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Solving linear and nonlinear Volterra Fuzzy Integral Equations System via Differential Transform

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A B S T R A C T

In this study, we consider solving the second kind Volterra Fuzzy Integral Equations System in two cases of linear and nonlinear by using a semi-analytic method, called Differential Transform Method(DTM). In this algorithm the first we convert a Volterra Fuzzy Integral Equations System into two crisp Integral Equations Systems of Volterra; then we solve each of them via DTM. If the equation has a solution in terms of the series expansion of known functions; this powerful method will catch the exact solution. Moreover the ability and efficiency of the algorithm are shown by solving some numerically examples.

1. Introduction

One of the important subjects in applied mathematics is Integral equations, which applied in several fields of sciences and engineering as: numerical analysis, biology, medical, pharmacy and etc; therefore many Scientists presented several methods to solve Integral equations. Often, in many applications some of the parameters in our problems are represented by fuzzy number rather than crisp, and also it is important to develop mathematical models and numerical procedures that would appropriately treat general fuzzy integral equations and solve them.

The concept of fuzzy number and arithmetic operations on them was first introduced by Zadeh [12], and later this topic was spreader by Mizumoto and Tanaka [18]. Dubios and Prade [8], had an important role in defining of fuzzy number concept and presented of fuzzy computational operations. Later they [9], introduced the fuzzy functions integration which has interested by many several scientists as: Goetschel and Voxman [11], Kalva [14]. Recently several numerical methods have been introduced to solve some models of linear and

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nonlinear fuzzy integral equations by Behzadi et al. [3], Zarebnia [24], Mirzaee et al. [17], Mohseni Moghadam and Saeedi [19], and others [5,6,7,25,26].

The concept of differential transform was first proposed and applied to solve linear and nonlinear equations in electric circuit analysis by Zhou [27]. The DTM obtains an analytical solution in the form of a polynomial. It is different from the traditional high-order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions and is computationally expensive for high order. In the DTM, the derivative is not computed directly; instead, the relative derivatives are calculated by an iteration procedure. New equations are obtained from the original equations by applying differential transform method.

The integral equations systems are important for studying and solving a large proportion of the problems in many topics in applied mathematics, mathematical economics and optimal control theory. Because these integral equations systems are often complex to solve explicitly, hence we require to obtain approximate solutions. Recently many Scientists have applied numerical algorithms to obtain the solution of integral equation systems, such as: Babolian et al. [1] applied the decomposition method to solve Ferdholm integral equations systems, Biazar et al. [4] to obtain solution of Volterra integral equations systems by Adomian method, Maleknejad and Shahrezaee [15] used Runge-Kutta method to solve Volterrea integral equations systems, Yusufoglu [23] applied Homotopy perturbation method to solve Ferdholm-Volterra integral equations system, and other works. Jafarian et al. [13] presented fuzzy form of integral equation systems; then they applied Taylor expansions Method to solve it. Now in this study, we consider solving Fuzzy Volterra Integral Equations System via Differential Transform Method (DTM).

The organization of this paper is as follows: In Section 2, we briefly recall the mathematical foundations of fuzzy calculus and required definitions of fuzzy setting theory; in Section 3 we explain Volterra integral equations system in two cases of crisp and fuzzy; the Differential Transform Method is introduced in Section 4; in Section 5 , we survey error estimation and convergence of the method; in section 6, we illustrate some numerical examples for this method; finally, conclusion are given in the end section.

2. Preliminaries

In this section some the basic notions used in fuzzy calculus are introduced. We start with definition a fuzzy number.

Definition 1. A arbitrary fuzzy number is represented by a fuzzy set $u: R^1 \to [0,1]$ which satisfies into following conditions

a: u is upper semicontinuous.

b: $u(x) = 0$ outside some interval $[c, d]$.

c: There are real numbers a and b, $c \le a \le b \le d$, for which

i) $u(x)$ is monotonically increasing on $[c, a]$,

ii) $u(x)$ is monotonically decreasing on $[b, d]$,

iii) $u(x) = 1$ for $a \le x \le b$.

The set of all fuzzy numbers are denoted by E^1. An alternative definition or parametric form of a fuzzy number which yields the same as E^1 is given by Kaleva [14].

Definition 2. The parametric form of a fuzzy number $\tilde{u}$ for $0 \le r \le 1$ is an ordered pair function $(\underline{u}(r), \overline{u}(r))$, which satisfies the following requirements:

a: $\underline{u}(r)$ is a bounded, continuous, monotonic increasing function,

b: $\overline{u}(r)$ is a bounded, continuous, monotonic decreasing function,

c: $\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$.

$(\underline{u}(r), \overline{u}(r))$, for all $0 \le r \le 1$, are called the r-cut sets of $\tilde{u}$.

Definition 3. For arbitrary Fuzzy numbers $\tilde{u} = (\underline{u}(r), \overline{u}(r))$, $\tilde{v} = (\underline{v}(r), \overline{v}(r))$ and $k > 0$, we define addition, subtraction, scalar product by k and multiplication are as following:

Subtraction:

$$\begin{cases}(\underline{u+v})(r)=\underline{u}(r)+\underline{v}(r),\\(\overline{u+v})(r)=\overline{u}(r)+\overline{v}(r),\end{cases}$$

and

$$\begin{cases}(\underline{u-v})(r)=\underline{u}(r)-\overline{v}(r),\\(\overline{u-v})(r)=\overline{u}(r)-\underline{v}(r),\end{cases}$$

multiplication:

$$(\tilde{u}\tilde{v})(r)=\begin{cases}(\underline{uv})(r)=min\{\underline{u}(r)\underline{v}(r),\underline{u}(r)\overline{v}(r),\overline{u}(r)\underline{v}(r),\overline{u}(r)\overline{v}(r)\},\\(\overline{uv})(r)=max\{\underline{u}(r)\underline{v}(r),\underline{u}(r)\overline{v}(r),\overline{u}(r)\underline{v}(r),\overline{u}(r)\overline{v}(r)\}\end{cases}$$

scalar product:

$$(k\tilde{u})(r)=\{\begin{matrix}(k(\underline{u})(r),k(\overline{u}(r)));k\geq 0,\\(k(\overline{u})(r),k(\underline{u}(r)));k<0.\end{matrix}$$

Definition 4. For arbitrary Fuzzy numbers $\tilde{u}=(\underline{u}(r),\overline{u}(r))$ and $\tilde{v}=(\underline{v}(r),\overline{v}(r))$, we use the distance

$$D(\tilde{u},\tilde{v})=\max\{\sup_{0\leq r\leq 1}|\overline{u}(r)-\overline{v}(r)|,\sup_{0\leq r\leq 1}|\underline{u}(r)-\underline{v}(r)|\},$$

and (E^1,D) is a complete metric space [8].

Definition 5. Take $\tilde{f}:[a,b]\rightarrow E^1$ for each partition $p=\{x_0,x_1,\cdots,x_n\}$ of $[a,b]$ and for arbitrary $\varepsilon_i;x_{i-1}\leq\varepsilon_i\leq x_i$, $1\leq i\leq n$, and

$$\begin{cases}\lambda=\max_{1\leq i\leq n}|x_i-x_{i-1}|,\\R_p=\sum_{i=1}^{n}f(\varepsilon_i)(x_i-x_{i-1}).\end{cases}$$

The definition integral of $\tilde{f}(x)$ over [a,b] is

$$\int_a^b\tilde{f}(x)dx=\lim_{\lambda\rightarrow 0}R_p,$$

provided that this $limit$ exists in the metric D.
We let the parametric form of $\tilde{f}(x)$ be:

$$\tilde{f}(x;r)=(\underline{f}(x,r),\overline{f}(x,r)),$$

Now if the fuzzy function $\tilde{f}(x;r)$ is continuous in the metric D, the definite integral exists [8]; furthermore,

$$\begin{cases}\underline{\int_a^b f(x,r)dx}=\int_a^b\underline{f}(x,r)dx,\\\overline{\int_a^b f(x,r)dx}=\int_a^b\overline{f}(x,r)dx.\end{cases}$$

Moreover, the fuzzy integral can be defined by using the Lebesgue-type approach [9]. In that case $\tilde{f}(x;r)$ is continuous, both approaches yield the same value. More details about the properties of the fuzzy integral are given in [8, 9].

Lemma 1. If $\tilde{f}$ and $\tilde{g}: [a,b] \subseteq R \to E^1$ are fuzzy continuous functions, then the function $F: [a,b] \to R_+$ by $F(x) = D(\tilde{f}(x), \tilde{g}(x))$ is continuous on $[a,b]$, and [19]

$$D(\int_a^b \tilde{f}(x)dx, \int_a^b \tilde{g}(x)dx) \leq \int_a^b D(\tilde{f}(x), \tilde{g}(x))dx.$$

3. Fuzzy Integral equations System

The basic definition of integral equation is given in [14].

Definition 6. The Fredholm integral equation of the second kind is

$$F(x) = G(x) + \lambda(HU)(x), \tag{1}$$

where

$$(HU)(x) = \int_a^b H(s,x)F(s)ds, a \leq x \leq b \tag{2}$$

In (1), $H(s,x)$ is an arbitrary kernel function over the square$a \leq s, x \leq b$ and $G(x)$is a function of x :$a \leq x \leq b$. If $H(s,x) = 0$, $s > x$, we obtain the Volterra integral equation

$$F(x) = G(x) + \lambda \int_a^x H(s,x)F(s)ds, a \leq x \leq b \tag{3}$$

Moreover, if $G(x)$ be a crisp function, then the solution of the above equation is crisp as well. Also if $G(x)$ be a fuzzy function, we have fuzzy integral equation of the second kind which may only process fuzzy solutions. Sufficient conditions for the existence and uniqueness of the solution of the second kind equation, where $G(x)$ is a fuzzy function, are given in [17, 20].

Definition 7. The second kind fuzzy linear Volterra integral equations system is in the form

$$\begin{pmatrix} F_1(x) \\ \vdots \\ F_i(x) \\ \vdots \\ F_m(x) \end{pmatrix} = \begin{pmatrix} G_1(x) + \sum_{j=0}^{m} [\lambda_{1j} \int_a^x U_{1j}(s,x)ds] \\ \vdots \\ G_i(x) + \sum_{j=0}^{m} [\lambda_{ij} \int_a^x U_{ij}(s,x)ds] \\ \vdots \\ G_m(x) + \sum_{j=0}^{m} [\lambda_{mj} \int_a^x U_{mj}(s,x)ds] \end{pmatrix} \tag{4}$$

where $a \leq s \leq x \leq b$ and $\lambda_{ij} \neq 0$ (for $i,j = 1,\dots,m$) are real constants. Moreover, in system (4), the fuzzy function $G_i(x)$ and kernel $H_{ij}(s,x)$ are given and assumed to be sufficiently differentiable with respect to all their arguments on the interval $a \leq s, x \leq b$. Also we assume that the kernel function $H_{ij}(s,x) \in L^2([a,b] \times [a,b])$ and

$$F(x) = \begin{pmatrix} F_1(x) \\ \vdots \\ F_i(x) \\ \vdots \\ F_m(x) \end{pmatrix}$$

is the solution to be determined.

Now let, for $0 \leq r \leq 1$, $a \leq x \leq b$ and $1 \leq i \leq m$ be parametric form of functions:

$$\tilde{F}_i(x;r) = [\underline{F}_i(x;r), \overline{F}_i(x;r)],$$

$$\tilde{G}_i(x;r) = [\underline{G}_i(x;r), \overline{G}_i(x;r)],$$

and

$$\tilde{U}_{ij}(s,x;r) = [\underline{U}_{ij}(s,x;r), \overline{U}_{ij}(s,x;r)].$$

To simplify, we assume that $\lambda_{ij} > 0$ (for $i, j = 1, \ldots, m$). In order to design a numerical scheme for solving (4), we write the parametric form of the given fuzzy integral equations system as follows:

$$\tilde{F}(x;r) = \begin{pmatrix} \tilde{F}_1(x;r) \\ \vdots \\ \tilde{F}_i(x;r) \\ \vdots \\ \tilde{F}_m(x;r) \end{pmatrix} = \begin{pmatrix} [\underline{F}_1(x;r), \overline{F}_1(x;r)] \\ \vdots \\ [\underline{F}_i(x;r), \overline{F}_i(x;r)] \\ \vdots \\ [\underline{F}_m(x;r), \overline{F}_m(x;r)] \end{pmatrix}$$

$$= \begin{pmatrix} [\underline{G}_1(x;r) + \sum_{j=0}^{m} [\lambda_{1j} \int_a^x \underline{U}_{1j}(s,x)ds], \overline{G}_1(x;r) + \sum_{j=0}^{m} [\lambda_{1j} \int_a^x \overline{U}_{1j}(s,x)ds]] \\ \vdots \\ [\underline{G}_i(x;r) + \sum_{j=0}^{m} [\lambda_{ij} \int_a^x \underline{U}_{ij}(s,x)ds], \overline{G}_i(x;r) + \sum_{j=0}^{m} [\lambda_{ij} \int_a^x \overline{U}_{ij}(s,x)ds]] \\ \vdots \\ [\underline{G}_m(x;r) + \sum_{j=0}^{m} [\lambda_{mj} \int_a^x \underline{U}_{mj}(s,x)ds], \overline{G}_m(x;r) + \sum_{j=0}^{m} [\lambda_{mj} \int_a^x \overline{U}_{mj}(s,x)ds]] \end{pmatrix}$$

where for $0 \leq i \leq m$;

$$\underline{U}_{ij} = \begin{cases} H_{ij}(s,x)\underline{F}_j(s), & H_{ij}(s,x) \geq 0, \\ H_{ij}(s,x)\overline{F}_j(s), & H_{ij}(s,x) < 0. \end{cases}$$

and

$$\overline{U}_{ij} = \begin{cases} H_{ij}(s,x)\overline{F}_j(s), & H_{ij}(s,x) \geq 0, \\ H_{ij}(s,x)\underline{F}_j(s), & H_{ij}(s,x) < 0. \end{cases}$$

4. Differential Transform Method

In this section, we consider representing the Differential Transformation Method(DTM). For the function $f(x)$, the differential transformation of Kth is defined as follows [16]:

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0}, \tag{5}$$

where $f(x)$ and $F(k)$ are original function and derivative translate of k-th respectively. The differential inverse transform of $F(k)$ is defined as:

$$f(x) = \sum_{k=0}^{\infty} F(k)(x - x_0)^k, \tag{6}$$

From Equations (5) and (6) we have:

$$f(x) = \sum_{k=0}^{\infty} \left[\frac{(x-x_0)^k}{k!}\frac{d^k f(x)}{dx^k}\right]_{x=x_0}. \tag{7}$$

The following theorems can be deduced form Equations (5), (6) and (7).

Theorem 1.

$$if\ \ f(x) = g(x) \pm h(x) \Longrightarrow F(k) = G(k) \pm H(k).$$

Theorem 2.

$$if\ \ f(x) = ag(x) \Longrightarrow F(k) = aG(k).$$

Theorem 3.

$$if\ \ f(x) = \frac{d^m g(x)}{dx^m} \Longrightarrow F(k) = \frac{(k+m)!}{k!} G(k+m).$$

Theorem 4.

$$if\ \ f(x) = g(x)h(x) \Longrightarrow F(k) = \sum_{i=0}^{k} G(i)H(k-i).$$

Theorem 5.

$$if\ f(x) = x^n \Longrightarrow F(k) = \delta(k-n) = \begin{cases} 1, & k = n\,, \\ 0, & otherwise. \end{cases}$$

Theorem 6.

$$if\ \ f(x) = exp(\lambda x) \Longrightarrow F(k) = \frac{\lambda^k}{k!}.$$

Theorem 7.

$$if\ \ f(x) = sin(wx+\alpha) \Longrightarrow F(k) = \frac{w^k}{k!} sin(\frac{\pi k}{2} + \alpha).$$

Theorem 8.

$$if\ \ f(x) = cos(wx+\alpha) \Longrightarrow F(k) = \frac{w^k}{k!} cos(\frac{\pi k}{2} + \alpha).$$

Theorem 9.

$$if\ \ f(x) = \int_{x_0}^{x} g(t)dt \Longrightarrow F(k) = \frac{G(k-1)}{k}, \forall k \geq 1, and F(0) = 0.$$

Theorem 10.

$$if\ \ f(x) = \int_{x_0}^{x} g(t)h(t)dt \Longrightarrow F(k) = \sum_{r=0}^{k-1} G(r) \frac{H(k-r-1)}{k}, F(0) = 0.$$

Theorem 11.

$$if\ \ f(x) = g(x) \int_{x_0}^{x} h(t)dt \Longrightarrow F(k) = \sum_{r=0}^{k-1} G(r) \frac{H(k-r-1)}{k-r}, F(0) = 0.$$

Theorem 12. If

$$f(x) = (g_1(x)g_2(x)\cdots g_{n-1}(x)g_n(x)) \int_{x_0}^{x} h_1(t)h_2(t)\cdots h_{m-1}(t)h_m(t)dt,$$

then:

$$F(k) = \sum_{k_{m+n-1}=1}^{k} \sum_{k_{m+n-2}=1}^{k_{m+n-1}} \cdots \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} \frac{1}{k_m} H_1(k_1-1)H_2(k_2-k_1) \times \cdots$$
$$\times H_{m-1}(k_{m-1}-k_{m-2})H_m(k_m-k_{m-1})G_1(k_{m+1}-k_m)G_2(k_{m+2}-k_{m+1}) \times \cdots$$
$$\times G_{n-1}(k_{m+n-1}-k_{m+n-2})G_n(k-k_{m+n-1}).$$

5. Error Estimation

We consider the general form of a Fuzzy integral equations system, for $0 \leq r \leq 1$ as:

$$\tilde{F}(x;r) = \begin{pmatrix} \tilde{F}_1(x;r) \\ \vdots \\ \tilde{F}_n(x;r) \end{pmatrix}$$

where

$$\begin{cases} \tilde{F}_1(x;r) = \tilde{G}_1(x;r) + \lambda_1 \int_0^x H_1(x,s)\tilde{F}_1(s;r)ds, \\ \vdots \\ \tilde{F}_m(x;r) = \tilde{G}_m(x;r) + \lambda_m \int_0^x H_m(x,s)\tilde{F}_m(s;r)ds. \end{cases} \qquad (8)$$

Theorem 13. Let $H_1(x,s),\dots,H_m(x,s)$ have been continuous for $a \leq x,s \leq b$, and $\tilde{G}_1(x;r),\dots,\tilde{G}_m(x;r)$ are continuous of x, $a \leq x \leq b$, respectively. If

$$\begin{cases} \lambda_1 < \dfrac{1}{M_1(b-a)}, \\ \vdots \\ \lambda_m < \dfrac{1}{M_m(b-a)}, \end{cases}$$

where

$$\begin{cases} M_1 = \max\limits_{a\leq x,s\leq b} |H_1(x,s)|, \\ \vdots \\ M_m = \max\limits_{a\leq x,s\leq b} |H_m(x,s)|, \end{cases}$$

then the iterative procedure,

$$if k = 0 \Longrightarrow \begin{cases} \tilde{F}_{1,0}(x;r) = \tilde{G}_1(x;r), \\ \vdots \\ \tilde{F}_{m,0}(x;r) = \tilde{G}_m(x;r), \end{cases}$$

$$for \quad k > 0 \Longrightarrow \begin{cases} \tilde{F}_{1,k}(x;r) = \tilde{G}_1(x;r) + \lambda_1 \displaystyle\int_0^x H_1(x,s)\tilde{F}_{1,(k-1)}(s;r)ds, \\ \vdots \\ \tilde{F}_{m,k}(x;r) = \tilde{G}_m(x;r) + \lambda_m \displaystyle\int_0^x H_m(x,s)\tilde{F}_{m,(k-1)}(s;r)ds, \end{cases}$$

converges to the unique solution of (8). Specially,

$$\begin{cases} \sup\limits_{a\leq x\leq b} D(\tilde{F}_1(x;r),\tilde{F}_{1,k}(x;r)) \leq \dfrac{{L_1}^k}{1-L_1} \sup\limits_{a\leq x\leq b} D(\tilde{F}_{1,0}(x;r),\tilde{F}_{1,1}(x;r)), \\ \vdots \\ \sup\limits_{a\leq x\leq b} D(\tilde{F}_m(x;r),\tilde{F}_{m,k}(x;r)) \leq \dfrac{{L_2}^k}{1-L_m} \sup\limits_{a\leq x\leq b} D(\tilde{F}_{m,0}(x;r),\tilde{F}_{m,1}(x;r)), \end{cases}$$

where

$$\begin{cases} L_1 = \lambda_1 M_1(b-a), \\ \vdots \\ L_m = \lambda_m M_m(b-a), \end{cases}$$

Throughout this paper, we consider Volterra integral Equation (8) with $\lambda_1,\dots,\lambda_m > 0$, where

$$\begin{cases}\tilde{F}_1(x;r), \tilde{G}_1(x;r) \in {L_1}^2[0,1),\\ \vdots \\ \tilde{F}_m(x;r), \tilde{G}_2(x;r) \in {L_m}^2[0,1),\end{cases}$$

and

$$\begin{cases}H_1(x,s) \in {L_1}^2([0,1)\times[0,1)),\\ \vdots \\ H_m(x,s) \in {L_m}^2([0,1)\times[0,1)),\end{cases}$$

(coefficients of $\tilde{F}_1(x;r),\dots,\tilde{F}_m(x;r)$ in the interval $[0,1)$ from the know functions $\tilde{F}_1(x;r),\dots,\tilde{F}_m(x;r)$ and kernel $H_1(x;s),\dots,H_m(x;s)$).

6. Numerical examples

In this section, we present the application of the Differential Transform Method (DTM) to solve linear and nonlinear Fuzzy Integral equations systems.

Example 1. We consider the linear Fuzzy Integral equations system

$$\begin{cases}\tilde{F}_1(x;r) = \tilde{G}_1(x;r) + x\tilde{F}_2(x;r) + 2\int_0^x [x\tilde{F}_1(s;r) + \tilde{F}_2(s;r)]ds,\\ \tilde{F}_2(x;r) = \tilde{G}_2(x;r) - \frac{1}{2}(x^2+x)\tilde{F}_1(x;r) - \frac{1}{2}\int_0^x [\tilde{F}_1(s;r) - \tilde{F}_2(s;r)]ds,\end{cases}$$

where

$$\tilde{G}_1(x;r): \begin{cases}\underline{G}_1(x;r) = -\frac{2}{3}r^5x^4 - \frac{4}{3}rx^4 + r^5x^2 - 4rx^2 + 2x^2,\\ \overline{G}_1(x;r) = 2r^3x^4 - 4x^4 - 3r^3x^2 + 2rx^2,\end{cases}$$

and

$$\tilde{G}_2(x;r): \begin{cases}\underline{G}_2(x;r) = \frac{1}{2}r^5x^4 + rx^4 + \frac{2}{3}r^5x^3 + \frac{4}{3}rx^3 - \frac{3}{4}rx^2 + \frac{1}{4}x^2 + 3rx - x,\\ \overline{G}_2(x;r) = -\frac{3}{2}r^3x^4 + 3x^4 - 2r^3x^3 + 4x^3 + \frac{1}{4}rx^2 - \frac{3}{4}x^2 + 3x - rx.\end{cases}$$

The exact solution, for $0 \le r \le 1$, by using direct method, is

$$\tilde{F}_{exa.}(x;r) = \begin{pmatrix}\tilde{F}_1(x;r)\\ \tilde{F}_2(x;r)\end{pmatrix} = \begin{pmatrix}[\underline{F}_1(x;r), \overline{F}_1(x;r)]\\ [\underline{F}_2(x;r), \overline{F}_2(x;r)]\end{pmatrix} = \begin{pmatrix}[r^5x^2 + 2rx^2, -3r^3x^2 + 6x^2]\\ [3rx - x, -rx + 3x]\end{pmatrix}$$

First, we calculate the solution of $\underline{F_1}(k)$ and $\underline{F_2}(k)$ where:

$$\begin{cases} \underline{F_1}(x;r) = \underline{G_1}(x;r) + x\underline{F_2}(x;r) + 2\int_0^x [x\underline{F_1}(s;r) + \underline{F_2}(s;r)]ds, \\ \underline{F_2}(x;r) = \underline{G_2}(x;r) - \frac{1}{2}(x^2+x)\underline{F_1}(x;r) - \frac{1}{2}\int_0^x [\underline{F_1}(s;r) - \underline{F_2}(s;r)]ds, \end{cases}$$

and

$$\begin{cases} \underline{G_1}(x;r) = -\frac{2}{3}r^5x^4 - \frac{4}{3}rx^4 + r^5x^2 - 4rx^2 + 2x^2, \\ \underline{G_2}(x;r) = \frac{1}{2}r^5x^4 + rx^4 + \frac{2}{3}r^5x^3 + \frac{4}{3}rx^3 - \frac{3}{4}rx^2 + \frac{1}{4}x^2 + 3rx - x. \end{cases}$$

We have by using theorems of DTM, the following relation for $\underline{F_1}(k)$ and $\underline{F_2}(k)$:

$$\begin{cases} \underline{F_1}(k) = \underline{G_1}(k) + \sum_{h=0}^{k} \delta(h-1)\underline{F_2}(k-h) + 2\sum_{h=0}^{k-1} \delta(h-1)\frac{\underline{F_1}(k-h-1)}{k-h} + 2\frac{\underline{F_2}(k-1)}{k}, \\ \underline{F_2}(k;r) = \underline{G_2}(k;r) - \frac{1}{2}\sum_{h=0}^{k} \delta(h-2)\underline{F_1}(k-h) - \frac{1}{2}\sum_{h=0}^{k} \delta(h-1)\underline{F_2}(k-h) - \frac{1}{2}\frac{\underline{F_1}(k-1)}{k} + \frac{1}{2}\frac{\underline{F_2}(k-1)}{k}, \end{cases}$$

where:

$$\begin{cases} \underline{G_1}(k) = -\frac{2}{3}r^5\delta(k-4) - \frac{4}{3}r\delta(k-4) + r^5\delta(k-2) - 4r\delta(k-2) + 2\delta(k-2), \\ \underline{G_2}(k) = \frac{1}{2}r^5\delta(k-4) + r\delta(k-4) + \frac{2}{3}r^5\delta(k-3) + \frac{4}{3}r\delta(k-3) - \frac{3}{4}r\delta(k-2) + \frac{1}{4}\delta(k-2) + 3r\delta(k-1) - \delta(k-1), \end{cases}$$

Consequently, we find:

$$\begin{cases} \underline{F_1}(0) = 0, \\ \underline{F_2}(0) = 0, \end{cases} \quad \begin{cases} \underline{F_1}(1) = 0, \\ \underline{F_2}(1) = 3r-1, \end{cases} \quad \begin{cases} \underline{F_1}(2) = r^5+2r, \\ \underline{F_2}(2) = 0, \end{cases} \quad \begin{cases} \underline{F_1}(k) = 0; for: k \geq 3, \\ \underline{F_2}(k) = 0; for: k \geq 3. \end{cases}$$

Then, by using Equation (6), we obtain the approximate solution:

$$\begin{cases} \underline{F_1}(x;r) = (r^5+2r)x^2, \\ \underline{F_2}(x;r) = (3r-1)x, \end{cases} \tag{9}$$

We can to solve $\overline{F}_1(x;r)$,$\overline{F}_2(x;r)$ similar to $\underline{F_1}(x;r)$, $\underline{F_2}(x;r)$ respectively; then, we find:

$$\begin{cases} \overline{F}_1(0) = 0, \\ \overline{F}_2(0) = 0, \end{cases}$$

$$\begin{cases} \overline{F}_1(1) = 0, \\ \overline{F}_2(1) = 3-r, \end{cases}$$

$$\begin{cases}\overline{F}_1(2) = -3r^3 + 6,\\ \overline{F}_2(2) = 0,\end{cases}$$

$$\begin{cases}\overline{F}_1(k) = 0; for: k \geq 3,\\ \overline{F}_2(k) = 0; for: k \geq 3.\end{cases}$$

Then, by using Equation (6), we obtain the approximate solution:

$$\begin{cases}\overline{F}_1(x;r) = (6 - 3r^3)x^2,\\ \overline{F}_2(x;r) = (3 - r)x,\end{cases} \tag{10}$$

Therefore by using (9) and (10), we have:

$$\tilde{F}_{app.}(x;r) = \begin{pmatrix}\tilde{F}_1(x;r)\\ \tilde{F}_2(x;r)\end{pmatrix} = \begin{pmatrix}[r^5x^2 + 2rx^2, -3r^3x^2 + 6x^2]\\ [3rx - x, -rx + 3x]\end{pmatrix}$$

Thus, we obtain:

$$\tilde{F}_{app.}(x;r) = \tilde{F}_{exa.}(x;r).$$

Example 2. We consider the following nonlinear Fuzzy Integral equations system:

$$\begin{cases}\tilde{F}_1(x;r) = \tilde{G}_1(x;r) + \int_0^x [(\tilde{F}_1(s;r))^2 + (\tilde{F}_2(s;r))^3]ds,\\ \tilde{F}_2(x;r) = \tilde{G}_2(x;r) + \int_0^x [(\tilde{F}_1(s;r))^3 - (\tilde{F}_2(s;r))^2]ds,\end{cases}$$

where

$$\tilde{G}_1(x;r): \begin{cases}\underline{G}_1(x;r) = rx^2 - \frac{1}{5}r^2x^5 - \frac{1}{10}r^3x^{10},\\ \overline{G}_1(x;r) = 2x^2 - rx^2 - \frac{4}{5}x^5 + \frac{4}{5}rx^5 - \frac{1}{5}r^2x^5 - \frac{27}{10}x^{10} + \frac{27}{5}rx^{10} - \frac{18}{5}r^2x^{10} + \frac{4}{5}r^3x^{10},\end{cases}$$

and

$$\tilde{G}_2(x;r): \begin{cases}\underline{G}_2(x;r) = rx^3 - \frac{1}{7}r^3x^7 + \frac{1}{7}r^2x^7,\\ \overline{G}_2(x;r) = 3x^3 - 2rx^3 + \frac{1}{7}x^7 - \frac{2}{7}r^2x^7 + \frac{1}{7}r^3x^7.\end{cases}$$

The exact solution, for $0 \le r \le 1$, is

$$\tilde{F}_{exa.}(x;r) = \begin{pmatrix} \tilde{F}_1(x;r) \\ \tilde{F}_2(x;r) \end{pmatrix} = \begin{pmatrix} [\underline{F}_1(x;r), \overline{F}_1(x;r)] \\ [\underline{F}_2(x;r), \overline{F}_2(x;r)] \end{pmatrix} = \begin{pmatrix} [rx^2, 2x^2 - rx^2] \\ [rx^3, 3x^3 - 2rx^3] \end{pmatrix}$$

First we calculate the solution of $\underline{F}_1(k)$ and $\underline{F}_2(k)$ where:

$$\begin{cases} \underline{F}_1(x;r) = \underline{G}_1(x;r) + \int_0^x [(\underline{F}_1(s;r))^2 + (\underline{F}_2(s;r))^3]ds, \\ \underline{F}_2(x;r) = \underline{G}_2(x;r) + \int_0^x [(\underline{F}_1(s;r))^3 - (\underline{F}_2(s;r))^2]ds, \end{cases}$$

and

$$\begin{cases} \underline{G}_1(x;r) = rx^2 - \frac{1}{5}r^2x^5 - \frac{1}{10}r^3x^{10}, \\ \underline{G}_2(x;r) = rx^3 - \frac{1}{7}r^3x^7 + \frac{1}{7}r^2x^7. \end{cases}$$

We have by using theorems of DTM, the following relation for $\underline{F}_1(k)$ and $\underline{F}_2(k)$:

$$\begin{cases} \underline{F}_1(k) = \underline{G}_1(k) + \sum_{h=0}^{k-1} \underline{F}_1(h)\frac{\underline{F}_1(k-h-1)}{k} + \frac{1}{k}\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2} \underline{F}_1(k_1)\underline{F}_1(k_2-k_1)\underline{F}_1(k-k_2-1), \\ \underline{F}_2(k;r) = \underline{G}_2(k;r) + \frac{1}{k}\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2} \underline{F}_1(k_1)\underline{F}_1(k_2-k_1)\underline{F}_1(k-k_2-1) - \sum_{h=0}^{k-1} \underline{F}_2(h)\frac{\underline{F}_2(k-h-1)}{k}, \end{cases}$$

where:

$$\begin{cases} \underline{G}_1(k) = r\delta(k-2) - \frac{1}{5}r^2\delta(k-5) + \frac{1}{10}r^3\delta(k-10), \\ \underline{G}_2(k) = r\delta(k-3) - \frac{1}{7}r^3\delta(k-7) + \frac{1}{7}r^2\delta(k-7), \end{cases}$$

Consequently, we find:

$$\begin{cases} \underline{F}_1(0) = 0, \\ \underline{F}_2(0) = 0,' \end{cases} \begin{cases} \underline{F}_1(1) = 0, \\ \underline{F}_2(1) = 0,' \end{cases} \begin{cases} \underline{F}_1(2) = r, \\ \underline{F}_2(2) = 0,' \end{cases} \begin{cases} \underline{F}_1(3) = 0, \\ \underline{F}_2(3) = r,' \end{cases} \begin{cases} \underline{F}_1(k) = 0; for: k \ge 4, \\ \underline{F}_2(k) = 0; for: k \ge 4. \end{cases}$$

Then, by using Equation (6), we obtain the approximate solution:

$$\begin{cases} \underline{F}_1(x;r) = rx^2, \\ \underline{F}_2(x;r) = rx^3, \end{cases} \tag{11}$$

We can solve $\overline{F}_1(x;r)$,$\overline{F}_2(x;r)$ similar to $\underline{F}_1(x;r)$, $\underline{F}_2(x;r)$ respectively; then, we find:

$$\begin{cases}\overline{F}_1(0)=0,\\ \overline{F}_2(0)=0,\end{cases}$$

$$\begin{cases}\overline{F}_1(1)=0,\\ \overline{F}_2(1)=0,\end{cases}$$

$$\begin{cases}\overline{F}_1(2)=2-r,\\ \overline{F}_2(2)=0,\end{cases}$$

$$\begin{cases}\overline{F}_1(3)=0,\\ \overline{F}_2(3)=3-2r,\end{cases}$$

$$\begin{cases}\overline{F}_1(k)=0; for: k\geq 4,\\ \\ \overline{F}_2(k)=0; for: k\geq 4.\end{cases}$$

Then, by using Equation (6), we obtain the approximate solution:

$$\begin{cases}\overline{F}_1(x;r)=(2-r)x^2,\\ \\ \overline{F}_2(x;r)=(3-2r)x^3,\end{cases} \tag{12}$$

Therefore by using (11) and (12), we have:

$$\tilde{F}_{app.}(x;r)=\begin{pmatrix}\tilde{F}_1(x;r)\\ \\ \tilde{F}_2(x;r)\end{pmatrix}=\begin{pmatrix}[rx^2, 2x^2-rx^2]\\ \\ [rx^3, 3x^3-2rx^3]\end{pmatrix}$$

Thus, we obtain:

$$\tilde{F}_{app.}(x;r)=\tilde{F}_{exa.}(x;r).$$

7. Conclusion

In this paper, a linear and nonlinear fuzzy integral equations system of Volttera was converted into two linear and nonlinear crisp Integral Equations Systems of Volttera; then we successfully applied Differential Transform Method to obtain the solution of each of the systems. Differential transform method is different from the traditional high-order Taylor series method, which requires the symbolic computation of necessary derivatives of the data function and is computationally expensive for higher-order. The present technique in comparison with other numerical and traditional methods has the efficiency that all of the calculations can be made by simple manipulations; moreover presented examples in this article are authenticated that the DTM requires the least calculations among other algorithms; also another main advantage of this method is solving an integral equations system without using any integration. As a result, the DTM can be applied to solve many complicated linear and nonlinear integral equations system.

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