbers. Suppose all inputs, outputs and intermediate measures are TFN:

$$\tilde{x}_{ij}=(\tilde{x}^l_{ij},\tilde{x}^m_{ij},\tilde{x}^r_{ij}) \quad , \quad \tilde{x}'_{hj}=(\tilde{x}'^l_{hj},\tilde{x}'^m_{hj},\tilde{x}'^r_{hj}) \quad , \quad \tilde{x}''_{fj}=(\tilde{x}''^l_{fj},\tilde{x}''^m_{fj},\tilde{x}''^r_{fj})$$
$$\tilde{z}_{dj}=(\tilde{z}^l_{dj},\tilde{z}^m_{dj},\tilde{z}^r_{dj}) \quad , \quad \tilde{y}_{rj}=(\tilde{y}^l_{rj},\tilde{y}^m_{rj},\tilde{y}^r_{rj})$$

In which, m represents the mean value and r, l represent the right and left spreads corresponding to the data, respectively. Note that in this paper, ~ indicates that the inputs, intermediate measures and outputs are fuzzy. Also, $\tilde{1}=(1^l,1^m,1^r)=(1,1,1)$. The proposed model is as follows:

$$\begin{aligned}
&\tilde{E}^{s*}_o=\max\ \tilde{E}^{\mathrm{I}}_o\times\tilde{E}^{\mathrm{II}}_o=\frac{\sum_{d=1}^{D}w_d\tilde{z}_{do}}{\sum_{i=1}^{m}v_i\tilde{x}_{io}+\sum_{h=1}^{H}p_h\alpha'_{ho}\tilde{x}'_{ho}}\times\frac{\sum_{r=1}^{s}u_r\tilde{y}_{ro}}{\sum_{d=1}^{D}w_d\beta_{do}\tilde{z}_{do}+\sum_{h=1}^{H}p_h(1-\alpha'_{ho})\tilde{x}'_{ho}+\sum_{f=1}^{F}c_f\tilde{x}''_{fo}}\\
&s.t\qquad \frac{\sum_{d=1}^{D}w_d\tilde{z}_{dj}}{\sum_{i=1}^{m}v_i\tilde{x}_{ij}+\sum_{h=1}^{H}p_h\alpha'_{hj}\tilde{x}'_{hj}}\le\tilde{1},\qquad j=1,\dots,n\\
&\qquad\qquad \frac{\sum_{r=1}^{s}u_r\tilde{y}_{rj}}{\sum_{d=1}^{D}w_d\beta_{dj}\tilde{z}_{dj}+\sum_{h=1}^{H}p_h(1-\alpha'_{hj})\tilde{x}'_{hj}+\sum_{f=1}^{F}c_f\tilde{x}''_{fj}}\le\tilde{1},\qquad j=1,\dots,n\\
&\qquad\qquad \alpha'_{hj},\beta_{dj}\in[0,1],\qquad h=1,\dots,H\quad d=1,\dots,D\quad j=1,\dots,n\\
&\qquad\qquad u_r,v_i,w_d,p_h,c_f\ge 0,\qquad r=1,\dots,s\quad i=1,\dots,m\quad d=1,\dots,D\quad h=1,\dots,H\quad f=1,\dots,F
\end{aligned}\tag{2}$$

In this model, $\tilde{E}_o^{\mathrm{I}}, \tilde{E}_o^{\mathrm{II}}$ represent the efficiency of the first and second stages, respectively. Also, the overall efficiency of the system (denoted by $\tilde{E}_o^s$) is considered as a product of the efficiency of the stages. Also, the first and second constraints set ensure that the efficiency of the stages (and whole system) lines to be at or under unity. Parameters β_{dj}, α'_{hj} have been used to determine the portion of the second stage for the use of intermediate measure $\tilde{z}_{dj}$ and both stages in the use of shared input $\tilde{x}'_{hj}$, respectively. And, the corresponding weights of $d-$th intermediate measure in the stage1 and the stage2 are equally considered as w_d .

Weights v_i, u_r are assumed to be the $i-$th initial input weight of the stage1 and the $r-$th final output weight of the stage2, respectively. Also, the weights corresponding to the $f-$th input $\tilde{x}''_{fj}$ and the $h-$th shared input $\tilde{x}'_{hj}$ are considered as c_f, p_h . The proposed model is a nonlinear model that is not converted to a linear form by using Charnes-Cooper transformation [2].

Therefore, we propose the leader-follower approach (non-cooperative approach) to solve this model. With no loss of without losing generality, the first stage is considered as the leader stage and the second stage as the follower stage. Hence, firstly, we calculate the efficiency of stage1 by solving the following model:

$$
\begin{aligned}
&\tilde{E}_o^{\mathrm{I}*} = \max \ \frac{\sum_{d=1}^{D} w_d \tilde{z}_{do}}{\sum_{i=1}^{m} v_i \tilde{x}_{io} + \sum_{h=1}^{H} p_h \alpha'_{ho} \tilde{x}'_{ho}} \\
&s.t \quad \frac{\sum_{d=1}^{D} w_d \tilde{z}_{dj}}{\sum_{i=1}^{m} v_i \tilde{x}_{ij} + \sum_{h=1}^{H} p_h \alpha'_{hj} \tilde{x}'_{hj}} \le \tilde{1}, \qquad j=1,\dots,n \\
&\alpha'_{hj} \in [0,1], \qquad h=1,\dots,H \quad j=1,\dots,n \\
&v_i, w_d, p_h \ge 0, \qquad i=1,\dots,m \quad d=1,\dots,D \quad h=1,\dots,H
\end{aligned} \tag{3}
$$

Now, we use $\alpha-$cut technique to solve this fuzzy model. For this purpose, $\alpha-$cut intervals of inputs, intermediate measures and outputs are as follows [38]:

$$
\begin{aligned}
&(\tilde{x}_{ij})_\alpha = [\tilde{x}_{ij\alpha}^L, \tilde{x}_{ij\alpha}^U] = [\tilde{x}_{ij}^m - \tilde{x}_{ij}^l(1-\alpha), \tilde{x}_{ij}^m + \tilde{x}_{ij}^r(1-\alpha)] \\
&(\tilde{x}'_{hj})_\alpha = [\tilde{x}'^L_{hj\alpha}, \tilde{x}'^U_{hj\alpha}] = [\tilde{x}'^m_{hj} - \tilde{x}'^l_{hj}(1-\alpha), \tilde{x}'^m_{hj} + \tilde{x}'^r_{hj}(1-\alpha)] \\
&(\tilde{x}''_{fj})_\alpha = [\tilde{x}''^L_{fj\alpha}, \tilde{x}''^U_{fj\alpha}] = [\tilde{x}''^m_{fj} - \tilde{x}''^l_{fj}(1-\alpha), \tilde{x}''^m_{fj} + \tilde{x}''^r_{fj}(1-\alpha)] \\
&(\tilde{z}_{dj})_\alpha = [\tilde{z}_{dj\alpha}^L, \tilde{z}_{dj\alpha}^U] = [\tilde{z}_{dj}^m - \tilde{z}_{dj}^l(1-\alpha), \tilde{z}_{dj}^m + \tilde{z}_{dj}^r(1-\alpha)] \\
&(\tilde{y}_{rj})_\alpha = [\tilde{y}_{rj\alpha}{}^L, \tilde{y}_{rj\alpha}^U] = [\tilde{y}_{rj}^m - \tilde{y}_{rj}^l(1-\alpha), \tilde{y}_{rj}^m + \tilde{y}_{rj}^r(1-\alpha)]
\end{aligned} \tag{4}
$$

Noted that $\alpha-$cut of fuzzy number $\tilde{1}$ can be considered as interval $[1,1]$. By applying intervals (4) to model (3), the following interval model is obtained:

$$[\tilde{E}_o^{\mathrm{I}(L)*},\tilde{E}_o^{\mathrm{I}(U)*}]=\max \frac{\sum_{d=1}^{D} w_d[\tilde{z}_{do\alpha}^L,\tilde{z}_{do\alpha}^U]}{\sum_{i=1}^{m} v_i[\tilde{x}_{io\alpha}^L,\tilde{x}_{io\alpha}^U]+\sum_{h=1}^{H} p_h\alpha'_{ho}[\tilde{x}'^L_{ho\alpha},\tilde{x}'^U_{ho\alpha}]}$$
$$\begin{aligned} s.t \quad & \frac{\sum_{d=1}^{D} w_d[\tilde{z}_{dj\alpha}^L,\tilde{z}_{dj\alpha}^U]}{\sum_{i=1}^{m} v_i[\tilde{x}_{ij\alpha}^L,\tilde{x}_{ij\alpha}^U]+\sum_{h=1}^{H} p_h\alpha'_{hj}[\tilde{x}'^L_{hj\alpha},\tilde{x}'^U_{hj\alpha}]}\le[1,1], \qquad j=1,\dots,n \\ & \alpha'_{hj}\in[0,1], \qquad h=1,\dots,H \quad j=1,\dots,n \\ & v_i,w_d,p_h\ge 0, \qquad i=1,\dots,m \quad d=1,\dots,D \quad h=1,\dots,H \end{aligned} \tag{5}$$

In this model, the first constraint set ensures that the efficiency of stage1 does not exceed the value of one. This model is converted to a linear form using the following transformation.

Suppose $t=\frac{1}{\sum_{i=1}^{m} v_i[\tilde{x}_{io\alpha}^L,\tilde{x}_{io\alpha}^U]+\sum_{h=1}^{H} p_h\alpha'_{ho}[\tilde{x}'^L_{ho\alpha},\tilde{x}'^U_{ho\alpha}]}$. Then: $tv_i=v'_i$, $tp_h=p'_h$, $p'_h\alpha'_{hj}=q_{hj}$, $tw_d=w'_d$.

Therefore, the following linear model is obtained:

$$\begin{aligned} [\tilde{E}_o^{\mathrm{I}(L)*},\tilde{E}_o^{\mathrm{I}(U)*}]=\max\ & \sum_{d=1}^{D} w'_d[\tilde{z}_{do\alpha}^L,\tilde{z}_{do\alpha}^U] \\ s.t \quad & \sum_{i=1}^{m} v'_i[\tilde{x}_{io\alpha}^L,\tilde{x}_{io\alpha}^U]+\sum_{h=1}^{H} q_{ho}[\tilde{x}'^L_{ho\alpha},\tilde{x}'^U_{ho\alpha}]=[1,1] \\ & \sum_{d=1}^{D} w'_d[\tilde{z}_{dj\alpha}^L,\tilde{z}_{dj\alpha}^U]-\sum_{i=1}^{m} v'_i[\tilde{x}_{ij\alpha}^L,\tilde{x}_{ij\alpha}^U]-\sum_{h=1}^{H} q_{hj}[\tilde{x}'^L_{hj\alpha},\tilde{x}'^U_{hj\alpha}]\le 0, \qquad j=1,\dots,n \\ & 0\le q_{hj}\le p_h \qquad h=1,\dots,H \quad j=1,\dots,n \\ & v'_i,w'_d,p_h\ge 0, \qquad i=1,\dots,m \quad d=1,\dots,D \quad h=1,\dots,H \end{aligned} \tag{6}$$

Interval model (6) is easily solved with an optimistic and pessimistic approach.

To calculate the upper bound of the $\alpha-$cut for stage1 of DMU_o ($\tilde{E}_o^{\mathrm{I}(U)*}$), it is assumed that DMU_o has the most favorable conditions and other $DMUs$ have unfavorable conditions (or: the worst condition). In other words, DMU_o consumes the lowest inputs ($\tilde{x}_{io\alpha}^L,\tilde{x}'^L_{ho\alpha}$) for producing the largest output ($\tilde{z}_{do\alpha}^U$).

Moreover, in the other $DMUs$, the worst values of inputs ($\tilde{x}_{io\alpha}^U,\tilde{x}'^U_{ho\alpha}$) are consumed to produce the lowest amount of output ($\tilde{z}_{do\alpha}^L$). Therefore, for obtaining $\tilde{E}_o^{\mathrm{I}(U)*}$, the following model is proposed:

$$\begin{aligned} \tilde{E}_o^{\mathrm{I}(U)*}=\max\ & \sum_{d=1}^{D} w'_d\tilde{z}_{do\alpha}^U \\ s.t \quad & \sum_{i=1}^{m} v'_i\tilde{x}_{io\alpha}^L+\sum_{h=1}^{H} q_{ho}\tilde{x}'^L_{ho\alpha}=1 \\ & \sum_{d=1}^{D} w'_d\tilde{z}_{do\alpha}^U-\sum_{i=1}^{m} v'_i\tilde{x}_{io\alpha}^L-\sum_{h=1}^{H} q_{hj}\tilde{x}'^L_{ho\alpha}\le 0, \\ & \sum_{d=1}^{D} w'_d\tilde{z}_{dj\alpha}^L-\sum_{i=1}^{m} v'_i\tilde{x}_{ij\alpha}^U-\sum_{h=1}^{H} q_{hj}\tilde{x}'^U_{hj\alpha}\le 0, \qquad j=1,\dots,n(j\ne o) \\ & 0\le q_{hj}\le p_h \qquad h=1,\dots,H \quad j=1,\dots,n \\ & v'_i,w'_d,p_h\ge 0, \qquad i=1,\dots,m \quad d=1,\dots,D \quad h=1,\dots,H \end{aligned} \tag{7}$$

Now to find the lower bound of the $\alpha-$cut for stage1 of DMU_o ($\tilde{E}_o^{\mathrm{I}(U)*}$), we consider the most unfavorable conditions for DMU_o. Indeed, DMU_o generates the smallest values of outputs by using the largest amount of inputs. And also, other DMUs consume the lowest amounts of inputs to produce the worst output values. Hence, can be obtained for solving the following model:

$$\begin{aligned}
\tilde{E}_o^{\mathrm{I}(L)*} = \max\ & \sum_{d=1}^{D} w'_d \tilde{z}_{do\alpha}^{L} \\
s.t\quad & \sum_{i=1}^{m} v'_i \tilde{x}_{io\alpha}^{U} + \sum_{h=1}^{H} q_{ho} \tilde{x}'^{U}_{ho\alpha} = 1 \\
& \sum_{d=1}^{D} w'_d \tilde{z}_{do\alpha}^{L} - \sum_{i=1}^{m} v'_i \tilde{x}_{io\alpha}^{U} - \sum_{h=1}^{H} q_{hj} \tilde{x}'^{U}_{ho\alpha} \le 0, \\
& \sum_{d=1}^{D} w'_d \tilde{z}_{dj\alpha}^{U} - \sum_{i=1}^{m} v'_i \tilde{x}_{ij\alpha}^{L} - \sum_{h=1}^{H} q_{hj} \tilde{x}'^{L}_{hj\alpha} \le 0, \qquad j=1,\dots,n (j \ne o) \\
& 0 \le q_{hj} \le p_h \qquad h=1,\dots,H \quad j=1,\dots,n \\
& v'_i, w'_d, p_h \ge 0, \qquad i=1,\dots,m \quad d=1,\dots,D \quad h=1,\dots,H
\end{aligned} \tag{8}$$

Therefore, the efficiency of stage1 (leader stage) is obtained as $[\tilde{E}_o^{\mathrm{I}(L)*}, \tilde{E}_o^{\mathrm{I}(U)*}]$.

Definition 3. In stage1, DMU_o is $\alpha-cut$ efficient in the lower bound if and only if $\tilde{E}_o^{\mathrm{I}(L)*} = 1$.

Definition 4. In stage1, DMU_o is efficient in the upper bound if and only if $\tilde{E}_o^{\mathrm{I}(U)*} = 1$.

Now, in order to measure the follower stage (stage2) efficiency, the following model is suggested:

$$\begin{aligned}
\tilde{E}_o^{\mathrm{II}*} = \max\ & \frac{\sum_{r=1}^{s} u_r \tilde{y}_{ro}}{\sum_{d=1}^{D} w_d \beta_{do} \tilde{z}_{do} + \sum_{h=1}^{H} p_h (1-\alpha'_{ho}) \tilde{x}'_{ho} + \sum_{f=1}^{F} c_f \tilde{x}''_{fo}} \\
s.t\quad & \frac{\sum_{d=1}^{D} w_d \tilde{z}_{dj}}{\sum_{i=1}^{m} v_i \tilde{x}_{ij} + \sum_{h=1}^{H} p_h \alpha'_{hj} \tilde{x}'_{hj}} \le \tilde{1}, \qquad j=1,\dots,n \\
& \frac{\sum_{r=1}^{s} u_r \tilde{y}_{rj}}{\sum_{d=1}^{D} w_d \beta_{dj} \tilde{z}_{dj} + \sum_{h=1}^{H} p_h (1-\alpha'_{hj}) \tilde{x}'_{hj} + \sum_{f=1}^{F} c_f \tilde{x}''_{fj}} \le \tilde{1}, \qquad j=1,\dots,n \\
& \frac{\sum_{d=1}^{D} w_d \tilde{z}_{do}}{\sum_{i=1}^{m} v_i \tilde{x}_{io} + \sum_{h=1}^{H} p_h \alpha'_{ho} \tilde{x}'_{ho}} = [\tilde{E}_o^{\mathrm{I}(L)*}, \tilde{E}_o^{\mathrm{I}(U)*}] \\
& \alpha'_{hj}, \beta_{dj} \in [0,1], \qquad h=1,\dots,H \quad d=1,\dots,D \quad j=1,\dots,n \\
& u_r, v_i, w_d, p_h, c_f \ge 0, \qquad r=1,\dots,s \quad i=1,\dots,m \quad d=1,\dots,D \quad h=1,\dots,H \quad f=1,\dots,F
\end{aligned} \tag{9}$$

Now, by applying intervals (4) to model (9), the following interval model is obtained:

$$[\tilde{E}_o^{\mathrm{II}(L)*},\tilde{E}_o^{\mathrm{II}(U)*}]=\max\ \frac{\sum_{r=1}^{s}u_r[\tilde{y}_{ro\alpha}{}^{L},\tilde{y}_{ro\alpha}^{U}]}{\sum_{d=1}^{D}w_d\beta_{do}[\tilde{z}_{do\alpha}^{L},\tilde{z}_{do\alpha}^{U}]+\sum_{h=1}^{H}p_h(1-\alpha'_{ho})[\tilde{x}'^{L}_{ho\alpha},\tilde{x}'^{U}_{ho\alpha}]+\sum_{f=1}^{F}c_f[\tilde{x}''^{L}_{fo\alpha},\tilde{x}''^{U}_{fo\alpha}]}$$

$$s.t\qquad \frac{\sum_{d=1}^{D}w_d[\tilde{z}_{dj\alpha}^{L},\tilde{z}_{dj\alpha}^{U}]}{\sum_{i=1}^{m}v_i[\tilde{x}_{ij\alpha}^{L},\tilde{x}_{ij\alpha}^{U}]+\sum_{h=1}^{H}p_h\alpha'_{hj}[\tilde{x}'^{L}_{hj\alpha},\tilde{x}'^{U}_{hj\alpha}]}\le[1,1],\qquad j=1,\dots,n$$

$$\frac{\sum_{r=1}^{s}u_r[\tilde{y}_{rj\alpha}{}^{L},\tilde{y}_{rj\alpha}^{U}]}{\sum_{d=1}^{D}w_d\beta_{dj}[\tilde{z}_{dj\alpha}^{L},\tilde{z}_{dj\alpha}^{U}]+\sum_{h=1}^{H}p_h(1-\alpha'_{hj})[\tilde{x}'^{L}_{hj\alpha},\tilde{x}'^{U}_{hj\alpha}]+\sum_{f=1}^{F}c_f[\tilde{x}''^{L}_{fj\alpha},\tilde{x}''^{U}_{fj\alpha}]}\le[1,1],\quad j=1,\dots,n$$

$$\frac{\sum_{d=1}^{D}w_d[\tilde{z}_{do\alpha}^{L},\tilde{z}_{do\alpha}^{U}]}{\sum_{i=1}^{m}v_i[\tilde{x}_{ij\alpha}^{L},\tilde{x}_{ij\alpha}^{U}]+\sum_{h=1}^{H}p_h\alpha'_{ho}[\tilde{x}'^{L}_{ho\alpha},\tilde{x}'^{U}_{ho\alpha}]}=[\tilde{E}_o^{\mathrm{I}(L)*},\tilde{E}_o^{\mathrm{I}(U)*}]$$

$$\alpha'_{hj},\beta_{dj}\in[0,1],\qquad h=1,\dots,H\quad d=1,\dots,D\quad j=1,\dots,n$$

$$u_r,v_i,w_d,p_h,c_f\ge 0,\qquad r=1,\dots,s\quad i=1,\dots,m\quad d=1,\dots,D\quad h=1,\dots,H\quad f=1,\dots,F$$

(10)

Note that in this model, the efficiency of stage 2 is calculated while the efficiency of the leader stage (stage1) is unchanged. Also, the first and second constraints set also ensure that the efficiency of the first and second stages does not exceed the value of one.

By using transformation $t=\frac{1}{\sum_{d=1}^{D}w_d\beta_{do}[\tilde{z}_{do\alpha}^{L},\tilde{z}_{do\alpha}^{U}]+\sum_{h=1}^{H}p_h(1-\alpha'_{ho})[\tilde{x}'^{L}_{ho\alpha},\tilde{x}'^{U}_{ho\alpha}]+\sum_{f=1}^{F}c_f[\tilde{x}''^{L}_{fo\alpha},\tilde{x}''^{U}_{fo\alpha}]}$, $tv_i=v'_i$, $tp_h=p'_h$, $p'_h\alpha'_{hj}=q_{hj}$, $tw_d=w'_d$, $w'_d\beta_{dj}=w''_{dj}$ $tc_f=c'_f$ $tu_r=u'_r$, model (10) can be converted into the following linear form:

$$
\begin{aligned}
&[\tilde{E}_o^{\mathrm{II}(L)*},\tilde{E}_o^{\mathrm{II}(U)*}]=\max\ \sum_{r=1}^{s}u'_r[\tilde{y}_{ro\alpha}{}^{L},\tilde{y}_{ro\alpha}^{U}]\\
&s.t \qquad \sum_{d=1}^{D}w''_{do}[\tilde{z}_{do\alpha}^{L},\tilde{z}_{do\alpha}^{U}]+\sum_{h=1}^{H}(p'_h[\tilde{x}'^{L}_{ho\alpha},\tilde{x}'^{U}_{ho\alpha}]-q_{ho}[\tilde{x}'^{L}_{ho\alpha},\tilde{x}'^{U}_{ho\alpha}])+\sum_{f=1}^{F}c'_f[\tilde{x}''^{L}_{fo\alpha},\tilde{x}''^{U}_{fo\alpha}]=[1,1]\\
&\qquad \frac{\sum_{d=1}^{D}w'_d[\tilde{z}_{dj\alpha}^{L},\tilde{z}_{dj\alpha}^{U}]}{\sum_{i=1}^{m}v'_i[\tilde{x}_{ij\alpha}^{L},\tilde{x}_{ij\alpha}^{U}]+\sum_{h=1}^{H}q_{hj}[\tilde{x}'^{L}_{hj\alpha},\tilde{x}'^{U}_{hj\alpha}]}\le[1,1],\quad j=1,\dots,n\\
&\qquad \frac{\sum_{r=1}^{s}u'_r[\tilde{y}_{rj\alpha}{}^{L},\tilde{y}_{rj\alpha}^{U}]}{\sum_{d=1}^{D}w''_{dj}[\tilde{z}_{dj\alpha}^{L},\tilde{z}_{dj\alpha}^{U}]+\sum_{h=1}^{H}(p'_h[\tilde{x}'^{L}_{hj\alpha},\tilde{x}'^{U}_{hj\alpha}]-q_{hj}[\tilde{x}'^{L}_{hj\alpha},\tilde{x}'^{U}_{hj\alpha}])+\sum_{f=1}^{F}c_f[\tilde{x}''^{L}_{fj\alpha},\tilde{x}''^{U}_{fj\alpha}]}\le[1,1],\quad j=1,\dots,n\\
&\qquad \frac{\sum_{d=1}^{D}w'_d[\tilde{z}_{do\alpha}^{L},\tilde{z}_{do\alpha}^{U}]}{\sum_{i=1}^{m}v_i[\tilde{x}_{io\alpha}^{L},\tilde{x}_{io\alpha}^{U}]+\sum_{h=1}^{H}p_h\alpha'_{ho}[\tilde{x}'^{L}_{hj\alpha},\tilde{x}'^{U}_{hj\alpha}]}=[\tilde{E}_o^{\mathrm{I}(L)*},\tilde{E}_o^{\mathrm{I}(U)*}]\\
&\qquad 0\le \mathrm{q}_{hj}\le p'_h,\qquad h=1,\dots,H\quad j=1,\dots,n\\
&\qquad 0\le w''_{dj}\le w'_d,\qquad d=1,\dots,D\quad j=1,\dots,n\\
&\qquad u'_r,v'_i,w'_d,p'_h,c'_f\ge 0,\quad r=1,\dots,s\quad i=1,\dots,m\quad d=1,\dots,D\quad h=1,\dots,H\quad f=1,\dots,F
\end{aligned}
\tag{11}
$$

Similarly, the upper bound of the α-cut for stage 2 of DMU_o is obtained from solving the following model:

$$
\begin{aligned}
&\tilde{E}_o^{\mathrm{II}(U)*}=\max\ \sum_{r=1}^{s}u'_r\tilde{y}_{ro\alpha}^{U}\\
&s.t \qquad \sum_{d=1}^{D}w''_{do}\mathrm{z}_{do}+\sum_{h=1}^{H}(p'_h-q_{ho})\tilde{x}'^{L}_{ho\alpha}+\sum_{f=1}^{F}c'_f\tilde{x}''^{L}_{fo\alpha}=1\\
&\qquad \sum_{d=1}^{D}w'_d\mathrm{z}_{do}-\sum_{i=1}^{m}v'_i\tilde{x}_{io\alpha}^{L}-\sum_{h=1}^{H}q_{ho}\tilde{x}'^{L}_{ho\alpha}\le 0,\\
&\qquad \mathrm{z}^{L}{}_{dj\alpha}\le\tilde{\mathrm{z}}_{dj}\le\mathrm{z}^{U}{}_{dj\alpha},\qquad d=1,\dots,D\quad j=1,\dots,n\\
&\qquad 0\le \mathrm{q}_{hj}\le p'_h,\qquad h=1,\dots,H\quad j=1,\dots,n\\
&\qquad 0\le w''_{dj}\le w'_d,\qquad d=1,\dots,D\quad j=1,\dots,n\\
&\qquad u'_r,v'_i,w'_d,p'_h,c'_f\ge 0,\quad r=1,\dots,s\quad i=1,\dots,m\quad d=1,\dots,D\quad h=1,\dots,H\quad f=1,\dots,F
\end{aligned}
\tag{12}
$$

Actually, in order to calculate $\tilde{E}_o^{\mathrm{II}(U)*}$, the efficiency of stage 1 is in the best conditions (i.e., $\tilde{E}_o^{\mathrm{I}(U)*}$). Also, similar to model (7), the most favorable condition is considered for DMU_o. In this model, it must be noted that intermediate measures are used as an interval ($\mathrm{z}^{L}{}_{dj\alpha}\le\tilde{\mathrm{z}}_{dj}\le\mathrm{z}^{U}{}_{dj\alpha}$). Actually, the data are as interval and intermediate measures are the output of the first stage and the inputs of the second stage. Hence, we consider $\mathrm{z}^{L}{}_{dj\alpha}\le\tilde{\mathrm{z}}_{dj}\le\mathrm{z}^{U}{}_{dj\alpha}$ as a constraint and solve this model to obtain the optimal value for intermediate measures.

Now, we use transformation $w'_d\mathrm{z}_{dj}=\hat{w}_{dj}$, $w''_{dj}\mathrm{z}_{dj}=\hat{w}'_{dj}$. The constraints can be transformed to $w'_d\mathrm{z}^{L}{}_{dj\alpha}\le\hat{w}_{dj}\le w'_d\mathrm{z}^{U}{}_{dj\alpha}$, $w''_{dj}\mathrm{z}^{L}{}_{dj\alpha}\le\hat{w}'_{dj}\le w''_{dj}\mathrm{z}^{U}{}_{dj\alpha}$. Hence, the following model is obtained:

$$
\begin{aligned}
&\tilde{E}_o^{\mathrm{II}(U)*} = \max \sum_{r=1}^{s} u'_r \tilde{y}_{ro\alpha}^{U} \\
&s.t \quad \sum_{d=1}^{D} \hat{w}'_{do} + \sum_{h=1}^{H} (p'_h - q_{ho}) \tilde{x}'^{L}_{ho\alpha} + \sum_{f=1}^{F} c'_f \tilde{x}''^{L}_{fo\alpha} = 1 \\
&\sum_{d=1}^{D} \hat{w}_{do} - \sum_{i=1}^{m} v'_i \tilde{x}_{io\alpha}^{L} - \sum_{h=1}^{H} q_{ho} \tilde{x}'^{L}_{ho\alpha} \le 0, \\
&\sum_{d=1}^{D} \hat{w}_{dj} - \sum_{i=1}^{m} v'_i \tilde{x}_{ij\alpha}^{U} - \sum_{h=1}^{H} q_{hj} \tilde{x}'^{U}_{hj\alpha} \le 0, \qquad j = 1,\dots,n\,(j \ne o) \\
&\sum_{r=1}^{s} u'_r \tilde{y}_{ro\alpha}^{U} - \sum_{d=1}^{D} \hat{w}'_{do} - \sum_{h=1}^{H} (p'_h - q_{hj}) \tilde{x}'^{L}_{ho\alpha} - \sum_{f=1}^{F} c'_f \tilde{x}''^{L}_{fo\alpha} \le 0, \\
&\sum_{r=1}^{s} u'_r \tilde{y}_{rj\alpha}{}^{L} - \sum_{d=1}^{D} \hat{w}'_{dj} - \sum_{h=1}^{H} (p'_h - q_{hj}) \tilde{x}'^{U}_{hj\alpha} - \sum_{f=1}^{F} c'_f \tilde{x}''^{U}_{fj\alpha} \le 0, \quad j = 1,\dots,n\,(j \ne o) \\
&\sum_{d=1}^{D} \hat{w}_{do} = \tilde{E}_o^{\mathrm{I}(U)*} \times \Big(\sum_{i=1}^{m} v'_i \tilde{x}_{io\alpha}^{L} + \sum_{h=1}^{H} q_{ho} \tilde{x}'^{L}_{ho\alpha}\Big) \\
&w'_d \mathrm{z}^{L}{}_{dj\alpha} \le \hat{w}_{dj} \le w'_d \mathrm{z}^{U}{}_{dj\alpha}, \qquad d = 1,\dots,D \quad j = 1,\dots,n \\
&w''_{dj} \mathrm{z}^{L}{}_{dj\alpha} \le \hat{w}'_{dj} \le w''_{dj} \mathrm{z}^{U}{}_{dj\alpha}, \qquad d = 1,\dots,D \quad j = 1,\dots,n \\
&0 \le \mathrm{q}_{hj} \le p'_h, \qquad h = 1,\dots,H \quad j = 1,\dots,n \\
&0 \le w''_{dj} \le w'_d, \qquad d = 1,\dots,D \quad j = 1,\dots,n \\
&\hat{w}_{dj}, \hat{w}'_{dj} \ge 0, \qquad d = 1,\dots,D \quad j = 1,\dots,n \\
&u'_r, v'_i, w'_d, p'_h, c'_f \ge 0, \quad r = 1,\dots,s \quad i = 1,\dots,m \quad d = 1,\dots,D \quad h = 1,\dots,H \quad f = 1,\dots,F
\end{aligned}
\tag{13}
$$

Therefore, if $(u'^*_r, v'^*_i, w'^*_d, p'^*_h, c'^*_f, \mathrm{q}^*{}_{ho}, w''^*_{do}, \hat{w}^*_{do}, \hat{w}'^*_{do})$ be an optimal solution of model (13), the upper bound of the $\alpha-$cut for whole system corresponding to DMU_o is defined as follows:

$$\tilde{E}_o^{(U)*} = \tilde{E}_o^{\mathrm{I}(U)*} \times \tilde{E}_o^{\mathrm{II}(U)*}$$

Definition 5. In stage2, DMU_o is efficient in the upper bound if and only if $\tilde{E}_o^{\mathrm{II}(U)*} = 1$.

Definition 6. DMU_o is overall efficient in the upper bound if and only if $\tilde{E}_o^{(U)*} = 1$.

Also, model (13) calculates the lower bound of the α-cut for stage 2 of DMU_o :

$$\begin{aligned}
&\tilde{E}_o^{\mathrm{II}(L)*} = \max \ \sum_{r=1}^{s} u_r' \tilde{y}_{ro\alpha}^{L} \\
&s.t \qquad \sum_{d=1}^{D} w_{do}'' \mathrm{z}_{do} + \sum_{h=1}^{H} (p_h' - q_{ho}) \tilde{x}_{ho\alpha}'^{U} + \sum_{f=1}^{F} c_f' \tilde{x}_{fo\alpha}''^{U} = 1 \\
&\qquad \sum_{d=1}^{D} w_d' \mathrm{z}_{do} - \sum_{i=1}^{m} v_i' \tilde{x}_{io\alpha}^{U} - \sum_{h=1}^{H} q_{ho} \tilde{x}_{ho\alpha}'^{U} \le 0, \\
&\qquad \sum_{d=1}^{D} w_d' \mathrm{z}_{dj} - \sum_{i=1}^{m} v_i' \tilde{x}_{ij\alpha}'^{L} - \sum_{h=1}^{H} q_{hj} \tilde{x}_{hj\alpha}'^{L} \le 0, \qquad j = 1,\dots,n (j \ne o) \\
&\qquad \sum_{r=1}^{s} u_r' \tilde{y}_{ro\alpha}^{L} - \sum_{d=1}^{D} w_{do}'' \mathrm{z}_{do} - \sum_{h=1}^{H} (p_h' - q_{hj}) \tilde{x}_{ho\alpha}'^{U} - \sum_{f=1}^{F} c_f' \tilde{x}_{fo\alpha}''^{U} \le 0, \\
&\qquad \sum_{r=1}^{s} u_r' \tilde{y}_{rj\alpha}^{U} - \sum_{d=1}^{D} w_{dj}'' \mathrm{z}_{dj} - \sum_{h=1}^{H} (p_h' - q_{hj}) \tilde{x}_{hj\alpha}^{L} - \sum_{f=1}^{F} c_f' \tilde{x}_{fj\alpha}''^{L} \le 0, \quad j = 1,\dots,n (j \ne o) \\
&\qquad \sum_{d=1}^{D} w_d' \mathrm{z}_{do} = \tilde{E}_o^{\mathrm{I}(L)*} \times (\sum_{i=1}^{m} v_i' \tilde{x}_{io\alpha}^{U} + \sum_{h=1}^{H} q_{ho} \tilde{x}_{ho\alpha}'^{U}) \\
&\qquad \mathrm{z}^{L}{}_{dj\alpha} \le \tilde{\mathrm{z}}_{dj} \le \mathrm{z}^{U}{}_{dj\alpha}, \qquad d = 1,\dots,D \quad j = 1,\dots,n \\
&\qquad 0 \le \mathrm{q}_{hj} \le p_h', \qquad h = 1,\dots,H \quad j = 1,\dots,n \\
&\qquad 0 \le w_{dj}'' \le w_d', \qquad d = 1,\dots,D \quad j = 1,\dots,n \\
&\qquad u_r', v_i', w_d', p_h', c_f' \ge 0, \quad r = 1,\dots,s \quad i = 1,\dots,m \quad d = 1,\dots,D \quad h = 1,\dots,H \quad f = 1,\dots,F
\end{aligned} \tag{14}$$

It should be noted that the efficiency of stage 1 is considered in the worst status (i.e., $\tilde{E}_o^{\mathrm{I}(L)*}$). Also, DMU_o has the most unfavorable conditions. Also, we use the transformation $w_d' \mathrm{z}_{dj} = \hat{w}_{dj}$, $w_{dj}'' \mathrm{z}_{dj} = \hat{w}_{dj}'$ for linearization of model (14):

$$\begin{aligned}
&\tilde{E}_o^{\mathrm{II}(L)*} = \max \ \sum_{r=1}^{s} u_r' \tilde{y}_{ro\alpha}^{L} \\
&s.t \qquad \sum_{d=1}^{D} \hat{w}_{do}' + \sum_{h=1}^{H} (p_h' - q_{ho}) \tilde{x}_{ho\alpha}'^{U} + \sum_{f=1}^{F} c_f' \tilde{x}_{fo\alpha}''^{U} = 1 \\
&\qquad \sum_{d=1}^{D} \hat{w}_{do} - \sum_{i=1}^{m} v_i' \tilde{x}_{io\alpha}^{U} - \sum_{h=1}^{H} q_{ho} \tilde{x}_{ho\alpha}'^{U} \le 0, \\
&\qquad \sum_{d=1}^{D} \hat{w}_{dj} - \sum_{i=1}^{m} v_i' \tilde{x}_{ij\alpha}^{L} - \sum_{h=1}^{H} q_{hj} \tilde{x}_{hj\alpha}'^{L} \le 0, \qquad j = 1,\dots,n (j \ne o) \\
&\qquad \sum_{d=1}^{D} \hat{w}_{do} = \tilde{E}_o^{\mathrm{I}(L)*} \times (\sum_{i=1}^{m} v_i' \tilde{x}_{io\alpha}^{U} + \sum_{h=1}^{H} q_{ho} \tilde{x}_{ho\alpha}'^{U}) \\
&\qquad \sum_{r=1}^{s} u_r' \tilde{y}_{ro\alpha}^{L} - \sum_{d=1}^{D} \hat{w}_{do}' - \sum_{h=1}^{H} (p_h' - q_{hj}) \tilde{x}_{ho\alpha}'^{U} - \sum_{f=1}^{F} c_f' \tilde{x}_{fo\alpha}''^{U} \le 0, \\
&\qquad \sum_{r=1}^{s} u_r' \tilde{y}_{rj\alpha}{}^{U} - \sum_{d=1}^{D} \hat{w}_{dj}' - \sum_{h=1}^{H} (p_h' - q_{hj}) \tilde{x}_{hj\alpha}'^{L} - \sum_{f=1}^{F} c_f' \tilde{x}_{fj\alpha}''^{L} \le 0, \quad j = 1,\dots,n (j \ne o) \\
&\qquad w_d' \mathrm{z}^{L}{}_{dj\alpha} \le \hat{w}_{dj} \le w_d' \mathrm{z}^{U}{}_{dj\alpha}, \qquad d = 1,\dots,D \quad j = 1,\dots,n \\
&\qquad w_{dj}'' \mathrm{z}^{L}{}_{dj\alpha} \le \hat{w}_{dj}' \le w_{dj}'' \mathrm{z}^{U}{}_{dj\alpha}, \qquad d = 1,\dots,D \quad j = 1,\dots,n \\
&\qquad \hat{w}_{dj}, \hat{w}_{dj}' \ge 0, \qquad d = 1,\dots,D \quad j = 1,\dots,n \\
&\qquad 0 \le \mathrm{q}_{hj} \le p_h', \qquad h = 1,\dots,H \quad j = 1,\dots,n \\
&\qquad 0 \le w_{dj}'' \le w_d', \qquad d = 1,\dots,D \quad j = 1,\dots,n \\
&\qquad u_r', v_i', w_d', p_h', c_f' \ge 0, \quad r = 1,\dots,s \quad i = 1,\dots,m \quad d = 1,\dots,D \quad h = 1,\dots,H \quad f = 1,\dots,F
\end{aligned} \tag{15}$$

And also, if $(u_r'^*, v_i'^*, w_d'^*, p_h'^*, c_f'^*, q^*_{ho}, w_{do}''^*, \hat{w}^*_{do}, \hat{w}'^*_{do})$ be an optimal solution of model (15), the lower bound of the $\alpha-$cut for whole system of DMU_o is defined as follows:

$$\tilde{E}_o^{(L)*} = \tilde{E}_o^{\mathrm{I}(L)*} \times \tilde{E}_o^{\mathrm{II}(L)*}$$

Definition7. In stage2, DMU_o is efficient in the lower bound if and only if $\tilde{E}_o^{\mathrm{II}(L)*} = 1$.

Definition8. DMU_o is overall efficient in the lower bound if and only if $\tilde{E}_o^{(L)*} = 1$.

Finally, we can conclude that the following theorems:

Theorem 2. DMU_o is overall efficient in the upper bound ($\tilde{E}_o^{(U)*} = 1$) if and only if $\tilde{E}_o^{\mathrm{I}(U)*} = \tilde{E}_o^{\mathrm{II}(U)*} = 1$.

Theorem3. DMU_o is overall efficient in the lower bound ($\tilde{E}_o^{(L)*} = 1$) if and only if $\tilde{E}_o^{\mathrm{I}(L)*} = \tilde{E}_o^{\mathrm{II}(L)*} = 1$.

If stage 2 is the most important stage from the point of view of DM, we can use the similar procedure for stage 2 and calculate the efficiency of stage 1 while the efficiency of stage 2 is unchanged.

It must be noted that, this paper, for the first time, used extended two-stage systems (Figure 2) in the presence of triangular fuzzy data.

Actually, in practice, many systems have an internal structure, it is appropriate to use this proposed approach in evaluating the efficiency of extended two-stage systems (Figure 2) (which are the triangular fuzzy type). The portion of stages in the use of these inputs and outputs is also determined using these models.

4. Case study

In this section, we will explain proposed models (7), (9), (13) and (15). For this, we use the data of 15 Chinese industrial sectors [11]. Note that each real number can be considered as a TFN. Hence, we consider data have TFN structure. Each company (DMU) viewed as two-stage system with shared inputs, the part of intermediate measure as input of stage2 and additional inputs in stage2. In this evaluation, stage 1 use input "the intramural expenditure on R&D" and shared input "the full time equivalent of R&D personnel" to produce "the projects for new product" and "the number of patents in force" as intermediate measures. Then, stage 2 consume the part of these intermediate measures, additional input "expenditure on new products development" and shared inputs to produce the final output "the gross industrial output value of new products". Therefore, we will illustrate proposed models by applying these inputs, outputs and intermediate measures. For this, we firstly, calculate the intervals of the α -cut corresponding to inputs, outputs and intermediate measures. Suppose that $\alpha = 0.25$. We consider each real number as (x^l, x^m, x^r). In this case, the α -cut interval is defined as $(x^m - x^l(1-\alpha), x^m + x^r(1-\alpha))$ .Then; we calculate these intervals of inputs, intermediate measures and outputs. Then, we apply these intervals in models (7) and (8). Hence, the lower and upper bounds of the efficiency of the stage1 (leader stage) are reported in Table1:

In Table 1, the first column indicates the number of each DMU. And also, the columns 2 and 3 indicate the lower and the upper bounds of the efficiency of stage1. Intervals of the efficiency of stage1 are reported in column 4. Based on this table, DMU_9, DMU_{13} are efficient in the upper and the lower bound. Hence, thence are efficient. Also, DMU_5 is efficient in the upper bound of the efficiency. Other $DMUs$ are inefficient. Among inefficient $DMUs$, DMU_2 has the worst efficiency score in the upper (and the lower) bound of the efficiency. Also, the best upper (and the best lower) bound of the efficiency is belongs to DMU_{10} . Then, we calculated the efficiencies of the stage 2 (follower stage) while the efficiencies of the leader stage is unchanged. Therefore, the final results are reported in Table 2.

Table 1. The efficiencies of the stage 1

DMU	$\tilde{E}_o^{\mathrm{I}(L)*}$	$\tilde{E}_o^{\mathrm{I}(U)*}$	$[\tilde{E}_o^{\mathrm{I}(L)*}, \tilde{E}_o^{\mathrm{I}(U)*}]$
1	0.4927	0.5673	[0.4927,0.5673]
2	0.2472	0.3830	[0.2472,0.3830]
3	0.3163	0.4514	[0.3163,4514]
4	0.3841	0.4446	[0.3841,0.4446]
5	0.7653	1	[0.7653,1]
6	0.5073	0.6058	[0.5073,0.6058]
7	0.4213	0.5244	[0.4213,0.5244]
8	0.6647	0.7424	[0.6647,0.7424]
9	1	1	[1,1]
10	0.7332	0.8617	[0.7332,0.8617]
11	0.5420	0.5988	[0.5420,0.5988]
12	0.6501	0.8201	[0.6501,0.8201]
13	1	1	[1,1]
14	0.3393	0.4781	[0.3393,0.4781]
15	0.4791	0.4832	[0.4791,0.4832]

Table 2. The efficiencies of the stage 2

DMU	$\tilde{E}_o^{\mathrm{II}(L)*}$	$\tilde{E}_o^{\mathrm{II}(U)*}$	$[\tilde{E}_o^{\mathrm{II}(L)*}, \tilde{E}_o^{\mathrm{II}(U)*}]$	$[\tilde{E}_o^{(L)*}, \tilde{E}_o^{(U)*}]$	α'_{1o}	β_{1o}	β_{2o}
1	0.1740	0.5110	[0.1740,0.5110]	[0.0857,0.2898]	0.4652	0.3673	0.5355
2	0.1866	0.6020	[0.1866,0.6020]	[0.0461,0.2305]	0.3733	0.5892	0.7668
3	0.2321	0.6960	[0.2321,0.6960]	[0.0734,0.3141]	0.7335	0.6627	0.4328
4	0.3147	0.5634	[0.3147,0.5634]	[0.1208,0.1394]	0.5392	0.6472	0.5348
5	0.3136	0.7687	[0.3136,0.7687]	[0.2399,0.7687]	0.5048	0.3746	0.5240
6	0.2479	0.5695	[0.2479,0.5659]	[0.1257,0.3450]	0.6084	0.5733	0.6155
7	0.6386	1	[0.6386,1]	[0.2690,0.5244]	0.4396	0.5918	0.6643
8	0.3946	0.4340	[0.3946,0.4340]	[0.2622,0.3222]	0.5521	0.4150	0.7327
9	0.6719	0.7660	[0.6719,0.7660]	[0.6719,0.7660]	0.6654	0.7342	0.5626
10	0.7522	1	[0.7522,1]	[0.5515,0.8617]	0.4893	0.5520	0.7315
11	0.4993	0.7225	[0.4993,0.7225]	[0.2706,0.4326]	0.7043	0.6620	0.5578
12	0.3436	0.7377	[0.3436,0.7377]	[0.2233,0.6049]	0.5372	0.4597	0.4825
13	0.2362	0.7496	[0.2362,0.7496]	[0.2362,0.7496]	0.6735	0.6509	0.6537
14	0.3501	0.4599	[0.3501,0.4599]	[0.1187,0.2198]	0.5172	0.6036	0.5088
15	0.4275	0.6369	[0.4791,0.6369]	[0.2048,0.3076]	0.3968	0.7138	0.6266

Table 2 indicates the upper and the lower bounds of the efficiency corresponding to stage 2. Note that the upper (and the lower) bound of the efficiency of the whole system is the product of the upper (and the lower) bound of the efficiency of the stages i.e., $\tilde{E}_o^{(L)*} = \tilde{E}_o^{\mathrm{I}(L)*} \times \tilde{E}_o^{\mathrm{II}(L)*}$ and $\tilde{E}_o^{(U)*} = \tilde{E}_o^{\mathrm{I}(U)*} \times \tilde{E}_o^{\mathrm{II}(U)*}$. And also, the whole system is efficient if and only if each of the stages is efficient. Therefore, according to the obtained intervals efficiency of the stage 1 and stage 2, intervals $[\tilde{E}_o^{(L)*}, \tilde{E}_o^{(U)*}]$ are obtained. These results are listed in the column 4 of Table 2. Based on this table, in stage 2 (follower stage) DMU_7, DMU_{10} are efficient in the upper bound. All of other $DMUs$ are inefficient. Between inefficient $DMUs$ in stage2, in the upper bound, the best efficiency and the worst efficiency belongs to DMU_8, DMU_1 with scores 0.7687 and 0.4340, respectively. DMU_1, DMU_{10} have the worst and the best efficiency in the lower bound of the efficiency, with scores

0.1740, 0.7522 respectively. Hence, in stage2, DMU_1 has the worst efficiency in the upper and the lower bound.

Therefore, based on this table, we can conclude that all $DMUs$ are inefficient in the upper and the lower bound of the efficiency in whole system. The highest and the lowest efficiency of the lower bound of the efficiency is belongs to DMU_9 , DMU_2 , respectively. Also, DMU_4, DMU_{10} has the worst and the best efficiency in the upper bound of the efficiency with scores 0.1394 and 0.8617, respectively.

Also, the optimal values of the parameters α'_{1o} , β_{1o} and β_{2o} to determine the portion of each stage in the use of shared inputs and also the portion of the second stage in the use of intermediate measures (produced by the stage1) are listed in columns 5, 6 and 7. Also, in stage2, the value of the parameter α'_{1o} for efficient $DMUs$ in upper bound, DMU_7, DMU_{10} are 0.4396, 0.4893, respectively. Also, for these $DMUs$, the values of the parameters (β_{1o}, β_{2o}) are (0.5918, 0.6643), (0.5520, 0.7315), respectively.

The highest value of the parameter α'_{1o} belongs to DMU_2 . Note that this values for stage 2 is considered as $(1-\alpha'_{1o})$. Also, DMU_9, DMU_1 have the highest and the lowest value of the parameter β_{1o} (Which is the portion of stage1 in the production of intermediate measures as the final outputs), respectively. The highest and lowest values of the parameter β_{2o} belong to DMU_2, DMU_3 . It should be noted that values β_{1o} , β_{2o} for stage2, as the portion of stage 2 in use of the intermediate measures.

5. Conclusion

In real world, there are many production systems with internal structure such as network systems. Two-stage systems as special case of network systems are important in real life applications. For example, each bank branch can be considered as a two-stage system that stage 1 is considered as "attract resources" and stage 2 as" resources allocation". NDEA models are introduced to evaluate the efficiency of these systems in deterministic environment. One of the most important approaches in evaluating the performance of these systems is the cooperative and non-cooperative procedure. Also, in manufacturing processes, observations of inputs and outputs and intermediate measures may be in the form of fuzzy data. Hence, FNDEA models have been introduced to evaluate the performance of two-stage systems in the presence of fuzzy data. TFNs are especially importance due to the simplicity of calculations between fuzzy data. Therefore, in this paper, we focused on TFNs, and $\alpha-$cut approach to evaluate the efficiency of two-stage systems with shared inputs, the part of intermediate measures as inputs of stage2 and additional inputs in stage2. In order to solve the proposed non-linear model, by introducing a non-cooperative approach and assuming that one of the stages is more important (leader stage) from the manager's point of view, the efficiency of this stage was calculated. Since the data are interval, in order to calculate the upper efficiency of the stage1, DMU_o considered in the best condition and other $DMUs$ in the worst conditions. Also, in the calculation of the lower bound of the efficiency, DMU_o was considered in the worst status. Then, we obtained the upper and lower bounds of the efficiency of the follower stage while the efficiency of the leader stage was unchanged. Finally, the geometric mean of the upper (lower) bound efficiency of stages was considered as the upper (lower) bound of the overall efficiency. Finally, we used the data of 15 Chinese industrial sectors [20] to illustrate presented models. For future study, this technique can be applied to two-stage systems in presence of intuitionistic FNs. Also, the proposed approach can also be used in non-radial models to evaluate the efficiency of extended two-stage systems in presence of fuzzy data.

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