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On the Signless Laplacian Eigenvalues and Optimum SLE of Graph

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ABSTRACT

The signelss Laplacian eigenvalues of a graph are the roots of characteristic polynomial of the signless Laplacian matrix of it. The energy of graph is the sum of the absolute value of its eigenvalues. The Laplacian energy of graph is defined the sum of the absolute value of its Laplacian eigenvalues and 2m/n. In this paper, we obtained signless Laplacian spectrum of some special subgraphs of complete graph and then estimated some bounds for signless Laplacian Energy of some graphs.

1. Introduction

Let G be a graph of order n and with the vertex set $\{v_1, v_2, \dots, v_n\}$ and the edge set $E(G)$. The first Zagreb index $M_1(G)$ is defined as $\sum_{e=uv\in E(G)}[d(u)+d(v)]$ where $d(u)$ is degree of vertex v in G [6, 8].

The adjacency matrix of G is an n×n matrix $A(G)$ whose (i,j)-entry is 1 if v_i is adjacent to v_j and 0, otherwise. Assume that $D(G)$ is the $n\times n$ diagonal matrix whose (i,i)-entry is the degree of v_i. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are called the Laplacian matrix and signless Laplacian matrix of G, respectively[1, 2, 5, 7]. Since $A(G)$, $L(G)$ and $Q(G)$ are symmetric matrices, their eigenvalues are real numbers. Let $q_1 \le q_2 \le \cdots \le q_n$be the eigenvalues of $Q(G)$, i.e. the roots of $\varphi(G,q) = \det\big(qI_n - Q(G)\big)$. It is easy to see [3,4]:

$$\sum q_i = 2m\,, \sum {q_i}^2 = 2m + M_1(G).$$

The signless Laplacian energy of G defined as $SLE(G) = \sum_{i=1}^{n}\left|q_i - \frac{2m}{n}\right|$,where n and m are the number of vertices and edges of G respectively. A matching of a graph G is a set of edges without common vertices in G [11, 12].

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At first we need the following theorems:
Theorem 1 (Interalace) [2]**.** Let G be a graph and $e \in E(G)$. Then

$$q_1(G-e) \leq q_1(G) \leq q_2(G-e) \leq q_2(G) \leq \cdots \leq q_n(G-e) \leq q_n(G).$$

Theorem 2 [9]. If a_i and b_i, $1 \leq i \leq n$, are nonnegative real numbers, then

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \frac{n^2}{4}(M_1 M_2 - m_1 m_2),$$
where $M_1 = max_{1\leq i\leq n} a_i$, $M_2 = max_{1\leq i\leq n} b_i$, $m_1 = min_{1\leq i\leq n} a_i$ and $m_2 = min_{1\leq i\leq n} b_i$.

Theorem 3 [10**].** Suppose a_i and b_i, $1 \leq i \leq n$, are positive real numbers. Then

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leq \frac{1}{4}\left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right)^2 \left(\sum_{i=1}^{n} a_i b_i\right)^2,$$
where $M_1 = max_{1\leq i\leq n} a_i$, $M_2 = max_{1\leq i\leq n} b_i$, $m_1 = min_{1\leq i\leq n} a_i$ and $m_2 = min_{1\leq i\leq n} b_i$.

2. Signless Laplacian eigenvalues of some subgraphs of complete graph

In this section, we obtain some new results on the eigenvalues of subgraphs of K_n.
Lemma 1: If G is a subgraph of K_n with $n_1 < n$ vertices such that $K_n - \varepsilon(G)$ is a bipartite graph. Then the signless Laplacian of $K_n - \varepsilon(G)$ are as follows:

$$0, n-q_1(G), n-q_2(G), \ldots, n-q_{n_1}(G), \underbrace{n, \ldots, n}_{n-n_1-1 \text{ times}}$$
Proof: Assume $G_n = G \cup \{V(K_n) - V(G)\}$. Then $\overline{G_n} = K_n - \varepsilon(G)$. By [3] $\varphi(G_n, x) = \varphi(G, x).x^{n-n_1}$ and by [1] $\varphi(\overline{G_n}, x) = (-1)^n \frac{x}{x-n} \varphi(G_n, n-x)$. So

$$\varphi(\overline{G_n}, x) = (-1)^n x\, \varphi(G, n-x).(n-x)^{n-n_1-1}$$
Calculation of the roots of this polynomial implies the lemma.☐

Theorem 4. The signless Laplacian eigenvalues of $K_{n,n} - e$ are computed as follows:

$$0, \frac{3n-2-\sqrt{n^2+4n-4}}{2}, \underbrace{n, \ldots, n}_{n-3 \text{ times}}, \frac{3n-2+\sqrt{n^2+4n-4}}{2}$$
Therefore, $SLE(K_{n,n} - e) = n + 2 - {}^4/_n + \sqrt{n^2+4n-4}$.

Proof: Because of $K_{n,n}$ is a bipartite graph, its Laplacian and signless Laplacian eigenvalues are equal to each other [2], and we now that the signlessLaplacian eigenvalues of $K_{n,n}$ are:

$$0, \underbrace{n, n, \ldots, n}_{n-2 \text{ times}}, 2n$$
By Theorem 1, the signless Laplacian eigenvalues of $K_{n,n} - e$ satisfy the following inequalities:

$$0 = q_1 \leq q_2 \leq q_3 = n = \cdots = q_{2n-1} \leq q_{2n}.$$
Since $\sum q_i = 2m$, $\sum q_i^2 = 2m + M_1(G)$, $q_2 + q_{2n} = 3n-2$, $q_2^2 + q_{2n}^2 = 5n^2 - 4$, $q_2 q_{2n} = 2(n-1)^2$.

This concludes that $q_2 = \frac{3n-2-\sqrt{n^2+4n-4}}{2}$, $q_{2n} = \frac{3n-2+\sqrt{n^2+4n-4}}{2}$.
For the second part, we notice that $SLE\left(K_{n,n} - e\right) = \sum_{i=1}^{2n} \left|q_i - n + {}^{1}/_{n}\right| = n + 2 - {}^{4}/_{n} + \sqrt{n^2 + 4n - 4}$, proving the result.☐

Theorem 5. Suppose G is a graph. Then $|SLE(G-e) - SLE(G)| < 4$ and 4 is the best possible bound.

Proof. Define $q_i' = q_i(G-e)$. By Theorem 1, $q_i - q_i' \geq 0$and $\sum_{i=1}^{n}(q_i - q_i') = 2$. So, there exists i, $1 \leq i \leq n$, such that $q_i - q_i' > 0$. This impolies that

$$\sum\nolimits_{i=1}^{n} \left|q_i - q_i' - \frac{2}{n}\right| < \sum\nolimits_{i=1}^{n} (|q_i - q_i'| + \frac{2}{n})$$

Thus, we have:

$$\begin{aligned}
|SLE(G-e) - SLE(G)| &= \left|\sum_{i=1}^{n} \left|q_i - 2\frac{m}{n}\right| - \left|{q_i}' - 2\frac{m-1}{n}\right|\right| \\
&= \left|\sum_{i=1}^{n} \left(\left|q_i - 2\frac{m}{n}\right| - \left|{q_i}' - 2\frac{m-1}{n}\right|\right)\right| \\
&\leq \sum_{i=1}^{n} \left|\left|q_i - 2\frac{m}{n}\right| - \left|{q_i}' - 2\frac{m-1}{n}\right|\right| \\
&\leq \sum_{i=1}^{n} \left|q_i - q_i' - \frac{2}{n}\right| \\
&< \textstyle\sum_{i=1}^{n} \left(|q_i - q_i'| + \frac{2}{n}\right) = \sum_{i=1}^{n} \left(q_i - q_i' + \frac{2}{n}\right) = 4.
\end{aligned}$$

To complete the argument we construct a sequence $\{G_n\}_{n\geq 2}$ of graphs such that $|SLE(G-e) - SLE(G)| \to 4$. Define $G_n = \overline{K_n} + e$. Then $SLE(G_n) = 4 - \frac{4}{n}$ and $SLE(G_n - e) = 0$ and so $|SLE(G_n - e) - SLE(G_n)| = 4 - \frac{4}{n} \to 4$. This completes the argument.☐

3. Bounds on the signless Laplacian Energy of graphs

In this section, we get some bounds on the signless Laplacian Energy of graphs.

Theorem 6. Suppose zero is not eigenvalue of G. Then

$$E(G) \geq \frac{2\sqrt{2mn}\sqrt{a_1 a_n}}{a_1 + a_n},$$

where a_1 and a_n are minimum and maximum values of the set $\left\{|\lambda_i| \,\middle|\, 1 \leq i \leq n\right\}$. In particular, if G is bipartite and k-regular, then

$$SLE(G) = E(G) \geq \frac{2nk\sqrt{2a_1}}{a_1 + k}.$$

Proof. Suppose $\lambda_i, 1 \leq i \leq n$, are the eigenvalue of G. We also assume that $a_i = |\lambda_i|$, where $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_i = 1, 1 \leq i \leq n$. Apply Theorem 3 to show that

$$\sum_{i=1}^{n}|\lambda_i|^2 \sum_{i=1}^{n} 1^2 \leq \frac{1}{4}\left(\sqrt{\frac{|\lambda_n|}{|\lambda_1|}} + \sqrt{\frac{|\lambda_1|}{|\lambda_n|}}\right)^2 (\sum_{i=1}^{n}|\lambda_i|)^2.$$

Therefore, by $\sum_{i=1}^{n}|\lambda_i|^2 = 2m$ and a simple calculation,

$$E(G) \geq \frac{2\sqrt{2mn}\sqrt{a_1 a_n}}{a_1+a_n}.$$

To prove the second part, it is enough to notice that $a_n = k$ and $2m=nk$ for k-regular graphs. ☐

Theorem 7. Let G be a connected graph nonbipartite with the small stand largest positive signless Laplacian eigenvalues q_1 and q_n, respectively. Then

$$q_n/q_1 + q_1/q_n + 2 \geq \frac{n}{m^2}(2m + M_1(G)).$$

If G is a bipartite graph, then $q_n/q_2 + q_2/q_n + 2 \geq \frac{n-1}{m^2}(2m + M_1(G))$.

where q_2 is the smallest nonzero signless Laplacian eigenvalue of G.

In particular, if G is an n-vertex tree, then

$$q_n/q_2 + q_2/q_n \geq 3.$$

Proof. Suppose $a_i = 1$and $b_i = q_i, 1 \leq i \leq n$. Apply Theorem 2 to show that,

$$\sum 1^2 \sum {q_i}^2 \leq \frac{1}{4}\left(\sqrt{q_n/q_1} + \sqrt{q_1/q_n}\right)^2 (\sum q_i)^2.$$

Since $\sum {q_i}^2 = 2m + M_1(G)$,

$$\sqrt{q_n/q_1} + \sqrt{q_1/q_n} \geq (\sqrt{n}/m)\sqrt{2m + M_1(G)}.$$

Thus

$$q_n/q_1 + q_1/q_n + 2 \geq \frac{n}{m^2}\big(2m + M_1(G)\big).$$

If G is a bipartite graph then $q_1 = 0$. In this case, the signless Laplacian eigenvalues of a graph are equal to Laplacian eigenvalues of that graph. Then by [4] the result is proved. When G is a tree with at least three vertices, $d(u) + d(v) \geq 3$, we have $M_1(G) \geq 3(n-1)$ as desired.☐

Corollary 1. With notation of Theorem 7, when G is an n-vertex tree, then

$$q_n/q_2 + q_2/q_n \geq 4 - \frac{4}{n}.$$

Theorem 8. Suppose G is a graph. Then

$$SLE(G) \geq \sqrt{M_1 + 2m - \frac{4m^2}{n} + 2\binom{n}{2}\sqrt[n]{\phi(G,\frac{2m}{n})^2}}, \qquad (1)$$

With equality if and only if G is an empty graph. In particular, if G is a non-empty graph, then

$$SLE(G) > \sqrt{2m + 2\binom{n}{2}\sqrt[n]{\phi(G,\frac{2m}{n})^2}}. \qquad (2)$$

Moreover, if T is an n-vertex tree, $n \geq 3$, then

$$SLE(G) \geq \sqrt{M_1(T) + \left(\frac{n-1}{n}\right)(-2n+5)}. \quad (3)$$

Proof. Let $x = \sum_{i=1}^{n} \left|q_i - \frac{2m}{n}\right|$.then by the arithmetic mean inequality,

$$\begin{aligned} x^2 &= \sum_{i=1}^{n} \left|q_i - \frac{2m}{n}\right|^2 + 2 \sum_{i \neq j, i,j=1,2,\ldots,n} \left|q_i - \frac{2m}{n}\right| \left|q_j - \frac{2m}{n}\right| \\ &= M_1 + 2m - \frac{8m^2}{n} - \frac{4m^2}{n} + 2 \sum_{i \neq j, i,j=1,2,\ldots,n} \left|q_i - \frac{2m}{n}\right| \left|q_j - \frac{2m}{n}\right| \\ &\geq M_1 + 2m - \frac{4m^2}{n} + 2\binom{n}{2} \left(\prod_{i=1}^{n} \left|q_i - \frac{2m}{n}\right|^{2(n-1)} \right)^{1/n(n-1)} \\ &= M_1 + 2m - \frac{4m^2}{n} + 2\binom{n}{2} \left(\prod_{i=1}^{n} \left(q_i - \frac{2m}{n}\right) \right)^{2/n} \\ &= M_1 + 2m - \frac{4m^2}{n} + 2\binom{n}{2} \sqrt[n]{\phi(G, \tfrac{2m}{n})^2}. \quad (4) \end{aligned}$$

Equation (1) is a direct consequence of (4) with equality if and only if $\left|q_i - 2\frac{m}{n}\right| \left|q_j - 2\frac{m}{n}\right| = \left|q_r - 2\frac{m}{n}\right| \left|q_s - 2\frac{m}{n}\right|, 2 \leq i,j \leq n, 2 \leq r,s \leq n.$

We claim that these equalities hold if and only if $q_2 = q_3 = \cdots = q_n = 2\frac{m}{n}$. To prove this, we assume that one of q_i is equal to $2\frac{m}{n}$.

Then by a simple calculation, $q_1 = q_3 = \cdots = q_n = 2\frac{m}{n}$. Otherwise, $q_i \neq 2\frac{m}{n}$, $1 \leq i \leq n$. If $q_i, q_j \geq 2\frac{m}{n}$or $q_i, q_j \leq 2\frac{m}{n}$, then $q_i = q_j$. Otherwise $q_i + q_j = 4\frac{m}{n}$. Thus signless Laplacian eigenvalues of G are:

$$\underbrace{q_1, \ldots, q_1}_{k_1\ times}, \underbrace{q_2, \ldots, q_2}_{k_2\ times}$$

where $k_1 + k_2 = n$ and $q_1 + q_2 = 4\frac{m}{n}$.

Since $\sum q_i = 2m$, $k_1 q_1 + k_2 q_2 = 2m$, Thus

$$q_1 = 2\frac{m}{n}\left(\frac{2k_1 - n}{2k_1 - n}\right) = 2\frac{m}{n}.$$

Since $q_1 > 2\frac{m}{n}$, a contradiction. Thus in Equation (1) equality holds if and only if $q_1 = q_3 = \cdots = q_n = 2\frac{m}{n}$ if and only if G is an empty graph. It is a well-known fact that $M_1 \geq \frac{4m^2}{n^2}$. The next Equation is now derived from Equation (1) by this inequality.

To prove (3), suppose$\phi\left(G, \frac{2m}{n}\right) = 0$. Thus $\frac{2m}{n}$ is a signless laplacian eigenvalueand so it is an algebraic integer. So by a well-known result in algebraic number theory $\frac{2m}{n}$ is an integer. But for a tree we have $\frac{2m}{n} = \frac{2(n-1)}{n}$, a contradiction. Thus $\phi\left(G, \frac{2m}{n}\right) \neq 0$. Now, suppose $\phi(G,x) = x^n + b_{n-1}x^{n-1} + \cdots + b_1 x$. Then $\phi\left(G, \frac{2m}{n}\right) = \frac{2m}{n}^n + b_{n-1}\frac{2m}{n}^{n-1} + \cdots + b_1\frac{2m}{n} \neq 0.$

This implies that $n^n \left|\phi\left(G, \frac{2m}{n}\right)\right|$ is an integer and so $\left|\phi\left(G, \frac{2m}{n}\right)\right| \geq \frac{1}{n^n}$. This completes the third part of the Theorem.☐

4. Conclusion

In this study, we obtained signless Laplacian spectrum of some special subgraphs of complete graph and then estimated some bounds for signless Laplacian Energy of some graphs. We conclude that $2m/n$ is not a Laplacian characteristic polynomial root. Also, future studies can deal with both stochastic and fuzzy graphs.

Conflict of interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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