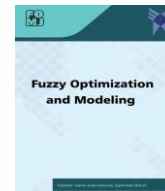




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## New Existence Results for Boundary Value Problems with Integral Conditions

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### ABSTRACT

In this paper, we investigate the existence and uniqueness of solution for fractional boundary value problem for nonlinear fractional differential equation with the integral boundary conditions

$$\begin{cases} {}^{ABC}D_t^\alpha u(t) = f(t, u(t)), & 0 < t < 1, \\ u(0) - \gamma_1 u(1) = \lambda_1 \int_0^1 g(s, u(s)) ds, \\ u'(0) - \gamma_2 u'(1) = \lambda_2 \int_0^1 h(s, u(s)) ds, \end{cases}$$

where  ${}^{ABC}D_t^\alpha$  denotes Caputo derivative of order  $\alpha$  by using the fixed point theory. We apply the contraction mapping principle and Krasnoselskii's fixed point theorem to obtain some new existence and uniqueness results. Two examples are given to illustrate the main results.

## 1. Introduction

In recent years, fractional calculus is one of the interest issues that attract many scientists, specially mathematics and engineering sciences. Many natural phenomena can be present by boundary value problems of fractional differential equations. Many authors in different field such as chemical physics, fluid flows, electrical networks, viscoelasticity, try to modeling of these phenomena by boundary value problems of fractional differential equations [1, 5]. For achieve extra information in fractional calculus, specially boundary value problems, reader can refer to more valuable papers or books that are written by authors [6, 7, 14, 15]. In boundary value problems, one of the most important factors that cause to write different papers is the variety of boundary condition selection. One of these situations is integral boundary conditions. Integral boundary conditions have various applications in applied

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fields such as underground water flow, blood flow problems and population dynamics (see [16, 18] for more details).

In this paper, we wish to survey the existence findings for the new type nonlinear Langevin equation involving two fractional orders as:

$$\begin{cases} {}_0^{ABC} D_t^\alpha u(t) = f(t, u(t)), & 0 < t < 1, \\ u(0) - \gamma_1 u(1) = \lambda_1 \int_0^1 g(s, u(s)) ds, \\ u'(0) - \gamma_2 u'(1) = \lambda_2 \int_0^1 h(s, u(s)) ds, \end{cases} \tag{1}$$

where  $0 < t < 2$ ,  ${}_0^{ABC} D_t^\alpha$  shows the  $\alpha$ -th Atanga- Baleanu fractional derivatives and  $f : [0,1] \rightarrow \mathbb{R}$  is a given continuous function.

**Lemma 1:** Given  $f \in C[0,1]$  and  $1 < \alpha \leq 2$ , the problem (1) is equivalent to

$$u(t) = N_1 + M_3 t + \int_0^1 G(t,s) f(s, u(s)) ds + N_6 \int_0^1 g(s, u(s)) ds + (N_3 + M_5 t) \int_0^1 h(s, u(s)) ds,$$

where

$$G(t,s) = \begin{cases} N_2(1-s)^{\alpha-2} + N_4(1-s)^{\alpha-2} + N_5(1-s)^{\alpha-1} + M_4 t(1-s)^{\alpha-2} + M_1 + M_4(t-s)^{\alpha-1}, & s \leq t \\ N_2(1-s)^{\alpha-2} + N_4(1-s)^{\alpha-2} + N_5(1-s)^{\alpha-1} + M_4 t(1-s)^{\alpha-2}, & s > t \end{cases} \tag{2}$$

$${}^{AB} I^\alpha {}_0^{ABC} D_t^\alpha u(t) = {}^{AB} I^\alpha f(t, u(t)), \quad 0 < t < 1, \tag{3}$$

So, we have

$$u(t) = c_1 + c_2 t + {}^{AB} I^\alpha f(t, u(t)), \quad 0 < t < 1. \tag{4}$$

Letting  $\beta = \alpha - 1$ ,

$$\begin{aligned} ({}^{AB} I^\alpha f)(t) &= I^1 ({}^{AB} I^\beta f)(t) = I^1 \left\{ \frac{1-\beta}{B(\alpha)} f + \frac{\beta}{B(\beta)} (I^\beta f) \right\}(t) \\ &= \frac{2-\alpha}{B(\alpha-1)} \int_0^t f(s) ds + \frac{\alpha-1}{B(\alpha-1)} I^{\beta+1} f(t) \\ &= \frac{2-\alpha}{B(\alpha-1)} \int_0^t f(s) ds + \frac{\alpha-1}{B(\alpha-1)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds. \end{aligned}$$

Therefore,

$$u(t) = c_1 + c_2 t + \frac{2-\alpha}{B(\alpha-1)} \int_0^t f(s, u(s)) ds + \frac{\alpha-1}{B(\alpha-1)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds. \tag{5}$$

By using the boundary condition in (1) we obtain

$$c_1 = N_1 + N_2 \int_0^1 (1-s)^{\alpha-2} f(s, u(s)) ds + N_3 \int_0^1 h(s, u(s)) ds \\ + N_4 \int_0^1 (1-s)^{\alpha-2} f(s, u(s)) ds + N_5 \int_0^1 (1-s)^{\alpha-1} f(s, u(s)) ds + N_6 \int_0^1 g(s, u(s)) ds,$$

and

$$c_2 = M_3 + M_4 \int_0^1 (1-s)^{\alpha-2} f(s, u(s)) ds + M_5 \int_0^1 h(s, u(s)) ds,$$

were

$$M_1 = \frac{2-\alpha}{B(\alpha-1)}, M_2 = \frac{\alpha-1}{B(\alpha-1)\Gamma(\alpha)}, M_3 = \frac{\gamma_2}{(1-\gamma_2)} M f(1, u(1)),$$

$$M_4 = \frac{\gamma_2(\alpha-1)}{(1-\gamma_2)(\alpha-1)\Gamma(\alpha-1)}, M_5 = \frac{\lambda_2}{1-\gamma_2},$$

and

$$N_1 = \frac{\gamma_1}{(1-\gamma_1)} M_3, N_2 = \frac{\gamma_1}{(1-\gamma_1)} M_4, N_3 = \frac{\gamma_1}{(1-\gamma_1)} M_5$$

$$N_4 = \frac{\gamma_1}{(1-\gamma_1)} M_1, N_5 = \frac{\gamma_1}{(1-\gamma_1)} M_2, N_6 = \frac{\lambda_1}{1-\gamma_1}.$$

## 2. Preliminaries

In this section, we study the existence and uniqueness solutions for FBVP (1).

Let  $E = (C[0,1]; \mathbb{R})$  be the Banach space of all continuous functions from  $[0,1]$  into  $\mathbb{R}$  with the norm  $\|u\| = \max_{t \in [0,1]} |u(t)|$ . Define operator  $T : E \rightarrow E$  as

$$Tu(t) = N_1 + M_3 t + \int_0^1 G(t,s) f(s, u(s)) ds + N_6 \int_0^1 g(s, u(s)) ds \quad (6)$$

$$+ (N_3 + M_5 t) \int_0^1 h(s, u(s)) ds, \quad (7)$$

Due to the Lemma 1, problem (1) is converted into a fixed point problem  $u = Tu$ . We behold, the initial problem (1) has solutions if the operator (3) has fixed points.

For convenience of presentation, we now present below hypothesis to be used in the rest of the paper:

$(H_1)$   $f : [0,1] \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuously differentiable function.

$(H_2)$  There exist  $a(t) \in C([0,1], [0, \infty))$  such that

$$i) \int_0^1 (1-s)^{\alpha-i-1} |\alpha(s)| ds < 1 - v_i, \quad i = 1, 2;$$

$$ii) \int_0^t (t-s)^{\alpha-1} |\alpha(s)| ds < 1 - v_3;$$

$$\text{iii) } \int_0^1 |\alpha(s)| ds < 1 - \nu_4.$$

(H<sub>3</sub>) For  $i = 1, 2, 3$

$$\text{i) } R_1 = \sup \int_0^1 (1-s)^{\alpha-1} |f(s, 0)| ds < \infty;$$

$$\text{ii) } R_2 = \sup \int_0^t (t-s)^{\alpha-1} |f(s, 0)| ds < \infty;$$

$$\text{iii) } R_3 = \sup_{t \in [0,1]} \int_0^t (t-s)^{\alpha-1} |f(s, 0)| ds < \infty.$$

(H<sub>4</sub>) For  $u, v \in C[0, \infty]$

$$\text{i) } |f(t, u(t)) - f(t, v(t))| \leq a(t) |u(t) - v(t)|.$$

(H<sub>5</sub>) There exists positive constants  $L_1$  and  $L_2$ , such that

$$\text{i) } |g(t, u(t)) - g(t, v(t))| \leq L_1(1-\nu) |u(t) - v(t)|.$$

$$\text{ii) } |h(t, u(t)) - h(t, v(t))| \leq L_2(1-\nu) |u(t) - v(t)|.$$

where  $u, v \in C[0, \infty]$  and  $\nu = \min \{\nu_i; i = 1, 2, 3, 4\}$ .

(H<sub>6</sub>) There exists positive constants  $p_i \in C([0, 1], \mathbb{R})$  and  $\psi_i \in C([0, 1] \times \mathbb{R}, \mathbb{R})$ , ( $i = 1, 2, 3$ ) such that

$$\text{i) } |f(t, u(t))| \leq p_1(t) \psi_1(|u|);$$

$$\text{ii) } |g(t, u(t))| \leq p_2(t) \psi_2(|u|);$$

$$\text{iii) } |h(t, u(t))| \leq p_3(t) \psi_3(|u|).$$

with  $\psi_i(|u|) \leq \sqrt{|u|}$ , ( $i = 1, 2, 3$ ).

### 3. Main results

In this section, we will establish the existence and uniqueness criteria of solutions for the initial problem (1). Then, the existence and uniqueness results for the fixed problem are obtained respectively via Theorem 3 and Theorem 2.

**Theorem 1:** Assume that (H<sub>1</sub>)–(H<sub>5</sub>) hold. Let

$$\rho_1 = |M_1| + |M_2| + |M_4| + |N_2| + |N_4| + |N_5| + |N_6| L_1 + (|N_3| + |M_5|) L_2$$

$$\rho_2 = |N_1| + |M_3| + (|N_2| + |N_4| + |M_4|) R_1 + |N_5| R_2 + |M_1| K_1 + |M_2| R_3 + |N_6| K_2 + (|N_3| + |M_5|) K_3$$

$(1-\nu)\rho_1 < 1$ . Then, IVP (1) has a unique solution on  $[0, 1]$ .

**Proof:** Consider  $B_\gamma = \{u \in E : \|u\|_* < \gamma\}$  with  $\gamma = \frac{\rho_2}{1-(1-\nu)\rho_1} > 0$ ,  $\forall u \in B_\gamma$ , we get

$$\begin{aligned}
|Tu(t)| &\leq |N_1| + |M_3| + (|N_2| + |M_4|) \left\{ \int_0^1 (1-s)^{\alpha-2} |f(s, u(s)) - f(s, 0)| ds + \int_0^1 (1-s)^{\alpha-2} |f(s, 0)| ds \right\} \\
&\quad + |N_5| \left\{ \int_0^1 (1-s)^{\alpha-1} |f(s, u(s)) - f(s, 0)| ds + \int_0^1 (1-s)^{\alpha-1} |f(s, 0)| ds \right\} \\
&\quad + |M_4| \left\{ \int_0^1 (1-s)^{\alpha-2} |f(s, u(s)) - f(s, 0)| ds + \int_0^1 (1-s)^{\alpha-2} |f(s, 0)| ds \right\} \\
&\quad + |M_1| \left\{ \int_0^t |f(s, u(s)) - f(s, 0)| ds + \int_0^t |f(s, 0)| ds \right\} \\
&\quad + |M_2| \left\{ \int_0^t (t-s)^{\alpha-1} |f(s, u(s)) - f(s, 0)| ds + \int_0^t (t-s)^{\alpha-1} |f(s, 0)| ds \right\} \\
&\quad + |N_6| \left\{ \int_0^1 |g(s, u(s)) - g(s, 0)| ds + \int_0^1 |g(s, 0)| ds \right\} \\
&\quad + (|N_3| + |M_5|) \left\{ \int_0^1 |h(s, u(s)) - h(s, 0)| ds + \int_0^1 |h(s, 0)| ds \right\} \\
&\leq |N_1| + |M_3| + (|N_2| + |N_4| + |M_4|) \|u\| \int_0^1 (1-s)^{\alpha-2} |a(s)| ds \\
&\quad + (|N_2| + |N_4| + |M_4|) \int_0^1 (1-s)^{\alpha-2} |f(s, 0)| ds \\
&\quad + |N_5| \|u\| \int_0^1 (1-s)^{\alpha-1} |a(s)| ds + |N_5| \int_0^1 (1-s)^{\alpha-1} |f(s, 0)| ds \\
&\quad + |M_1| \|u\| \int_0^t |a(s)| ds + |M_1| \int_0^t |f(s, 0)| ds \\
&\quad + |M_2| \|u\| \int_0^t (t-s)^{\alpha-1} |a(s)| ds + |M_2| \int_0^t (t-s)^{\alpha-1} |f(s, 0)| ds \\
&\quad + |N_6| \|u\| \int_0^1 L_1 ds + |N_6| \int_0^1 |g(s, 0)| ds \\
&\quad + (|N_3| + |M_5|) \|u\| \int_0^1 L_2 ds + (|N_3| + |M_5|) \int_0^1 |h(s, 0)| ds \\
&\leq |N_1| + |M_3| + (|N_2| + |N_4| + |M_4|) \|u\| (1-\nu_1) + (|N_2| + |N_4| + |M_4|) R_1 \\
&\quad + |N_5| + |M_2| \|u\| (1-\nu_2) + (|N_5| + |M_2|) R_2 + |M_2| \|u\| \nu_3 + |M_1| R_3 \\
&\quad + |N_6| \|u\| \nu_4 + |N_6| + |M_3| R + (|N_3| + |M_5|) \|u\| \nu_5 + (|N_3| + |M_5|) + |M_4| R_3 \\
&\leq (|M_1| + |M_2| + |M_4| + |N_2| + |N_4| + |N_5| + |N_6| L_1 + (|N_3| + |M_5|) L_2) (1-\nu) \|u\| \\
&\quad + |N_1| + |M_3| + (|N_2| + |N_4| + |M_4|) R_1 + |N_5| R_2 + |M_1| K_1 + |M_2| R_s \\
&\quad + |N_6| K_2 + (|N_3| + |M_5|) K_3
\end{aligned}$$

$$\begin{aligned}
&= (1-\nu) \|u\| \rho_1 + \rho_2 \\
&\leq \gamma,
\end{aligned}$$

which results  $TB_\gamma \subseteq B_\gamma$ . We have

$$\begin{aligned}
|Tu(t) - Tv(t)| &\leq (|N_2| + |N_4|) \int_0^1 (1-s)^{\alpha-2} |f(s, u(s)) - f(s, v(s))| ds \\
&\quad + |N_5| \int_0^1 (1-s)^{\alpha-1} |f(s, u(s)) - f(s, v(s))| ds \\
&\quad + |M_4| t \int_0^1 (1-s)^{\alpha-2} |f(s, u(s)) - f(s, v(s))| ds \\
&\quad + |M_1| \int_0^t |f(s, u(s)) - f(s, v(s))| ds \\
&\quad + |M_2| \int_0^t (t-s)^{\alpha-1} |f(s, u(s)) - f(s, v(s))| ds \\
&\quad + |N_6| \int_0^1 |g(s, u(s)) - g(s, v(s))| ds \\
&\quad + (|N_3| + |M_5| t) \int_0^1 |h(s, u(s)) - h(s, v(s))| ds \\
&\leq (|N_2| + |N_4|) t \int_0^1 (1-s)^{\alpha-2} |a(s)| |u(s) - v(s)| ds \\
&\quad + |N_5| \int_0^1 (1-s)^{\alpha-1} |a(s)| |u(s) - v(s)| ds \\
&\quad + |M_4| t \int_0^1 (1-s)^{\alpha-2} |a(s)| |u(s) - v(s)| ds \\
&\quad + |M_1| \int_0^1 |a(s)| |u(s) - v(s)| ds \\
&\quad + |M_2| \int_0^t (t-s)^{\alpha-1} |a(s)| |u(s) - v(s)| ds \\
&\quad + |N_6| \int_0^1 L_1(1-\nu) |u(s) - v(s)| ds \\
&\quad + (|N_3| + |M_5| t) \int_0^1 L_1(1-\nu) |u(s) - v(s)| ds \\
&\leq \rho_1(1-\nu) \|u - v\|.
\end{aligned}$$

Since  $\rho_1(1-\nu) < 1$ , it yields  $T$  is contraction mapping.  $\square$

**Theorem 2:** Assume that  $(H_1)$ ,  $(H_2)$  and  $(H_4)$ – $(H_6)$  hold, with

$$\rho_1 = (1-\nu) (|M_4| + |N_2| + |N_4| + |N_5| + |N_6| L_1 + (|N_3| + |M_5|) L_2) < 1. \tag{8}$$

Then IVP (1) has at least one solution on  $[0, 1]$ .

**Proof:** Consider  $B_\gamma = \{u \in E; \|u\| < \gamma\}$ , where

$$\sqrt{\gamma} \geq \max \{4\eta_1 \|p_1\|, 4\sqrt{\eta_2}, 4N_6 \|P_2\|, 4\eta_3 \|p_3\|\}$$

Now, we defines operator  $\mathbf{A}$  and  $\mathbf{B}$  on  $B_\gamma$  as

$$\mathbf{A}u(t) = M_1 \int_0^t f(s, u(s)) ds + M_2 \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds$$

$$\mathbf{B}u(t) = N_1 + M_3 t + (N_2 + N_4) \int_0^1 (1-s)^{\alpha-2} f(s, u(s)) ds$$

$$+ N_5 \int_0^1 (1-s)^{\alpha-1} f(s, u(s)) ds + M_4 t \int_0^1 (1-s)^{\alpha-2} f(s, u(s)) ds$$

$$+ N_6 \int_0^1 g(s, u(s)) ds + (|N_3| + |M_5|t) \int_0^1 h(s, u(s)) ds .$$

In first we show that  $\mathbf{A}$  is a compact operator on  $B_\gamma$ .

Since  $f$  is continuous, the operator  $\mathbf{A}$  is continuous

$$\begin{aligned} \|\mathbf{A}u\| &= \sup_{t \in [0,1]} \left| M_1 \int_0^t f(s, u(s)) ds + M_2 \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds \right| \\ &\leq \sup_{t \in [0,1]} M_1 \int_0^t |f(s, u(s))| ds + M_2 \sup_{t \in [0,1]} \int_0^t (t-s)^{\alpha-1} |f(s, u(s))| ds \\ &\leq \sup_{t \in [0,1]} M_1 \int_0^t P_1(s) \psi_1(|u(s)|) ds + M_2 \sup_{t \in [0,1]} \int_0^t (t-s)^{\alpha-1} P_1(s) \psi_1(|u(s)|) ds \\ &\leq M_1 \|P_1\| \psi_1(|u|) + M_2 \frac{\|P_1\| \psi_1(|u|)}{\alpha} \\ &\leq \left( M_1 + \frac{M_2}{\alpha} \right) \|P_1\| \psi_1(|u|) \\ &\leq \left( M_1 + \frac{M_2}{\alpha} \right) \|P_1\| \gamma, \end{aligned}$$

Which show that  $\mathbf{A}$  is uniformly bounded on  $B_\gamma$ .

Form  $(H_1)$ , we define  $\varpi = \sup \{ |f(t, u(t))|; t \in [0,1], u \in B_\gamma \}$ . Now, for  $t_1, t_2 \in (0,1), t_1 < t_2$ , we have

$$\begin{aligned} \|\mathbf{A}u(t_1) - \mathbf{A}u(t_2)\| &= \sup_{t \in [0,1]} \left| M_1 \int_0^{t_2} f(s, u(s)) ds + M_2 \int_0^{t_2} (t_2-s)^{\alpha-1} f(s, u(s)) ds \right. \\ &\quad \left. - M_1 \int_0^{t_1} f(s, u(s)) ds - M_2 \int_0^{t_1} (t_1-s)^{\alpha-1} f(s, u(s)) ds \right| \\ &\leq |M_1| \sup_{t \in [0,1]} \int_{t_1}^{t_2} |f(s, u(s))| ds + |M_2| \sup_{t \in [0,1]} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} |f(s, u(s))| ds \end{aligned}$$

$$\begin{aligned}
& + |M_2| \sup_{t \in [0,1]} \int_0^{t_1} \left( (t-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right) |f(s, u(s))| ds \\
& \leq \varpi |M_1| (t_2 - t_1) + \varpi \frac{|M_2|}{\alpha} (t_2 - t_1)^\alpha + \varpi \frac{|M_2|}{\alpha} \left| t_2^\alpha - t_1^\alpha - (t_2 - t_1)^\alpha \right|
\end{aligned}$$

that tends to zero as  $t_2 \rightarrow t_1$ . Therefore,  $\mathbf{A}$  is relatively compact on  $B_\gamma$ . By the Arzela-Ascoli theorem  $\mathbf{A}$  is compact on  $B_\gamma$ . Now, for  $u, v \in B_\gamma$ , we show that  $\mathbf{A}u + \mathbf{B}u \in B_\gamma$ .

$$\begin{aligned}
\|\mathbf{A}u + \mathbf{B}u\| &= \sup_{t \in [0,1]} \left| M_2 \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds + M_1 \int_0^t f(s, u(s)) ds \right. \\
& + N_1 + M_3 t + (N_2 + N_4) \int_0^1 (1-s)^{\alpha-2} f(s, u(s)) ds \\
& + N_5 \int_0^1 (1-s)^{\alpha-1} f(s, u(s)) ds + M_4 t \int_0^1 (1-s)^{\alpha-2} f(s, u(s)) ds \\
& \left. + N_6 \int_0^1 g(s, u(s)) ds + (N_3 + M_5 t) \int_0^1 h(s, u(s)) ds \right| \\
& + |M_2| \sup_{t \in [0,1]} \int_0^t (t-s)^{\alpha-1} |f(s, u(s))| ds + |M_1| \sup_{t \in [0,1]} \int_0^t |f(s, u(s))| ds \\
& + \sup_{t \in [0,1]} (|N_1| + |M_3|t) + (|N_2| + |N_4|) \int_0^1 (1-s)^{\alpha-2} |f(s, u(s))| ds \\
& + |N_5| \int_0^1 (1-s)^{\alpha-1} |f(s, u(s))| ds + \sup_{t \in [0,1]} |M_4| t \int_0^1 (1-s)^{\alpha-2} |f(s, u(s))| ds \\
& + N_6 \int_0^1 |g(s, u(s))| ds + \sup_{t \in [0,1]} (N_3 + M_5 t) \int_0^1 |h(s, u(s))| ds \\
& \leq \left( \frac{|M_2|}{\alpha} + |M_1| \right) \|p_1\| \psi_1(|u|) + |N_1| + |M_3| + \frac{|N_2| + |N_4|}{\alpha-1} \|p_1\| \psi_1(|u|) \\
& + \frac{|N_5|}{\alpha} \|p_1\| \psi_1(|u|) + \frac{|M_4|}{\alpha-1} \|p_1\| \psi_1(|u|) + |N_6| \|p_2\| \psi_2(|u|) \\
& + (|N_3| + |M_5|) \|p_3\| \psi_3(|u|) \\
& = \left( |M_2| \alpha + |M_1| + \frac{|N_1| + |N_4|}{\alpha-1} + \frac{|N_5|}{\alpha} + \frac{|M_4|}{\alpha-1} \right) \|p_1\| \psi_1(|u|) \\
& + |N_1| + |M_3| + |N_6| \|p_2\| \psi_2(|u|) + (|N_3| + |M_5|) \|p_3\| \psi_3(|u|) \\
& \leq \eta_1 \|p_1\| \sqrt{|u|} + \eta_2 \|p_2\| \sqrt{|u|} + \eta_3 \|p_3\| \sqrt{|u|} \\
& \leq \frac{\gamma}{4} + \frac{\gamma}{4} + \frac{\gamma}{4} + \frac{\gamma}{4} = \gamma.
\end{aligned}$$



Furthermore, from (3), similar to Theorem 3 we conclude that  $\beta$  is contraction mapping. Thus, all assumptions of Krasnoselskii's fixed point theorem are satisfied. Therefore, BVP (1) has at least one solution on  $[0,1]$ .  $\square$

#### 4. Conclusions

In this paper, we have successfully applied the contraction mapping principle and Krasnoselskii's fixed point theorem to obtain some new existence and uniqueness results for fractional integral boundary value problems of Atangana-Baleanu type.

Finally, it should be added that the suggested approach has the potentials to be applied to obtain results for other similar nonlinear problems of fractional order.

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#### References

- Ahmad, B., Nieto, J. J. (2009). Existence results for a coupled system of nonlinear fractional differential equations with three point boundary conditions. *Computers and Mathematics with Applications*, 58 (9), 1838-1843.
- Ahmad, B., Sivasundaram, S. (2010). On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order. *Applied Mathematics and Computation*, 217(2), 480-487.
- Bai, B. (2010). On positive solutions of a nonlocal fractional boundary value problems. *Nonlinear Analysis: Theory, Methods & Applications*, 72(2), 916-924.
- Baleanu, D., Diethelm, K., Scalas, E., Trujillo, J. J. (2012). *Fractional calculus models and numerical methods* (Series on Complexity, Nonlinearity and Chaos). World Scientific.
- Benchohara, M., Hamani, S., Ntouyas, S.K. (2009). Boundary value problems for differential equations with fractional order and nonlocal conditions. *Nonlinear Analysis: Theory, Methods & Applications*, 71(7-8), 2391-2396.
- Boucherif, A. (2009). Second order value problems with integral boundary conditions. *Nonlinear Analysis: Theory, Methods & Applications*, 70(1), 364-371.
- Darzi, R., Mohammadzadeh, B., Neamaty, A., Baleanu, D. (2013). On the existence and uniqueness of solution of a nonlinear fractional differential equations. *Journal of Computer Analysis and Applications*, 15(1), 152-162.
- Ding, Y., Xu, J., Fu, Z. (2019). Positive Solutions for a system of fractional integral boundary value problems of Riemann-Liouville Type involving semipositone nonlinearities. *Mathematics*, 7, 970, doi:10.3390/math7100970.
- Hamani, S., Benchohara, M., Graef, J.R. (2010). Existence results for boundary value problems with nonlinear fractional inclusions and integral conditions. *Electronic Journal of Differential Equations*, 20, 1-16.
- Kilbas, A.A., Srivastava, H.M., Trujillo, J.J. (2006). *Theory and application of fractional differential equations*. Elsevier, Netherlands.
- Liu, X., Jia, M., Wu, B. (2009). Existence and uniqueness of solution for fractional differential equations with integral boundary conditions. *Electronic Journal of Qualitative Theory of Differential Equations*, 69, 1-10.
- Mahmudov, N., Awadalla, M., Abuassba, K. (2017). Hadamard and Caputo-Hadamard FDE's with Three Point Integral Boundary Conditions. *Nonlinear Analysis and Differential Equations*, 5, 271-282.
- Miller, I. (1993). *An introduction to the fractional calculus and fractional differential equation*. John Wiley and Sons, New York.
- Ntouyas, S.K., Etemad, S. (2015). On the existence of solutions for fractional differential inclusions with sum and integral boundary conditions. *Applied Mathematics and Computation*, 266 (1), 235-243.
- Podlubny, J. (1999). *Fractional differential equations*. Academic Press, San Diego.
- Qarout, D., Ahmad, B., Alsaedi, A. (2016). Existence theorems for semi-linear Caputo fractional differential equations with nonlocal discrete and integral boundary conditions. *Fractional Calculus and Applied Analysis*, 19(2), 463-479.
- Ross, B. (Ed.). (1975). The fractional calculus and its application, in: *Lecture Notes in Mathematics*. vol.475, Springer-Verlag, Berlin.
- Salem, H. H. A. (2011). Fractional order boundary value problems with integral boundary condition involving Pettis integral, *Acta Mathematica Scientia*, 31(2) 661-672.
- Wang, G., Pei, K., Agarwal, R.P., Zhang, L., Ahmad, B. (2018). Nonlocal Hadamard fractional boundary value problem with Hadamard integral and discrete boundary conditions on a half-line. *Journal of Computational and Applied Mathematics*, 343 (1), 230-239.
- Zou, Y., He, G. (2017). Existence of solutions to integral boundary value problems of fractional differential equations at resonance. *Journal of Function Spaces*, 2785937, doi.org/10.1155/2017/2785937.



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